ADAPTIVE ROBUST CONTROL OF UNCERTAIN SYSTEMS
WITH TIME-VARYING STATE DELAY
Jian-Xin Xu, Qing-Wei Jia and Tong-Heng Lee

ABSTRACT

In this paper, a new adaptive robust control scheme is developed for a class of uncertain dynamical systems with time-varying state delay, unknown parameters and disturbances. By incorporating adaptive techniques into the robust control method, we propose a continuous adaptive robust controller which guarantees the uniform boundedness of the system and at the same time, the regulating error enters an arbitrarily designated zone in a finite time. The proposed controller is independent of the time-delay, hence it is applicable to a class of dynamical systems with uncertain time delays. The paper includes simulation studies demonstrating the performance of the proposed control scheme.

KeyWords: Adaptive robust control, time-varying state delay, nonlinear uncertain systems.

I. INTRODUCTION

Time-varying delay has been often encountered in various engineering systems. Since the existence of such delay is frequently a source of instability, considerable attention has been paid to the study of systems with delay [1-7].

Cheres et al. [1] proposed mini-max controllers for time-varying dynamical systems with uncertain parameters, disturbance, and a bounded time-varying state delay. It is shown that the proposed mini-max controllers can guarantee the uniform asymptotic stability of this class of uncertain time-delay systems. Because the so-called min-max controller is discontinuous, in [1] a continuous controller is employed to take the place of the min-max controller in the practical implementation. Thus, the asymptotic stability result can no longer be guaranteed practically. Continuous linear and nonlinear state feedback controllers are proposed in [2], where uncertain delay systems similar to [1] are considered. Uniform ultimate boundedness and uniform asymptotic stability can be guaranteed respectively. Note that the control methods proposed both in [1] and [2] have to solve the Riccati equation, whose analytical solutions are generally difficult to find and numerical solutions are time-consuming to obtain. It is also worth noting that all the controllers in [1,2] are designed based on the bounds of the unknown parameters, system disturbances and time-delay states. Hence the controller design may be conservative and may incur large control action.

In this paper, an adaptive robust control scheme is proposed for a class of uncertain dynamical systems with time-varying state delay, which includes the systems considered in [1-3] as its subsets. To reduce the robust control efforts, we incorporate adaptive techniques to deal with the unknown parameters as well as the unknown gains in the upper bounding function. The adaptive methods are designed based on the bounds of the unknown parameters as well as the unknown gains in the upper bounding function. The adaptive methods are designed based on the bounds of the unknown parameters as well as the unknown gains in the upper bounding function. The adaptive methods are designed based on the bounds of the unknown parameters as well as the unknown gains in the upper bounding function.

Since the time delay in the system states is assumed to be a non-negative continuous function with unknown bound, memory-based robust control methods fail to work. Analogous to [1] and [2], the Razumikhin-type theorem [5] is used to address this problem, but more general system uncertainties, nonlinearities as well as adaptive mechanisms are taken into account. Without
assuming that a bounding parameter \( \vartheta \) in the Razumikhin-type theorem is known and sufficiently small as in [2], we estimate the unknown parameter \( \vartheta \) by adaptation. As a consequence, the proposed controller is completely independent of the time-delay.

In the adaptive robust scheme, a \( \mu \)-modification method [8] is adopted to cease the adaptations and produce continuous controller. It is shown that the proposed controller can guarantee the uniform boundedness of the system and at the same time, the regulating error enters an arbitrarily designated zone in a finite time.

This paper is organized as follows. Section 2 describes the class of nonlinear uncertain dynamical systems with time-varying state delay to be controlled. Section 3 gives the design procedure of the adaptive robust controller. Section 4 gives the stability analysis. Simulation-based studies on the proposed method are given in Section 5.

**II. PROBLEM FORMULATION**

Consider a class of uncertain dynamical system described by

\[
\dot{x}(t) = f(x, t) + A_0 p(t) f_p(x(t - h(t)), t) + B_0 [I + E(x, p, t)] u(t) + d_1(x, p, t)
\]

\[+ d_2(x, p, t)\]

