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NONBOUNDING n-C.E. Q-DEGREES

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Abstract. We prove that for any noncomputable c.e. set A there is a noncomputable c.e. set $B \subseteq A$ such that for every noncomputable c.e. set W we have $W \not\leq_Q A - B <_Q A$. We show that if c.e. Q-degrees $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair in the c.e. Q-degrees, then $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair in the Σ_2^0 Q-degrees.

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In this paper we study the Q-degrees of n-computable enumerable (n-c.e.) sets. Tennenbaum (as quoted by Rogers [8, p. 159]) defined the notion of Q-reducibility on sets of natural numbers as follows: a set A is Q-reducible to a set B (written as $A \leq_Q B$) if there is a computable function f such that for every $x \in \omega$ (where ω denotes the set of natural numbers),

$$x \in A \Longleftrightarrow W_{f(x)} \subseteq B.$$

In this case we say that $A \leq_Q B$ via f. The relation of Q-reducibility is reflexive and transitive, so that it generates a degree structure on the subsets of ω. It is not difficult to show that in general Q-reducibility is incomparable with Turing reducibility $\leq_T$. On c.e. sets we have that if $A \leq_Q B$, then $A \leq_T B$; the converse implication does not always hold: this easily follows from the observation that if $A \leq_Q B$, then $\overline{A}$ is c.e. in B, where $\overline{A}$ denotes the complement of A.

Our notation and terminology are standard, and can be found e.g. in [8] or [10].

A set A is n-c.e. if there is a computable function $f(s,x)$ such that for every x:

$$f(0,x) = 0,$$
$$A(x) = \lim_s f(s,x),$$
$$\big|\{s : f(s,x) \neq f(s+1,x)\}\big| \leq n.$$

Here the symbol $|X|$ denotes the cardinality of a given set X. The 2-c.e. sets are also known as d-c.e. sets as they are differences of c.e. sets.

A degree $\mathbf{a}$ is called an n-c.e. degree for $n \geq 1$ if it contains an n-c.e. set, and it is called a properly n-c.e. degree if it contains an n-c.e. set but no m-c.e. set for any $m < n$.

It is known [1] that in n-c.e. sets (even for the case $n = 2$) T-reducibility is incomparable with Q-reducibility. Therefore, the development of the structural

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theory of Q-degrees of n-c.e. sets compared to their T-degrees becomes one of the interesting directions in the study of Q-degrees of n-c.e. sets.

It is well-known that for any n-c.e. set ($n > 1$) A of properly n-c.e. T-degree there exists an $(n-1)$-c.e. set B such that $B <_T A$ (this is called Lachlan's proposition). Below in Theorem 1 we show that for any noncomputable c.e. set A there is a noncomputable c.e. set $B \subseteq A$ such that the Q-degree of $A - B$ bounds no noncomputable c.e. Q-degrees.

Theorem 1. *For any noncomputable c.e. set A there is a noncomputable c.e. set $B \subseteq A$ such that for every noncomputable c.e. set W we have*

$$W \nleq_Q A - B <_Q A.$$

The proof of this theorem is based on the following

Lemma 1. *Let A be a noncomputable c.e. set, B be an immune set, then $A \nleq_Q B$.*

Proof. Let A, B be as in the statement of the lemma, and let f be a computable function such that for any x

$$x \in A \Longleftrightarrow W_{f(x)} \subseteq B.$$

Then the c.e. set $\bigcup_{x \in A} W_{f(x)}$ is a subset of B. By the immunity of the set B we have

$$\Big| \bigcup_{x \in A} W_{f(x)} \Big| < \infty.$$

Then the set

$$R = \overline{\bigcup_{x \in A} W_{f(x)}},$$

is computable and

$$\overline{A} = \big\{x : \ W_{f(x)} \cap R \neq \varnothing \big\}.$$

Therefore $\overline{A}$ is a c.e. set, a contradiction. □

Definition ([5]). Given c.e. sets $B \subseteq A$, B is a major subset of A (written $B \subset_m A$) if $A - B$ is infinite and for every c.e. set W,

$$\overline{A} \subseteq^* W \Longrightarrow \overline{B} \subseteq^* W.$$

Lachlan [5] proved that for every noncomputable c.e. set A there exists a c.e. set B such that $B \subset_m A$.

