A remark to the $L^2$ boundedness of parametric Marcinkiewicz integral

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Recently, A. Al-Salman (2011) [1] gave a characterization of the $L^2$ boundedness of the parametric Marcinkiewicz integral $\mu_{\Omega}^\rho$ when $\rho$ is a positive real number. In this note, using a quite elementary method, we give a sufficient and necessary condition for the $L^2(\mathbb{R}^n)$ boundedness of $\mu_{\Omega}^\rho$ when $\rho$ is a complex number.

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1. Introduction

Let $\mathbb{R}^n$ ($n \geq 2$) be the $n$-dimensional Euclidean space and $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \quad (1.1)$$

where $x'=x/|x|$ for any $x \neq 0$. For $\rho \in \mathbb{C}$ with $\text{Re} \rho > 0$, we define the parametric Marcinkiewicz integral operator $\mu_{\Omega}^\rho$ by

$$\mu_{\Omega}^\rho(f)(x) := \left( \int_0^1 \left( \int_{|y| < t} \frac{\Omega(y')f(x-y)}{|y|^{n-\rho}} dy \right)^2 dt \right)^{1/2}. \quad (1.2)$$

When $\rho = 1$, we denote $\mu_{\Omega}^1$ by $\mu_\Omega$. It is well known that the operator $\mu_\Omega$ was first defined by Stein [12] in 1958. Stein proved that if $\Omega$ is continuous and satisfies a $\text{Lip}_\alpha$ ($0 < \alpha \leq 1$) condition on $S^{n-1}$, then $\mu_\Omega$ is the operator of type $(p,p)$ for $1 < p \leq 2$ and of weak type $(1,1)$. In [2] Benedek, Calderón and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then $\mu_\Omega$ is of type $(p,p)$ for $1 < p < \infty$. Then $L^p$ ($1 < p < \infty$) boundedness of $\mu_{\Omega}^\rho$ was first studied by Hörmander [9] for real $\rho$ in 1960, and later studied by Sakamoto and Yabuta [11] for complex number $\rho$ in 1999 when the kernel $\Omega \in \text{Lip}_\alpha(S^{n-1})$. In 2002, Ding, Lu and Yabuta [5] gave the $L^2$ boundedness of $\mu_{\Omega}^\rho$ when $\Omega \in L(\log L)(S^{n-1})$.

Recently, A. Al-Salman [1] gave a characterization of the $L^2$ boundedness of $\mu_{\Omega}^\rho$ when $\rho$ is a positive real number.

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Theorem A. Suppose that \( \rho > 0 \) and \( \Omega \in L^1(S^{n-1}) \) satisfying (1.1). Then \( \mu^\rho_\Omega \) is bounded on \( L^2(\mathbb{R}^n) \) if and only if
\[
\sup_{\xi' \in S^{n-1}} \left| \int \int \Omega(\xi') \overline{\Omega}(\xi') \chi_{(1,\infty)} \left( \frac{|\xi' \cdot z'|}{|\xi' \cdot y'|} \right) \log \frac{1}{|\xi' \cdot y'|} d\sigma(y) d\sigma(z) \right| < \infty. \tag{1.3}
\]

In this note, we give also a sufficient and necessary condition for the \( L^2(\mathbb{R}^n) \) boundedness of \( \mu^\rho_\Omega \) when \( \rho \) is a complex number. Here we would like to point that our condition (1.4) seems simpler than (1.3) in form at least. In particular, the method of proving our result is pretty much elementary.

Theorem 1. Let \( \rho \in \mathbb{C} \) with Re \( \rho > 0 \) and \( \Omega \in L^1(S^{n-1}) \) satisfying (1.1). Then \( \mu^\rho_\Omega \) is bounded on \( L^2(\mathbb{R}^n) \) if and only if
\[
\sup_{\xi' \in S^{n-1}} \left| \int \int \Omega(\xi') \overline{\Omega}(\xi') \log \left( \frac{2}{|\xi' \cdot y'|^2 + |\xi' \cdot z'|^2} \right)^{1/2} d\sigma(y) d\sigma(z) \right| < \infty. \tag{1.4}
\]

The proof of Theorem 1 will be given in the next section. Here we give some remarks on this conclusion.

