A Class of Unitary Constellations for Differential Space-Time Modulation

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Abstract—We present a new approach to design unitary constellations for differential space-time modulation. By exploiting the rotational invariance of unitary matrices and the property of M-ary PSK symbols, the proposed constellations of size $L$ are derivable from a smaller set of $D = L/M$ unitary matrices (via rotations defined by M-ary PSK symbols) which, in turn, are parameterized by two unitary matrices $U_1$ and $U_2$. The proposed constellations share some good property of group constellations thereby considerably simplifying the optimization of $U_1$ and $U_2$. They also possess many other good properties which, when exploited, allow for simplified maximum likelihood decoding. Extensive numerical examples are provided to demonstrate that the proposed constellations in general have larger diversity products than other constellations.

Index Terms—Differential unitary space-time (DUST) modulation, diversity product

I. INTRODUCTION

Differential unitary space-time modulation [1], [2] is very attractive to multiple-antenna wireless systems since it eliminates the need for costly channel estimation. For a system with a constellation size $L$ and $N_t$ transmit antennas, we need to determine $L N_t$-by-$N_t$ unitary matrices. To make the design tractable, it is common practice to impose certain structure, such as cyclic group [1], upon the candidate set of matrices. It is shown [3] that the cyclic group codes \(^1\) are optimal group codes when $L$ is an integer power of 2. In [4], all finite fixed-point-free (full diverse) group codes are completely characterized. Research efforts have been extended to non-group codes resulting in improved cyclic codes [5], Cayley codes [7], and codes from amicable designs [6]. Cyclic codes are less restrictive allowing for constellation size $L$ to be arbitrary, at the cost of relatively inferior performance. Other group codes have better performance but considerable restrictions on the constellation size $L$ and the number of transmit antennas. The performance of non-group codes can be even better since they have more degrees of freedom in structure. A possible drawback is increased decoding complexity. A natural question is the likelihood to construct a non-group constellation that has better performance, relatively simple decoding, and little restriction on constellation size $L$ and number of transmit antennas $N_t$. In this paper, we propose a class of $M$-PSK modulated DUST (termed M-DUST) constellations which are constructed from a small set of unitary matrices via rotations defined by $M$-PSK.

II. DIFFERENTIAL UNITARY SPACE-TIME MODULATION

Consider a communication system with $N_t$ transmit and $N_r$ receive antennas, which transmits and receives signals on a block basis. A transmitted information block can be represented by one of $L N_t \times N_t$ unitary matrices $\{V_1, V_2, \ldots, V_L\}$. Without loss of generality, we consider the $\tau$th block ($\tau = 1, 2, \ldots$). Assume that an $N_t \times N_t$ unitary signal matrix $S_{\tau}$ is transmitted through $N_t$ antennas over $N_t$ symbol duration. The DUST transmission matrix can be expressed as

$$S_{\tau} = V_{z_{\tau}} S_{\tau-1}$$

(1)

where the matrix $V_{z_{\tau}} \in \{V_1, V_2, \ldots, V_L\}$ stands for the transmitted information in the $\tau$th block.

We assume that channels linking each pair of transmit and receive antennas suffer from independent and identically distributed (i.i.d.) flat Rayleigh fading. Thus, if we denote the channel matrix observed at block $\tau$ by an $N_t \times N_r$ matrix $H_{\tau}$, its entries follow independent complex Gaussian distribution each having zero mean and unit variance. We further assume slowly fading channels such that $H_{\tau} = H_{\tau-1}$. The $N_t \times N_r$ received signal matrix $X_{\tau}$ is related to fading channels by

$$X_{\tau} = \sqrt{\rho} S_{\tau} H_{\tau} + W_{\tau}, \quad \tau = 1, 2, \ldots$$