If $B \subset_m A$, then $A - B$ is an immune set. Indeed, assume that $B \subset_m A$ and $A - B$ is not immune. Let W be an infinite c.e. set such that $W \subseteq A - B$. Choose any infinite computable set $R \subseteq W$. Then $\overline{A} \subset^* \overline{R}$ and $\overline{B} \not\subset^* \overline{R}$, a contradiction.

We are now ready to finish the proof of Theorem 1.

Proof of Theorem 1. Let A be any noncomputable c.e. set and, by the above remark, let $B \subseteq A$ be a c.e. set such that $A - B$ is immune, then by Lemma 1, for any noncomputable c.e. set W we have $W \not\leq_Q A - B$. By [1, Theorem 1] $A - B <_Q A$. □

It follows from Theorem 1 that the partial orderings of T- and Q-degrees of d-c.e. sets are different since by Lachlan's proposition each noncomputable d-c.e. set in T-degrees bounds some noncomputable c.e. sets.

Corollary 1. *For any noncomputable c.e. Q-degree* $\mathbf{a}$ *there is a properly d-c.e. Q-degree* $\mathbf{b} < \mathbf{a}$ *which bounds in Q-degrees no noncomputable c.e. Q-degrees.*

Proof. Immediate. □

Let $A - B$ be a d-c.e. set with c.e. sets $B \subseteq A$. Then any splitting of A into two disjoint c.e. sets A_0 and A_1 splits $A - B$ into two d-c.e. $A_0 - B$ and $A_1 - B$. In [2, Proposition 2] it is proved that both of these d-c.e. sets are Q-reducible to $A - B$, and in [2, Theorem 3] it is shown that we can choose A_0 and A_1 so that $A_0 - B$ is Q-incomparable with $A_1 - B$. It follows from this that any nonzero d-c.e.Q-degree bounds a spittable d-c.e. degree [2, Corollary 5].

It is proved in [1] that for any $n \geq 2$ there is a $(2n)$-c.e. set M of properly $(2n)$-c.e. Q-degree such that for any c.e. set W, if $W \leq_Q M$, then W is computable.

Corollary 2. *For any $n \geq 1$ and noncomputable c.e. set A there are noncomputable c.e. sets $A = A_1 \supseteq \cdots \supseteq A_{2n} \supseteq A_{2n+1}$ such that if*

$$M = (A_1 - A_2) \cup \cdots \cup (A_{2n-1} - A_{2n})$$

and

$$N = (A_1 - A_2) \cup \cdots \cup (A_{2n-1} - A_{2n}) \cup A_{2n+1},$$

then

(a) *for any c.e. set W it follows from $W \leq_Q M$ that W is computable;*

(b) *there is a noncomputable c.e. set W such that $W \leq_Q N$;*

(c) *$M \leq_Q N$ and $N \not\leq_Q M$.*

Proof. Let A be a noncomputable c.e. set, $n \geq 1$, and let $A = A_1 \supseteq \cdots \supseteq A_{2n} \supseteq A_{2n+1}$, where each A_i is c.e. and each $A_{2i-1} - A_{2i}$, is immune, and take

$$M = (A_1 - A_2) \cup \cdots \cup (A_{2n-1} - A_{2n})$$

and

$$N = (A_1 - A_2) \cup \cdots \cup (A_{2n-1} - A_{2n}) \cup A_{2n+1}.$$

(a) By Lemma 1, for the proof it is enough to show that M is immune. Suppose that there is an infinite c.e. set E such that $E \subseteq M$. Then there is a greatest i, $1 \leq i \leq n$, such that $E \cap (A_{2i-1} - A_{2i})$ is infinite and