where \( x = [x_1, x_2, \ldots, x_n]^T \in \mathcal{R}^n \) is the measurable state vector of the system, \( u(t) \in \mathcal{R}^m \) is the control input vector of the system, \( B_0 \in \mathcal{R}^{n \times m} \) is the input distribution matrix with full column rank; \( f(x, t, \in \mathcal{R}^n \) is a nonlinear function; \( p \in \mathcal{P} \) is an unknown system parameter vector and \( \mathcal{P} \) is the set of admissible system parameters; \( E(x, p, t) \) represents all possible perturbations of the input distribution matrix; \( d_1(x, p, t), d_2(x, p, t) \in \mathcal{R}^n \) are system uncertainties which may be nonlinear functions of the state \( x \), unknown parameter \( p \) and time \( t \); the time delay \( h(t) \) is a bounded and continuous function, i.e., \( 0 \leq h(t) \leq h' \), where \( h' \) is a nonnegative constant which may be unknown. The initial condition for (1) is given by

\[
x(t) = \phi(t), \ t \in [t_0 - h', t_0]
\]

where \( \phi(t) \) is a continuous function on \([t_0 - h', t_0]\).

In this paper we make the following assumptions:

**Assumption A1.** For the chosen function vector \( g(x, t) \), there exists a \( C^1 \) Lyapunov function \( V_i(x, t) \); \( \mathcal{R}^n \times \mathcal{R} \Rightarrow \mathcal{R}^+ \) and \( \mathcal{R}_C \) class functions \( \varphi_i \) (\( i = 1, 2, 3 \)), such that

\[
\varphi_i(\|x\|) \leq V_i(x, t) \leq \varphi_i(\|x\|)
\]

\[
\partial V_i / \partial t + \partial V_i / \partial x d(x, t) \leq -\varphi_i(\|x\|),
\]

where \( \| \cdot \| \) is the Euclidean norm for vectors and the spectral norm for matrices. Note that this assumption assures that the zero state of the system described by

\[
x = g(x, t)
\]

is uniformly asymptotically stable.

**Assumption A2.** There exists a function vector \( v(x, t) \) \( \in \mathcal{R}^n \) such that

\[
f(x, t) - g(x, t) = B_0 v(x, t).
\]

**Remark 1.** Assumption A2 restricts the structure of \( g(x, t) \). It should be noted that this assumption can always be satisfied if the system is with enough inputs [9]. When the system (1) is linear, then the matching condition can be interpreted in a more precise way. Suppose \( f(x, t) = A x \) and the pair \((A, B_0)\) is stabilizable. Then one may choose \( g(x, t) = (A + B_0 K) x \), where \( K \) is a gain matrix which is chosen such that all the eigenvalues of \( A + B_0 K \) are with negative real parts. In such a case, it is easily to verify that Assumption A2 is guaranteed with \( v = -K x \).

**Assumption A3.** For \( E \in \mathcal{R}^{m \times n} \), there exist known bounding functions \( r_{max}(x, t) \) and \( r_{min}(x, t) \) such that

\[
\forall t \in [t_0 - h', \infty) \ \forall x \in \mathcal{D} \ \forall p \in \mathcal{P}
\]

\[
r_{max} \geq \lambda (\frac{1}{2} E + \frac{1}{2} E^T) \geq r_{min} > -1
\]

where \( \lambda (\cdot) \) indicates the eigenvalues of \( \cdot \) and \( \mathcal{D} \) is a compact subset of \( \mathcal{R}^n \) in which the solution of (1) uniquely exists.

**Assumption A4.** The following matching condition [1], [2], [3] holds

\[
A_0(p, t) = B_0(t) H_0(p, t)
\]

where

\[
\|H_0(p, t)\| \leq \rho_0(t)
\]

with \( \rho_0(t) > 0 \) a known function. Moreover

\[
\|f_i(x(t - h(t)), t)\| \leq \eta(x, t) \|x(t - h(t))\|
\]

where \( \eta(x, t) \) is a positive function.
Assumption A5. The structured uncertainty \( d_i(x, p, t) = [d_{i1}(x, p, t), d_{i2}(x, p, t), \ldots, d_{im}(x, p, t)]^T \in \mathcal{K}^m \) can be expressed as

\[
d_i(x, p, t) = \Theta(p)\xi(x, t)
\]

\[
\Theta = \text{diag}(\theta_1^T, \ldots, \theta_n^T)
\]

\[
\xi = [\xi_1^T, \xi_2^T, \ldots, \xi_m^T]^T
\]