(2)

where $W_{\tau}$ is an $N_t \times N_r$ additive noise matrix whose entries are independent zero-mean Gaussian variable of unit variance, and $\rho$ signifies the signal-to-noise (SNR) ratio at each receive antenna. At the receiver, non-coherent maximum likelihood (ML) decoding is used to produce an estimate

$$\hat{z}_{\tau} = \arg \min_{l=1,2,\ldots,L} \| X_{\tau} - V_l X_{\tau-1} \|^2_F$$

$$= \arg \max_{l=1,2,\ldots,L} \Re \left\{ \Tr \left[ V_l X_{\tau-1} X_{\tau}^\dagger \right] \right\}$$

(3)

where $\| \cdot \|_F$, $\Tr(\cdot)$, $\Re(\cdot)$, $\dagger$ denote Frobenius norm, trace, real, and complex conjugate transpose, respectively.
III. M-PSK MODULATED DUST CONSTELLATIONS

The design of DUST constellations is difficult requiring the determination of a large set of \( N_t \times N_t \) unitary matrices \( \{V_1, V_2, \ldots, V_L\} \) even for a moderate \( L \). A novel technique for constellation construction is therefore proposed here. We choose \( L \) such that \( L = M \times D \). The new scheme starts from a basic set of \( D \) \( N_t \times N_t \) unitary matrices \( \{G_1, G_2, \cdots, G_D\} \). The basic set is then rotated by each of the \( M \)-PSK symbols \( \{\exp(j2\pi m/M), m = 0, 1, \cdots, M-1\} \). The resulting M-DUST constellation of size \( L \) is given by

\[
\mathcal{C} = \{C_{m,d}\} = \{\exp(j2\pi m/M)G_d\}
\]

where \( G_d \) is chosen as

\[
G_d = U_1^dU_2^d, \quad d = 1, 2, \cdots, D
\]

whit both \( U_1 \) and \( U_2 \) being \( N_t \times N_t \) unitary matrices. As such, each entry of \( C \) can be expressed as

\[
C_{m,d} = \exp\left(j\frac{2\pi}{M}m\right)G_d = \exp\left(j\frac{2\pi}{M}m\right)U_1^dU_2^d
\]

where \( m = 0, 1, \cdots, M-1 \) and \( d = 1, 2, \cdots, D \).

It is clear that the proposed M-DUST constellations are fully characterized by the parameter \( M \) and two basis unitary matrices \( U_1 \) and \( U_2 \). Before pursuing the issue of choosing \( M \) properly and \( U_1 \) and \( U_2 \) optimization, we define the following term for subsequent use.

**Definition 1 (Eigenvalue Spectrum and Multiplicity):** For an M-DUST constellation described by \( L \) \( N_t \times N_t \) unitary matrices, we denote the vector of \( N_t \) eigenvalues associated with \( C_{m',d'} \) and \( C_{m,d} \) such that

\[
\Lambda_{(m',d'),(m,d)} = \text{eig}\left( (C_{m',d'} - C_{m,d})\right)\text{eig}\left( (C_{m',d'} - C_{m,d})\right)
\]

where \( \text{eig}(Z) \) means the eigenvalues of \( Z \). All possible eigenvalue vectors \( \{\Lambda_{(m',d'),(m,d)}\} \) constitute the eigenvalue/distance spectrum of the M-DUST constellation. The number of pairs in the constellation with the same eigenvalue vector is called multiplicity of this eigenvalue vector.

IV. PROPERTIES OF M-DUST CONSTELLATIONS

Let us first investigate some properties of M-DUST constellations. Proofs can be found in [14].