$$F = E \cap \left[(A_{2i+1} - A_{2i+2}) \cup \cdots \cup (A_{2n-1} - A_{2n})\right]$$

is finite. Then

$$E \cap (A_{2i-1} - A_{2i}) = (E \cap A_{2i-1}) - F,$$

i.e. $E \cap (A_{2i-1} - A_{2i})$ is an infinite c.e. subset of the immune set $A_{2i-1} - A_{2i}$, a contradiction.

(b) Let $A_{2n} = \{f(x) : \ x \in \omega\}$ for some computable function f, and let h be a computable function such that for any x $W_{h(x)} = \{f(x)\}$. Define $W = f^{-1}(A_{2n+1})$. Then we have

$$x \in W \Longrightarrow f(x) \in A_{2n+1} \Longrightarrow W_{h(x)} \subseteq N,$$
$$x \notin W \Longrightarrow f(x) \notin A_{2n+1} \ \&\ f(x) \in A_{2n} \Longrightarrow W_{h(x)} \nsubseteq N.$$

Therefore $W \leq_Q N$.

If the set W is computable, then $A_{2n} - A_{2n+1}$ is c.e., since

$$A_{2n} - A_{2n+1} = \Big\{x : \ (\exists\, y)\big(x = f(y) \ \&\ y \notin W\big)\Big\}.$$

But $A_{2n} - A_{2n+1}$ is an immune set, a contradiction.

(c) $M \leq_Q N$ by [1, Theorem 1] and $N \nleq_Q M$ by (a) and (b). □

Definition ([4])**.** A set $A \subseteq \omega$ is *semirecursive* if there is a computable function f of two variables such that

1. $f(x, y) = x$ or $f(x, y) = y$;
2. $x \in A \ \vee\ y \in A \Longrightarrow f(x, y) \in A$.

Theorem 2. *Let A be a noncomputable c.e. semirecursive set. Then for any B, $\varnothing <_T B \leq_T A$, there is a set C with $C \equiv_T B$ such that $0 <_Q C \leq_Q A$.*

Proof. Let A, B be as in the statement of the theorem, and let Γ_B be the graph of the characteristic function of B. Since $B \leq_T A$, there is a c.e. regular set $W_{\rho(z)}$ (see, Rogers [8, Theorem IX.2]) such that for all x, y

$$\langle x, y\rangle \in \Gamma_B \Longleftrightarrow (\exists\, u)(\exists\, v)\Big[\langle x, y, u, v\rangle \in W_{\rho(z)} \ \&\ D_u \subseteq A \ \&\ D_v \subseteq \overline{A}\Big]. \qquad (*)$$

Since A is semirecursive, there are computable functions f and g such that (see Degtev [3])

$$(\forall\, u)\big[D_u \subseteq A \Longleftrightarrow f(u) \in A\big],$$
$$(\forall\, v)\big[D_v \subseteq \overline{A} \Longleftrightarrow g(v) \in \overline{A}\big].$$

For convenience, we first rewrite the property $(*)$ as follows:

$$x \in B \Longleftrightarrow (\exists\, u)(\exists\, v)\Big[\langle x, 1, u, v\rangle \in W_{\rho(z)} \ \&\ f(u) \in A \ \&\ g(v) \in \overline{A}\Big],$$
$$x \in \overline{B} \Longleftrightarrow (\exists\, u)(\exists\, v)\Big[\langle x, 0, u, v\rangle \in W_{\rho(z)} \ \&\ f(u) \in A \ \&\ g(v) \in \overline{A}\Big].$$