(7)

where \( \theta_i^T = [\theta_{i1}, \ldots, \theta_{in}] \) is an unknown parameter vector and \( \xi_i^T = [\xi_{i1}, \ldots, \xi_{im}] \) is a known function vector. The non-structured uncertainty \( d_j(x, p, t) = [d_{j1}(x, p, t), \ldots, d_{jm}(x, p, t)]^T \) is bounded such that

\[
\forall t \in [t_0 - h, \infty) \quad \forall x \in \mathcal{D} \quad \forall p \in \mathcal{P}
\]

\[
\|d_j(x, p, t)\| \leq \rho_j(x, q, t)
\]

(8)

where \( \rho_j(x, p, t) \) is an upper bounding function with unknown parameters (gains) \( q \in \mathcal{P} \). Here \( \rho_j(x, p, t) \) is differentiable and concave to \( q \), that is

\[
\rho_j(x, q, t) - \rho_j(x, q', t) \leq (q - q')^T \frac{\partial \rho_j}{\partial q} \bigg|_{q_1}
\]

(9)

**Remark 2.** [1] and [2] only consider the system uncertainty \( d \); together with a known upper bounding function, which is a particular case of (8) in the sense that \( q \) is known.

**Remark 3.** It should be noted that \( d_i(x, p, t) \) can be absorbed into \( d_j(x, p, t) \). However, it is obviously more conservative. This can be clearly shown through the following example. Assume that the structured uncertainty is \( d_i = \theta_i \xi_1 + \theta_2 \xi_2 \), with actual values \( \theta_1 = a, \theta_2 = -a \) and \( a \) is an unknown constant. Assume that the non-linear function \( \xi_2 = \xi_1 + \Delta \xi \), where \( \Delta \xi \ll \xi_1 \).

Then \( \|d_i\| \leq \|\theta_1\| \cdot \|\xi_1\| + \|\theta_2\| \cdot \|\xi_2\| = \gamma \|\xi_1\| \), where \( \gamma = \|\theta_1\| \cdot \|\xi_1\| + \|\theta_2\| \cdot \|\xi_2\| \). The upperbound parameter to be estimated is \( \gamma = \|\theta_1\| \cdot \|\xi_1\| + \|\theta_2\| \cdot \|\xi_2\| \). This implies that the actual uncertainty \( d_i = -a \Delta \xi \) has been amplified to its normed product \( \|a\| \cdot \|\xi_1\| + \|\theta_1\| \cdot \|\xi_2\| \), which is much larger than \( \|\theta_1\| \cdot \|\xi_1\| + \|\theta_2\| \cdot \|\xi_2\| \).\n
On the contrary, if the uncertainty is expressed by (7), the unknown parameters to be estimated is \( [a, -a]^T \). This means that, when the estimated parameters are near the true values, the estimated uncertainty of \( d_i \) will be able to approach the actual uncertainty \( a \Delta \xi \).

For simplicity, in this paper some time we use the abbreviation \( M \) in stead of \( M(\cdot) \), where \( M(\cdot) \) represents an arbitrary function or matrix used in this paper, and \( x_h \) instead of \( x(t - h(t)) \).

### III. The Adaptive Robust Controller

The adaptive robust control technique is used in this section to develop a controller which guarantees the global boundedness of the system. The design procedures of the adaptive robust control law are presented in details as follows.

Under the Assumption A2-5, the system (1) can be transformed into

\[
\dot{x} = f + B_d[l + E\dot{u} + H_f u_d + d_1 + d_3]
\]

\[
= g + B_d[l + E\dot{u} + v + H_f u_d + d_1 + d_3]
\]

(10)

Define the parameter error matrices as

\[
\bar{q} = q - \tilde{q}
\]

\[
\bar{\Theta} = \Theta - \tilde{\Theta}
\]

\[
\bar{\omega} = \omega - \tilde{\omega}
\]

\[
\bar{h} = h - \tilde{h}
\]

(11) (12) (13) (14)

where \( \tilde{q}, \Theta, \omega \) and \( h \) are the estimates of \( q, \Theta, \omega \) and \( h \), respectively. \( \omega \) and the bounding parameter \( h \) are defined in (27) and (36), respectively.