**Property 1 (Diversity Product):** Let \( I_{N_t} \) denote the \( N_t \times N_t \) identity matrix. The diversity product \( \zeta \) of an M-DUST constellation \( \{C_{m,d}\} \) can be simplified to

\[
\zeta = \min_{\Delta m_l, \Delta d} \zeta_{\Delta m_l, \Delta \Delta m_l, \Delta d}
\]

where

\[
\zeta_{\Delta m_l} = \left| \sin\left(\pi \Delta m_l\right)\right|
\]

with \( \Delta m_l = 1, 2, \cdots, M-1 \), and

\[
\zeta_{\Delta m_l, \Delta d} = \frac{1}{2} \left| \det\left( U_1^{\Delta d}U_2^{\Delta d} - \exp\left(j\frac{2\pi}{M}\Delta m_l\right)I_{N_t}\right)\right|^{\frac{1}{M}}
\]

for \( \Delta m_l = 0, 1, \cdots, M-1 \) and \( \Delta d = 1, 2, \cdots, D-1 \).

**Property 2 (Upper Bound on Diversity Product):** \( \zeta \) of M-DUST constellations is upper-bounded by

\[
\zeta \leq \min_{\Delta m_l = 1, 2, \cdots, M-1} \zeta_{\Delta m_l} = \sin\left(\frac{\pi}{M}\right)
\]

The diversity product is widely used for large SNR. When performance at lower SNRs is a major concern, the minimum Euclidean distance is a proper choice. Union bound on the block error rate, on the other hand, takes into account all error events/patterns thus characterizing the average performance, as opposed to the worst-case performance described by the diversity product or minimum Euclidean distance. The union bound is fully determined by the following distance/eigenvalue spectrum property.

**Property 3 (Eigenvalue Spectrum and Multiplicity):** Any M-DUST constellation \( \{C_{m,d}\} \) has only \( L-1 \) distinct vectors of eigenvalues which, according to their expressions and multiplicities, can be categorized into two classes. The first class is for two elements of \( (4) \) under the condition \( d = m', m \neq m \);

\[
\Lambda_{(m',d'),(m,d)} = 4\sin^2\left(\frac{\pi}{M}\Delta m_l\right)I_{N_t}
\]

where \( \Delta m_l = |m' - m| = 1, 2, \cdots, M-1 \) with corresponding multiplicity given by \( 2D(\Delta m_l - 1), 2D(\Delta m_l - 2), \cdots, 2D, \) respectively. The eigenvalue vector \( \Lambda_{(m',d'),(m,d)} \) depends on \( (m', m) \) only through \( \Delta m_l \), and is thus simply written as \( \Lambda^{I}_{\Delta m_l} \). The second class corresponds to any pair of elements of \( (4) \) under the condition \( d' \neq d \). Specifically, by letting \( \Delta d = |d' - d| = 1, 2, \cdots, M-1 \) and \( \Delta m_l = |m' - m| = 0, 1, \cdots, M-1 \), we have

\[
\Lambda_{(m',d'),(m,d)} = \text{eig}\left( U_1^{\Delta d}U_2^{\Delta d} - \exp\left(j\frac{2\pi}{M}\Delta m_l\right)I_{N_t}\right) = \Lambda^{II}_{\Delta m_l, \Delta d}
\]

which depends on \( (m', d) \) and \( (d', d) \) only through their differences \( \Delta m_l \) and \( \Delta d \) and is thus simply denoted by \( \Lambda^{II}_{\Delta m_l, \Delta d} \). The multiplicity is independent of \( \Delta m_l \) but varying with \( \Delta d \), as shown by \( 2M(\Delta d - 1) \) for \( \Delta d = 1, 2, \cdots, D-1 \).

Taking the two classes into account, there are a total of \( (M-1) + M(\Delta d - 1) = L-1 \) distinct eigenvalue vectors. For a general constellation of size \( L \), there can be \( L(L-1)/2 \) pairwise combinations of elements. Here we see that, though not a group constellation, the proposed M-DUST constellations share a good property of the former having \( L-1 \) distinct eigenvalue vectors. This property is especially useful in optimizing M-DUST constellations.

The eigenvalue spectrum property enables the following assertion.