Remark 1. From Walsh’s paper [14, p. 204] we see that if \( \rho \in \mathbb{C} \) with Re \( \rho > 0 \), then \( \mu^\rho_\Omega \) is \( L^2 \) bounded if and only if
\[
\Omega \in \mathcal{N}(S^{n-1}) := \left\{ \Omega \in L^1(S^{n-1}) \colon \left( \sup_{\xi' \in S^{n-1}} \int_0^1 \left| \int \Omega(\xi') d\sigma(y') \right|^2 \frac{dt}{t} \right)^{1/2} < \infty \right\}. \tag{1.5}
\]

Note that for any \( \xi' \in S^{n-1} \), the following equation holds:
\[
\int_0^1 \left| \int \Omega(\xi') d\sigma(y') \right|^2 \frac{dt}{t} = \int \left| \int \Omega(\xi') \overline{\Omega}(\xi') \log \frac{1}{\max\{|\xi' \cdot y'|, |\xi' \cdot z'|\}} d\sigma(y') d\sigma(z) \right|.
\]

By using the estimate \( \sqrt{(a^2 + b^2)/2} \leq \max\{|a|, |b|\} \leq \sqrt{a^2 + b^2} \) \( (a, b \in \mathbb{R}) \), it is immediate to see that our condition (1.4) is equivalent to Walsh’s condition (1.5). However, our proof method is very simple.

Remark 2. A well-known fact shows that (see [6, p. 271], for example), for the singular integral operator \( T_\Omega \) with \( \Omega \in L^1(S^{n-1}) \) satisfying (1.1), \( T_\Omega \) is bounded on \( L^2(\mathbb{R}^n) \) if and only if
\[
\Omega \in \mathcal{F}_1(\mathbb{R}^n) := \left\{ \Omega \in L^1(S^{n-1}) \colon \sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(\xi')| \log \frac{1}{|\xi' \cdot y'|} d\sigma(y') < \infty \right\}.
\]

Thus, by comparing the properties of \( T_\Omega \) and \( \mu^\rho_\Omega \), one may guess that \( \mu^\rho_\Omega \) is bounded on \( L^2(\mathbb{R}^n) \) if and only if
\[
\Omega \in \mathcal{F}_{1/2}(\mathbb{R}^n) := \left\{ \Omega \in L^1(S^{n-1}) \colon \sup_{\xi' \in S^{n-1}} \int_{S^{n-1}} |\Omega(\xi')| \left( \log \frac{1}{|\xi' \cdot y'|} \right)^{1/2} d\sigma(y') < \infty \right\}. \tag{1.6}
\]

However, by using Theorem 1 and a counterexample, we show that the above conjecture is negative.

Corollary 2. Let \( \rho \in \mathbb{C} \) with Re \( \rho > 0 \) and \( \Omega \in L^1(S^{n-1}) \) satisfying (1.1). Then (1.6) is only a sufficient condition for the \( L^2 \) boundedness of \( \mu^\rho_\Omega \), but not necessary.

Proof. The sufficiency follows immediately from Theorem 1 and the following fact:
\[
\log\left( \frac{2}{|\xi' \cdot y'|^2 + |\xi' \cdot z'|^2} \right)^{1/2} = \left( \log\left( \frac{2}{|\xi' \cdot y'|^2 + |\xi' \cdot z'|^2} \right)^{1/2} \right)^2 \leq \log^{1/2} \frac{\sqrt{2}}{|\xi' \cdot y'|} \log^{1/2} \frac{\sqrt{2}}{|\xi' \cdot z'|}.
\]
On the other hand, in two-dimensional case, let