Now we define two c.e. sets P_0 and P_1 as follows:

$$P_0 = \Big\{x : \ (\exists\, u)(\exists\, v)\big[\langle x, 0, u, v\rangle \in W_{\rho(z)} \ \&\ f(u) \in A\big]\Big\},$$
$$P_1 = \Big\{x : \ (\exists\, u)(\exists\, v)\big[\langle x, 1, u, v\rangle \in W_{\rho(z)} \ \&\ f(u) \in A\big]\Big\}.$$

Obviously,

$$B \subseteq P_1, \ \ \overline{B} \subseteq P_0 \ \ \&\ \ P_0 \cup P_1 = \omega.$$

Let C_0 and C_1 be computable sets such that

$$C_0 \subseteq P_0 \ \& \ C_1 \subseteq P_1 \ \& \ C_0 \cap C_1 = \varnothing \ \& \ C_0 \cup C_1 = \omega,$$

and let

$$C = (B \cap C_0) \cup (\overline{B} \cup C_1).$$

Let h_0 and h_1 be computable functions such that

$$W_{h_0(x)} = \Big\{ y : \ (\exists\, u)(\exists\, v)\big[\langle x, 0, u, v\rangle \in W_{\rho(z)} \ \&\ y = g(v)\big] \Big\},$$
$$W_{h_1(x)} = \Big\{ y : \ (\exists\, u)(\exists\, v)\big[\langle x, 1, u, v\rangle \in W_{\rho(z)} \ \&\ y = g(v)\big] \Big\},$$

and let h be a computable function such that

$$W_{h(x)} = \begin{cases} W_{h_0}(x), & \text{if } x \in C_0, \\ W_{h_1}(x), & \text{if } x \in C_1. \end{cases}$$

Then if $x \in C$ we have

$$x \in B \cap C_0 \Longrightarrow W_{h(x)} = W_{h_0(x)} \subseteq A.$$
$$x \in \overline{B} \cap C_1 \Longrightarrow W_{h(x)} = W_{h_1(x)} \subseteq A.$$

If $x \notin C$, then $x \in \overline{C} = (B \cap C_1) \cup (\overline{B} \cap C_0)$ and we have

$$x \in B \cap C_1 \Longrightarrow W_{h(x)} = W_{h_1(x)} \nsubseteq A.$$
$$x \in \overline{B} \cap C_0 \Longrightarrow W_{h(x)} = W_{h_0(x)} \nsubseteq A.$$

Therefore

$$(\forall\, x)\big[x \in C \Longleftrightarrow W_{h(x)} \subseteq A\big],$$

i.e. $C \leq_Q A$.

It remains to show that $B \equiv_T C$. For any x we have: If $x \in C_0$, then $x \in B \Longleftrightarrow x \in C$. If $c \in C_1$, then $x \in B \Longleftrightarrow x \in \overline{C}$. Therefore $B \equiv_T C$. □

Corollary 3. *Let A be a noncomputable c.e. semirecursive set. Then there is a Δ°_2 set C with $\varnothing <_Q C <_Q A$ such that for any noncomputable c.e. set W we have $W \nleq_Q C$.*

Proof. Let A be a noncomputable c.e. semirecursive set and $\mathbf{a} = \deg_T(A)$. Then there is a minimal T-degree $\mathbf{b}$, $\mathbf{b} < \mathbf{a}$ (Yates [11]). Let $B \in \mathbf{b}$, then $B <_T A$. By Theorem 2 there is a Δ°_2 set C such that

$$C \equiv_T B \ \&\ C \leq_Q A.$$

If there is a noncomputable c.e. set W such that $W \leq_Q C$, then $W \leq_T C \leq_T B$. Since $\mathbf{b}$ is a minimal T-degree, we have $W \equiv_T C$, a contradiction. □

We recall that in a poset $(P, \leq)$ with least element 0, a minimal pair is a pair of elements $\mathbf{a}$, $\mathbf{b}$ in P such that

$$\mathbf{a}, \mathbf{b} \neq 0 \ \ \& \ \ (\forall\, \mathbf{c} \in P)\big[\mathbf{c} \leq \mathbf{a} \ \&\ \mathbf{c} \leq \mathbf{b} \Longrightarrow \mathbf{c} = 0\big].$$

Definition ([9])**.** Given a set A, define the weak jump of A to be the set

$$H_A = \{e : W_e \cap A \neq \varnothing\}$$

and say that a set A is semilow if $H_A \leq_T \varnothing'$.