**Property 4 (Upper Bound on Minimum Euclidean Distance):**

The minimum Euclidean distance, defined by \( [5] \)

\[
\xi \triangleq \min_{(m',d') \neq (m,d)} \frac{1}{2\sqrt{N_t}} \| C_{m',d'} - C_{m,d} \| F
\]
has an upper bound

\[ \xi \leq \min_{\Delta m_1 = 1, 2, \ldots, M-1} \left| \sin \left( \frac{\pi}{M} \Delta m_1 \right) \right| = \sin \left( \frac{\pi}{M} \right). \]  

(14)

**Property 5 (Chernoff Union Bound on Block Error Rate):** The Chernoff bound on the pairwise error probability of mistaking \( C_{m',d'} \) for \( C_{m,d} \) is given by [4]

\[ \Pr(C_{m',d'} \rightarrow C_{m,d}) \leq \frac{1}{2} \prod_{n=1}^{N} \left[ 1 + \frac{\rho^2}{4(1 + 2\rho)} \lambda_n \right]^{-N}. \]  

(15)

where \( [\lambda_1, \lambda_2, \ldots, \lambda_{N}] = \Lambda(m',d')(m,d) \) are given by (12) and (13). Consequently, the Chernoff bound on block error rate can be expressed as

\[ P_e \leq \frac{1}{L} \sum_{(m,d)} \sum_{(m',d') \neq (m,d)} \Pr(C_{m',d'} \rightarrow C_{m,d}) = P_e^I + P_e^{II} \]

with

\[ P_e^I = \frac{D}{L} \sum_{\Delta m_1 = 1}^{M-1} (M - \Delta m_1) \left[ 1 + \frac{\rho^2}{4(1 + 2\rho)} \sin^2 \left( \frac{\pi}{M} \Delta m_1 \right) \right]^{-N}, \]  

(16)

\[ P_e^{II} = \frac{M}{L} \sum_{\Delta m_1 = 0}^{M-1} \sum_{\Delta d = 1}^{D-1} (D - \Delta d) \prod_{n=1}^{N} \left[ 1 + \frac{\rho^2}{4(1 + 2\rho)} \lambda_n^I \right]^{-N}, \]  

where \( [\lambda_1^I, \lambda_2^I, \ldots, \lambda_{N}^I] = \Lambda_{\Delta m_1, \Delta d}^I \) is given by (13).

**Property 6 (Simplified Maximum likelihood Decoding):** The number of complex matrix multiplications and trace operations required for ML decoding of M-DUST signals is \( D = L/M \), rather than \( L \).

Finally, a full-diversity special case is provided below, which can be regarded as a direct extension of M-PSK modulation to space-time modulation.

**Property 7 (Special Case: Full Diversity):** Assume \( M = L \) and \( D = 1 \). The corresponding M-DUST constellations have diversity product given by

\[ \zeta = \min_{\Delta e = 1, 2, \ldots, L-1} \left| \sin \left( \frac{\pi}{L} \Delta e \right) \right| = \sin \left( \frac{\pi}{L} \right) > 0. \]  

(17)

Parameter \( \Delta e \) is crucial to the design of M-DUST constellations. A large value of \( \Delta e \) will lower the upper bound on the diversity product as indicated in (11). On the other hand, increase in \( \Delta e \) implies a complexity reduction in detecting symbols from a M-DUST constellation since \( D = L/M \). Accordingly, the choice of \( \Delta e \) is a trade-off between the performance and ML decoding complexity. The upper bound on diversity product (11) is simply given by \( \sin(\pi/L) \). It serves as guidance in designing an M-DUST constellation to meet a target diversity product. It is also interesting to note that the diversity product \( \zeta \) and minimum Euclidean distance \( \xi \) have the same upper bound.