$$\Omega_0(x') = \begin{cases} \frac{1}{|\log|\theta|-1|^{1/2} \log \log |\theta|-\tau}, & |\theta| < 1/10, \\ -10 \left( \frac{1}{|\log|\theta|-1|^{1/2} \log \log |\theta|-\tau} \right), & 1/10 < |\theta| < 2/10, \\ 0, & 2/10 < |\theta| < \pi, \end{cases}$$

where $x' = (\cos \theta, \sin \theta) \in S^1 \subset \mathbb{R}^2$. Then it is easy to check that the function $\Omega_0$ defined above is in $L^1(S^1)$ and satisfies (1.1). In particular, one may verify that $\Omega_0$ satisfies

$$\sup_{\xi \in S^1} \int_{S^1} |\Omega_0(y') \log \left( \frac{2}{|\xi'|, |y'|^2 + |\xi'|, z'|^2} \right) |^{1/2} d\sigma(y') d\sigma(x') < \infty,$$

but

$$\sup_{\xi \in S^1} \int_{S^1} |\Omega_0(y')| \log^{1/2} \left( \frac{1}{|\xi'|, |y'|} \right) d\sigma(y') = \infty. \quad \square$$

**Remark 3.** Denote by $H^1(S^{n-1})$ the Hardy space on $S^{n-1}$. More precise,

$$H^1(S^{n-1}) = \left\{ \Omega \in L^1(S^{n-1}) : \sup_{0 < r < 1} \int_{S^{n-1}} \Omega(y') P_r(y') d\sigma(y') \right\}_{L^1(S^{n-1})} < \infty \right\},$$

where $P_{r'}(y')$ denotes the Poisson kernel on $S^{n-1}$ defined by

$$P_{r'}(y') = \frac{1 - r^2}{|r' - y'|^n}, \quad 0 \leq r < 1 \text{ and } x', y' \in S^{n-1}.$$

See [3] or [8] for the properties of $H^1(S^{n-1})$. Then by the above Remarks 1 and 2, we have the following containing relationship among some function classes on $S^{n-1}$:

$$H^1(S^{n-1}) \subset \mathcal{F}_1(S^{n-1}) \subset \mathcal{F}_{1/2}(S^{n-1}) \subset \mathcal{N}(S^{n-1}) \subset L^1(S^{n-1}),$$

and all inclusions are proper.

We need only to show the first inclusion. In fact, if $\Omega \in H^1(S^{n-1})$, then the singular integral operator $T_\Omega$ is bounded on $L^p$ for all $1 < p < \infty$ (see [4,10] or [8]). Hence $\Omega \in \mathcal{F}_1(S^{n-1})$ (see Remark 2). On the other hand, an example given in [7] shows that $H^1(S^{n-1})$ is a proper subset in $\mathcal{F}_1(S^{n-1})$.

2. **Proof of Theorem 1**

Write $\rho = \sigma + i \tau \in \mathbb{C}$ with $\sigma = \Re \rho > 0$. As done in [1], by Fubini’s theorem and the Plancherel theorem we get

$$\|\mu_\Omega^\rho(f)\|_2^2 = \int \left( \int_0^\infty \frac{1}{t^\rho} \int \frac{\Omega(y') f(x - y)}{|y'|^{n-\rho}} dy \right)^2 \frac{dt}{t} dx$$

$$= \int \left( \int_0^\infty \frac{1}{t^\rho} \int \frac{\Omega(y') f(x - y)}{|y'|^{n-\rho}} dx \right)^2 \frac{dt}{t}$$

$$= \frac{1}{(2\pi)^n} \int \left( \int_0^\infty \frac{1}{t^\rho} \int \frac{\Omega(y') e^{-iy \cdot \xi}}{|y'|^{n-\rho}} dy \right)^2 \frac{|\hat{f}(\xi)|^2}{|\xi|^2} d\xi \frac{dt}{t}$$