In [7] it is proved that if A and B are c.e. sets such that $\mathbf{a} = \deg_Q(A)$ and $\mathbf{b} = \deg_Q(B)$ form a minimal pair in the c.e. Q-degree and $\overline{A}$ and $\overline{B}$ are semilow, then $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair in the Q-degrees. The following theorem shows that the semilowness of $\overline{A}$ and $\overline{B}$ is unnecessary for Σ^0_2 sets.

Theorem 3. *If c.e. Q-degrees $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair in the c.e. Q-degrees, then $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair in the Σ^0_2 Q-degrees.*

This immediately follows from

Theorem 4. *If $\mathbf{a}$ and $\mathbf{b}$ are c.e. Q-degrees, then for every nonzero Σ^0_2 Q-degree $\mathbf{c}$ such that $\mathbf{c} \leq_Q \mathbf{a}, \mathbf{b}$, there exists a c.e. Q-degree $\mathbf{d}$ such that*

$$\mathbf{c} \leq_Q \mathbf{d} \leq_Q \mathbf{a}, \mathbf{b}.$$

Proof. Suppose that A and B are c.e. sets such that $\mathbf{a} = \deg_Q(A)$ and $\mathbf{b} = \deg_Q(B)$. Assume that $\mathbf{c}$ is a nonzero Σ^0_2 Q-degree such that $\mathbf{c} \leq_Q \mathbf{a}, \mathbf{b}$. Let $C \in \mathbf{c}$ be a Σ^0_2 set. It follows from $C \leq_Q A$ that $C \in \Pi^0_2$ (see [6, p. 282]). Then $C \in \Delta^0_2$ and by [7, Corollary 5] there exist computable functions f, g such that

$$(\forall\, x)\Big[\big[x \in C \Longleftrightarrow W_{f(x)} \subseteq A\big] \;\&\; \big[W_{f(x)} \text{ is finite}\big]\Big],$$

$$(\forall\, x)\Big[\big[x \in C \Longleftrightarrow W_{g(x)} \subseteq B\big] \;\&\; \big[W_{g(x)} \text{ is finite}\big]\Big].$$

Fix computable approximations $\{A_s\}_{s\in\omega}$ and $\{B_s\}_{s\in\omega}$ of A and B, respectively. Define a c.e. set D as follows:

$$D = \Big\{\langle x, t\rangle :\ (\exists\, s \geq t)\big[W_{f(x),s} \subseteq A_s \;\&\; W_{g(x),s} \subseteq B_s\big]\Big\}.$$

Then

$$x \in C \Longleftrightarrow (\forall\, t)\big[\langle x, t\rangle \in D\big].$$

Let

$$W_{\widetilde{f}(x)} = \big\{\langle x, t\rangle :\ t \in \omega\big\}.$$

Then

$$(\forall\, x)\big[x \in C \Longleftrightarrow W_{\widetilde{f}(x)} \subseteq D\big],$$

which gives $C \leq_Q D$.