V. DESIGN OF M-DUST CONSTELLATIONS

To design M-DUST constellations, we need to determine parameter \( M \) and to optimize unitary matrices \( U_1 \) and \( U_2 \). Once \( M \) is selected, the optimization problem can be expressed as

\[ (U_1, U_2)_{opt} = \arg \max_{U_1, U_2} \zeta \]

(18)

where the diversity product is utilized as our design criterion. So far, the constellation size \( L \) is assumed to have a factored form of \( L = M \times D \). This condition is not really a restriction since very often, \( L \) is an integer power of 2. If in certain cases where \( L \) is a prime, we can always replace \( L \) by a close non-prime number, such as \( L + 1 \).

A. Selection of \( M \)

Since \( M \) controls both the performance (characterized by the upper bound on diversity product) and decoding complexity (measured by the number of complex matrix multiplications and trace operations), it is necessary to make a judicious choice by taking into consideration both the performance and decoding complexity. A good suggestion is that \( M \) is selected as large as possible while keeping the upper diversity-product bound of candidate M-DUST constellations to meet the performance requirement.

B. Optimization of \( U_1 \) and \( U_2 \)

It remains to optimize unitary matrices \( U_1 \) and \( U_2 \). We perform optimization in two steps.

1) Givens Parameterization: From [8] (p. 297) we know that any \( N_i \times N_i \) unitary matrix can be completely characterized by \( N_i^2 \) rotation angles.

2) Hybrid Optimization: The objective function to be optimized is a highly nonlinear multimodulation function characterized by \( 2N_i^2 \) rotation angles. The conventional gradient-based method usually fails to yield satisfactory results since it may stick at a local optimum. Here we adopt a hybrid optimization approach, aiming to incorporate the advantages of both global and local search. The local search is based on the gradient method whereas the global search relies on a heuristic/stochastic algorithm such as a real-coded (i.e., with real variables) genetic algorithm (GA) [9]. GAs are population based parallel search with three basic operations; mutation, crossover, and selection. Interested readers are referred to [10]–[13] and the references therein for more details about GAs and their applications in communications. In this paper, a real-coded GA with the non-uniform mutation [9] (p. 293) and the BLX-\( \alpha \) crossover [9] (p. 288) is used. As illustrated in Fig. 1, where \( P \) is the population size, the hybrid optimization search can be divided into two phases:

**Phase 1: Coarse Search**

In the coarse-search phase (see Fig. 1(a)), only the real-coded GA is applied so that most local optima can be avoided.

**Phase 2: Fine Search**

In the fine-search phase (see Fig. 1(b)), a gradient-based search implemented by a MATLAB function \( \text{fmincon} \) is employed to improve the convergence speed of the GA.
In this section, the proposed M-DUST constellations, obtained by numerical optimization of diversity product $\varsigma$, are compared with DUST constellations available in the literature. We assume that ML decoding is used and the number of receive antennas is $1 (N_r = 1)$. As in Section II, we assume a block Rayleigh fading channel model with channel coefficients unchanged over two consecutive blocks. In certain cases, one may replace the constellation size $L$ (even if $L$ is not prime) by a close integer to make the factorization $L = M \times D$ more flexible. In some numerical examples, we can obtain global optima (equal to upper bound on diversity product), which will be marked by superscript $\ast$ in the following tables. The general superiority of the optimized M-DUST constellations in diversity product has been extensively verified through computer evaluation. Due to space limitations, however, only a small portion of optimized M-DUST constellations are provided here. More results are provided in [14].

A. Comparison with Cyclic, Quaternion, Orthogonal, and Parametric Constellations

Cyclic constellations are well studied [1], [2]. It is shown that cyclic constellations are optimal group constellations of size $2^p$ where $p$ is a positive integer [3]. Orthogonal and parametric constellations are designed for systems with two transmit antennas. Specifically, orthogonal constellations [4] (p. 2338) are derived from the famous Alamouti’s space-time block codes, and parametric constellations [5] are obtained by numerical optimization with respect to three parameters $k_1$, $k_2$, and $k_3$. Comparison of diversity product among M-DUST, cyclic, quaternion, orthogonal, and parametric constellations is shown in Table I, where the reference diversity products $\varsigma_{ref}$ are taken from Table I of [4] (p. 2339) for the cyclic, quaternion, and orthogonal constellations, and Table I of [5] (p. 2298) for the parametric constellations. It is observed that in general, M-DUST constellations have larger diversity products than other constellations.