$$= \frac{1}{(2\pi)^n} \int |\hat{f}(\xi)|^2 \left( \int_0^\infty \frac{1}{t^\rho} \int \frac{\Omega(y') e^{-iy \cdot \xi}}{|y'|^{n-\rho}} dy \right)^2 \frac{dt}{t} d\xi.$$
Setting $y = tsy'$, we get
\[
\frac{1}{t^\rho} \int_{|y| < t} \frac{\Omega(y)e^{-iy \xi}}{|y|^{n-\rho}} \, dy = \frac{1}{t^\rho} \int_{S^{n-1}} \Omega(y') \int_0^1 e^{-istsy' \xi} (ts)^{\rho-1} t \, ds \, d\sigma(y') \\
= \int_{S^{n-1}} \Omega(y') \int_0^1 e^{-istsy' \xi} s^{\rho-1} \, ds \, d\sigma(y').
\]
So, we have by using the cancellation property of $\Omega$
\[
m_{\Omega, \rho}(\xi) := \int_0^\infty \left( \frac{1}{t^\rho} \int_{|y| < t} \frac{\Omega(y)e^{-iy \xi}}{|y|^{n-\rho}} \, dy \right)^2 \frac{dt}{t} \\
= \int_{S^{n-1}} \int_{S^{n-1}} \Omega(y') \overline{\Omega(z')} \left( \int_0^1 \left( \int_{|y| < t} \frac{\Omega(y)e^{-iy \xi}}{|y|^{n-\rho}} \, dy \right)^2 \frac{dt}{t} \right) \, d\sigma(y') \, d\sigma(z') \\
= \lim_{\varepsilon \to 0, A \to +\infty} \int_{S^{n-1}} \int_{S^{n-1}} \Omega(y') \overline{\Omega(z')} \left( \int_0^1 \left( \int_{|y| < t} \frac{\Omega(y)e^{-iy \xi}}{|y|^{n-\rho}} \, dy \right)^2 \frac{dt}{t} \right) \, d\sigma(y') \, d\sigma(z') \\
\times (rs)^{\sigma-1} \left( \frac{r}{s} \right)^{i\tau} \, dr \, ds)
\]
As is discussed in Stein's book [13, pp. 40–41], we have
\[
\lim_{\varepsilon \to 0, A \to +\infty} \int_{\varepsilon}^A \left( e^{-it(ry' - sz')} \xi - \cos(t|\xi|) \right) \frac{dt}{t} = \log |\xi' \cdot (ry' - sz')|^{-1} - \frac{\pi}{2} i \text{sgn}(\xi' \cdot (ry' - sz'))
\]
and $\int_{\varepsilon}^A \left( e^{-it(ry' - sz')} \xi - \cos(t|\xi|) \right) \frac{dt}{t}$ is uniformly bounded in $\varepsilon$ and $A$. Therefore, setting
\[
K_\rho(\xi' \cdot y', \xi' \cdot z') = \int_0^1 \left( rs \right)^{\sigma-1} \left( \frac{r}{s} \right)^{i\tau} \log |\xi' \cdot (ry' - sz')|^{-1} \, dr ds,
\]
\[
H_\rho(\xi' \cdot y', \xi' \cdot z') = \frac{\pi}{2} \int_0^1 \left( rs \right)^{\sigma-1} \left( \frac{r}{s} \right)^{i\tau} \text{sgn}(\xi' \cdot (ry' - sz')) \, dr ds,
\]
we have by Lebesgue's dominated convergence theorem
\[
m_{\Omega, \rho}(\xi) = \int_{S^{n-1}} \int_{S^{n-1}} \Omega(y') \overline{\Omega(z')} \{K_\rho(\xi' \cdot y', \xi' \cdot z') - iH_\rho(\xi' \cdot y', \xi' \cdot z')\} \, d\sigma(y') \, d\sigma(z')
\]
with $\xi' = \xi/|\xi|$, $y' = y/|y|$, $z' = z/|z|$. Since clearly $H_\rho \in L^\infty(\mathbb{R}^2)$, in order to complete the proof of Theorem 1, it suffices to show that
\[
K_\rho(\xi' \cdot y', \xi' \cdot z') = \frac{1}{|\rho|^2} \log \left( \frac{2}{|\xi' \cdot y'|^2 + |\xi' \cdot z'|^2} \right)^{1/2} + J_\rho(\xi' \cdot z', \xi' \cdot y')
\]
with $J_\rho \in L^\infty(\mathbb{R}^2)$. For this end, we denote $a = \xi' \cdot y'$ and $b = \xi' \cdot z'$. Then we have
\[
\int_0^1 \left( rs \right)^{\sigma - 1} \left( \frac{1}{r} \right)^t \log |ar - bs|^{-1} \, dr \, ds
\]