Let f_1 be a computable function such that

$$W_{f_1(\langle x,t\rangle)} = \begin{cases} W_{f(x),n}, & \text{where } n = \min\big\{s :\ s \geq t \;\&\; W_{f(x),s} \subseteq A_s \;\&\; \langle x, t\rangle \in D_s\big\} \\ & \text{if } \langle x, t\rangle \in D, \\ W_{f(x)} & \text{otherwise.} \end{cases}$$

Then

$$\langle x, t\rangle \in D \Longrightarrow (\exists\, s \geq t)\big[W_{f(x),s} \subseteq A_s\big] \Longrightarrow W_{f_1(\langle x,t\rangle)} \subseteq A$$

and

$$\langle x,t\rangle \notin D \Longrightarrow x \notin C \Longrightarrow W_{f(x)} \not\subseteq A \Longrightarrow W_{f_1(\langle x,t\rangle)} \not\subseteq A.$$

Thus $D \leq_Q A$. In the same way we can prove that $D \leq_Q B$. □

Corollary 4. *Let* $\mathbf{a}$, $\mathbf{b}$ *be c.e. Q-degrees that form a minimal pair in the c.e. Q-degrees, and let* A, B, C *and* D *be c.e. sets such that* $A \in \mathbf{a}$, $B \in \mathbf{b}$, $C \subseteq A$ *and* $D \subseteq B$. *Then* $\deg_Q(A - C)$ *and* $\deg_Q(B - D)$ *form a minimal pair in the* Σ_2^0 *Q-degrees.*

Proof. By [1, Theorem 1] $A - C \leq_Q A$ and $B - D \leq_Q B$ and, by Theorem 3, $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair in the Σ_2^0 Q-degrees. If E is a noncomputable Σ_2^0 set and $E \leq_Q A - C$, $E \leq_Q B - D$, then $E \leq_Q A$ and $E \leq_Q B$, a contradiction. □

In [2, Theorem 6] it is proved that for any c.e. noncomputable set A there exist noncomputable c.e. sets A_0 and A_1 such that $A \oplus A_0 |_Q A \oplus A_1$ and A_0 and A_1 form a minimal pair in the c.e. Q-degrees.

From Theorem 4 and [2, Theorem 6] follows immediately the following

Corollary 5. *For any c.e. noncomputable set* A *there exist noncomputable c.e. sets* A_0 *and* A_1 *such that* $A \oplus A_0 |_Q A \oplus A_1$ *and* A_0 *and* A_1 *form a minimal pair in the* Σ_2^0 *Q-degrees.*

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References

1. M. M. Arslanov and R. Sh. Omanadze, Q-degrees of n-c.e. sets. *Illinois J. Math.* **51**(2007), No. 4, 1189–1206.
2. M. M. Arslanov, I. I. Batyrshin, and R. Sh. Omanadze, Structural properties of Q-degrees of n-c.e. sets. *Ann. Pure Appl. Logic* **156**(2008), No. 1, 13–20.
3. A. N. Degtev, Recursively enumerable sets and reducibility of truth-table type. (Russian) *Nauka. Fizmatlit., Moscow*, 1998.
4. C. G., Jockusch, Jr. Semirecursive sets and positive reducibility. *Trans. Amer. Math. Soc.* **131**(1968), 420–436.
5. A. H. Lachlan, On the lattice of recursively enumerable sets. *Trans. Amer. Math. Soc.* **130**(1968), 1–37.
6. P. Odifreddi, Classical recursion theory. *North-Holland Publishing Co., Amsterdam*, 1989.
7. R. Sh. Omanadze and A. Sorbi, Strong enumeration reducibilities. *Arch. Math. Logic* **45**(2006), No. 7, 869–912.
8. H. Rogers, Jr., Theory of recursive functions and effective computability. *McGraw-Hill Book Co., New York-Toronto, Ont.-London*, 1967.
9. R. I. Soare, Computational complexity, speedable and levelable sets. *J. Symbolic Logic* **42**(1977), No. 4, 545–563.

10. R. I. Soare, Recursively enumerable sets and degrees. A study of computable functions and computably generated sets. *Springer-Verlag, Berlin*, 1987.

11. C. E. M. Yates, Initial segments of the degrees of unsolvability. II. Minimal degrees. *J. Symbolic Logic* **35**(1970), 243–266.

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