B. Comparison with Fixed-Point-Free Group Constellations

In [4], all finite fixed-point-free (full diverse) group constellations are classified. Comparison between M-DUST and fixed-point-free constellations is listed in Table II, where the reference diversity products $\varsigma_{ref}$ are taken from Table III of [4] (p. 2357). We see that M-DUST constellations outperform the $G_{m,r}$ group constellations. The latter include the cyclic group constellations as a special case and are the least restrictive group constellations. On the other hand, when compared with other group constellations including $E_{m,r}$, $F_{m,r,t}$, $J_{m,r}$, and $K_{m,r,t}$, M-DUST constellations have similar diversity products or even larger. But in the special case of $N_t = 4$ and $L = 240$, the group constellation $K_{1,1,1}$ is clearly better than its M-DUST counterpart. This means that group constellations excluding $G_{m,r}$ generally have good performance. Their construction, however, needs to impose restrictions on key parameters such as constellation size $L$ and number of transmit antennas $N_t$ (see Table II of [4], p. 2345). For M-DUST constellations, there are no such strong limitations. Finally, it is interesting to see that for the case of $N_t = 2$ and $L = 120$, the M-DUST constellation has the same diversity product as the famous group constellation $J_{1,1}$ while enjoying a simplified ML decoding complexity $\left(\frac{L}{2} = \frac{1}{2}\right)$.

VI. NUMERICAL RESULTS AND DISCUSSIONS

In this section, the proposed M-DUST constellations, obtained by numerical optimization of diversity product $\varsigma$, are compared with DUST constellations available in the literature. We assume that ML decoding is used and the number of receive antennas is $1 (N_r = 1)$. As in Section II, we assume a block Rayleigh fading channel model with channel coefficients unchanged over two consecutive blocks. In certain cases, one may replace the constellation size $L$ (even if $L$ is not prime) by a close integer to make the factorization $L = M \times D$ more flexible. In some numerical examples, we can obtain global optima (equal to upper bound on diversity product), which will be marked by superscript $\ast$ in the following tables. The general superiority of the optimized M-DUST constellations in diversity product has been extensively verified through computer evaluation. Due to space limitations, however, only a small portion of optimized M-DUST constellations are provided here. More results are provided in [14].

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TABLE III

<table>
<thead>
<tr>
<th>( N_t )</th>
<th>( L )</th>
<th>( \zeta_{ref} )</th>
<th>( \zeta_{M-DUST} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>81</td>
<td>0.2417 (( L_A = 9 ))</td>
<td>0.3140 (( M = 9 ))</td>
</tr>
<tr>
<td>829</td>
<td>1089</td>
<td>0.0794 (( L_A = 33 ))</td>
<td>0.0930 (( M = 33 ))</td>
</tr>
<tr>
<td>3</td>
<td>57</td>
<td>0.4845 (( L_A = 23 ))</td>
<td>0.4401 (( M = 33 ))</td>
</tr>
<tr>
<td>20</td>
<td>1089</td>
<td>0.1863 (( L_A = 23 ))</td>
<td>0.2234 (( M = 10, L = 53 ))</td>
</tr>
<tr>
<td>2</td>
<td>512</td>
<td>0.3105 (( L_A = 11 ))</td>
<td>0.3000 (( M = 10, L = 290 ))</td>
</tr>
<tr>
<td>1</td>
<td>33</td>
<td>0.5580 (( S_3, 3 ))</td>
<td>0.5604 (( M = 3 ))</td>
</tr>
<tr>
<td>5</td>
<td>1369</td>
<td>0.2307 (( L_A = 37 ))</td>
<td>0.2310 (( M = 5, L = 1370 ))</td>
</tr>
</tbody>
</table>