\[
= \int_0^{\pi/4} \int_0^{1/\cos \theta} t^{2\sigma - 1} \log |\cos \theta - bt \sin \theta|^{-1} \, dt \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \, d\theta
\]

\[
+ \int_0^{\pi/2} \int_0^{1/\sin \theta} t^{2\sigma - 1} \log |\cos \theta - bt \sin \theta|^{-1} \, dt \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \, d\theta
\]

\[
= \int_0^{\pi/4} \int_0^{1/\cos \theta} t^{2\sigma - 1} \log t^{-1} \, dt \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \, d\theta
\]

\[
+ \int_0^{\pi/4} \frac{1}{2\sigma \cos^{2\sigma} \theta} \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \log |a^2 + b^2|^{-1/2} \, d\theta
\]

\[
+ \int_0^{\pi/4} \frac{1}{2\sigma \cos^{2\sigma} \theta} \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \log |\cos(\theta - \tan^{-1} b/a)|^{-1} \, d\theta
\]

\[
+ \int_0^{\pi/4} \frac{1}{2\sigma \sin^{2\sigma} \theta} \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \log |a^2 + b^2|^{-1/2} \, d\theta
\]

\[
+ \int_0^{\pi/4} \frac{1}{2\sigma \sin^{2\sigma} \theta} \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \log |\cos(\theta - \tan^{-1} b/a)|^{-1} \, d\theta
\]

\[
= \frac{1}{|\rho|^2} \log(a^2 + b^2)^{-1/2} + \int_0^{\pi/4} \frac{1/\cos \theta}{2\sigma \cos^{\sigma + 1} \theta} \left( \cos \theta \sin \theta \right)^{\sigma - 1} \left( \cot \theta \right)^t \log |\cos(\theta - \tan^{-1} b/a)|^{-1} \, d\theta
\]
We therefore complete the proof of Theorem 1.

Finally, it is easy to see that the function \( J_\rho(a, b) \) is bounded on \( \mathbb{R}^2 \). In fact, in the case \( \Re \rho = \sigma \geq 1 \), we have

\[
|J_\rho(a, b)| \leq \frac{\pi}{2} \int_0^{\infty} t^{\sigma - 1} \log \frac{1}{t} \, dt + \frac{2^{(\sigma+1)/2}}{\sigma} \int_0^{\pi/2} \log |\cos(\theta - \tan^{-1} b/a)|^{-1} \, d\theta
\]

And in the case \( 0 < \sigma < 1 \), we take \( \gamma > 0 \) with \( \gamma(1 - \sigma) < 1 \). Then we have

\[
|J_\rho(a, b)| \leq 2 \int_0^{\pi/4} \frac{2^{(\sigma+1)/2}}{\sigma} \left( \int_0^{\pi/4} \frac{2^{\sigma/2}}{\pi} \, d\theta \right)^{1/\gamma} \left( \int_0^{\pi/4} \log \gamma^{-1} |\cos(\theta - \tan^{-1} b/a)|^{-1} \, d\theta \right)^{1/\gamma'}
\]

\[
\leq C_\sigma + C_{\sigma, \gamma} \int_0^{\pi/4} \log \gamma^{-1} \frac{1}{\sin \theta} \, d\theta = C_{\sigma, \gamma} < \infty.
\]

We therefore complete the proof of Theorem 1.

References