TABLE IV

<table>
<thead>
<tr>
<th>( N_t )</th>
<th>( L )</th>
<th>( \zeta_{ref} )</th>
<th>( \zeta_{M-DUST} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>64</td>
<td>0.4083 (Amicable, 4-PSK)</td>
<td>0.4617 (( M = 4 ))</td>
</tr>
<tr>
<td>2</td>
<td>512</td>
<td>0.2208 (Amicable, 8-PSK)</td>
<td>0.2758 (( M = 8 ))</td>
</tr>
<tr>
<td>2</td>
<td>4096</td>
<td>7.4157 \times 10^{-3} \text{ (Cayley)}</td>
<td>4.9068 \times 10^{-2} \text{ (( M = 64 ))}</td>
</tr>
</tbody>
</table>

C. Comparison with Non-group Constellations

Given that the number of available group constellations is limited, non-group constellations are also discussed in [4]. However, due to the close connection between the non-group and group constellations, the number of available non-group constellations therein is also limited. Comparison between the M-DUST and non-group constellations is listed in Table III, where the reference diversity products \( \zeta_{ref} \) are taken from Table IV of [4] (p. 2358). We see that except for the special case of \( N_t = 3 \) and \( L = 57 \), the M-DUST constellations in general perform better than, or at least comparable with, the non-group constellations. In particular, for the cases of \( N_t = 2 \) and \( L = 81, 289, 1089 \), the M-DUST constellations outperform their counterparts and yet, are simple in decoding (\( \frac{D}{s} = \frac{1}{\sqrt{L}} \)).

D. Comparison with Amicable Constellations

In [6], unitary constellations are derived from Amicable orthogonal designs that have been used for the construction of orthogonal space-time block codes. We consider the case of \( N_t = 4 \). Amicable constellations are defined by eq. (17) of [6]. Comparison between the M-DUST and Amicable constellations is shown in Table IV in terms of diversity product. It can be seen that the M-DUST constellations outperform their counterparts, especially when the constellation size is large. It is worth noting that amicable constellations have linear decoding complexity. Unfortunately, the number of available amicable constellations is particularly limited due to the stringent requirements of Amicable orthogonal designs. The same fundamental limitation is also encountered in the design of orthogonal space-time block codes.

E. Comparison with Cayley Constellations

Cayley transform is used in [7] to construct unitary constellations from skew-Hermitian matrices. The Cayley constellations so obtained have a relatively simple ML decoding (sphere decoding) complexity. However, unlike the group and M-DUST constellations, the number of different eigenvalue vectors for a Cayley constellation of size \( L \) can be as large as \( L(L - 1)/2 \). To overcome the difficulty in performance evaluation, a statistical (rather than exhaustive but deterministic) criterion is used for the optimization of Cayley constellations [7]. This approach may incur a performance penalty. For comparison, we choose the Cayley constellation provided in [7] (p. 1495) with parameters \( Q = 4, r = 8 \), and \( N_t = 2 \). The results are shown in Table IV. It is clear that the M-DUST constellation outperforms the Cayley constellation while allowing for simple ML decoding (\( \frac{D}{s} = \frac{1}{\sqrt{L}} \)).

VII. Conclusions

We have proposed M-DUST constellations for differential space-time modulation, which are derivable from a set of \( M \)-PSK symbols and two basis unitary matrices \( U_1 \) and \( U_2 \). The most important feature of the M-DUST constellations is that the parameter \( M \) has conflicting influence on the error performance and ML decoding complexity. This feature, along with the simple upper-bound on diversity product of M-DUST constellations, can serve as a design guide for the selection of \( M \). Numerical examples show that in most cases the optimized M-DUST constellations outperform other alternatives while allowing for simplified ML decoding.

REFERENCES