The control law \( u \) are chosen to be

\[
u = u_i + u_v
\]

\[
u_i = -v - \Theta \xi - v_d
\]

(15) (16)

\[
u_i = -\frac{r^2}{1 + r_{\max}^2} \frac{\dot{u} \alpha}{\alpha} + \frac{k^2 \omega^2 \rho_3^2}{\omega \rho_3} \frac{\alpha}{\alpha} + \frac{\epsilon_1}{\epsilon_1}
\]

(17)

\[
v_i = \frac{\rho_3^2}{\rho_3^2} \frac{\alpha}{\alpha} + \frac{\epsilon_2}{\epsilon_2}
\]

\[
v_i = \frac{\epsilon_1}{\epsilon_1} + \frac{\epsilon_3}{\epsilon_3}
\]

(18)

where \( \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \) and \( k \) are positive constants. The corresponding adaptive laws, are given as

\[
\dot{\Theta} = \Gamma_1 (\alpha \xi^T - \mu_\Theta \tilde{\Theta})
\]

\[
\dot{\mu}_\Theta = \Gamma_1 \alpha \frac{\partial \Theta}{\partial q}
\]

\[
\dot{\omega} = \eta (\omega \rho_3) \frac{\alpha}{\alpha} - \mu_\omega \tilde{\omega}
\]

\[
\dot{\tilde{h}} = \gamma (\rho_2 \tilde{\omega} \tilde{h}) \frac{\alpha}{\alpha} - \mu_\omega \tilde{h}
\]

(19) (20) (21) (22)
where
\[
\begin{align*}
\dot{\rho}_d &= \rho_d(x, \hat{q}, t) \\
\alpha &\leq B_1^T \partial V_1 / \partial x \\
\dot{\rho}_h &= \eta \rho_h \\
\dot{q} &\leq \dot{q} + \frac{1}{\gamma_2} \left[ x + \sqrt{\lambda_{\max}(\Gamma_1^{-1})} \phi + \sqrt{\lambda_{\max}(\Gamma_2^{-1})} \hat{q} \right] \\
\dot{\omega} &\leq \frac{1}{2} \frac{1}{\gamma_3 \gamma_4} \left[ x + \sqrt{\lambda_{\max}(\Gamma_1^{-1})} \phi + \sqrt{\lambda_{\max}(\Gamma_2^{-1})} \hat{q} \right] \frac{1}{\gamma_3 \gamma_4} \left[ \gamma_1 \omega + \gamma_2 \hat{q} \right],
\end{align*}
\]

\( \gamma_1 \) and \( \gamma_2 \) are positive constants; \( \Gamma_i, i = 1, 2 \) are positive definite matrices chosen to be
\[
\Gamma_i = \text{diag}(\gamma_i, \gamma_i, \ldots, \gamma_i).
\]

\( \mu_i, i = 1, 2, 3, 4 \), which constitute the \( \mu \)-modification scheme, are defined as
\[
\mu_i = \begin{cases} 
    k_i \exp \left( -\frac{1}{2} |x|^2 \right) & x \in E_0 \\
    0 & \text{elsewhere}
\end{cases}
\]

where \( k_i, i = 1, 2, 3, 4 \) are positive constants, and
\[
E_0 = \{ x : |x| < \varepsilon_0 \}
\]

where \( \varepsilon_0 \) is a positive constant specifying the desired regulating error bound.

IV. CONVERGENCE ANALYSIS

For the above mentioned adaptive robust control controller (15)-(22), we have the following theorem.

**Theorem 1.** By properly choosing the function vector \( g(x, t) \) such that \( \phi(||x||) \geq k_0 V_1(x, t) \), where \( k_0 \) is a positive constant to be defined later, the proposed adaptive robust control law ensures that the system trajectory enters the set \( E_0 \) in a finite time. Moreover, the parameter estimation errors are bounded by the set
\[
S = \{ \Theta, \hat{q}, \hat{\omega}, \hat{\vartheta} : \text{trace} \{ \Theta^T \Theta \} + \frac{1}{2} |\hat{q}|^2 + |\hat{\omega}|^2 + |\hat{\vartheta}|^2 \leq k' \frac{1}{2} k_e e_0 \text{trace} \{ \Theta^T \Theta \} + \frac{1}{2} k_e e_0 |\hat{q}|^2 + \frac{1}{2} k_e e_0 |\hat{\omega}|^2 + \frac{1}{2} k_e e_0 |\hat{\vartheta}|^2 \}
\]

where \( k' \) is defined to be
\[
k' = \frac{1}{k'' \min \{ \lambda_{\min}(\Gamma_1^{-1}), \lambda_{\min}(\Gamma_2^{-1}), \vartheta_1^{-1}, \vartheta_2^{-1} \}}
\]

with
\[
k'' = \frac{2 \min \{ k_{\omega}, k_{\vartheta}, \delta, k_z \delta, k_{\vartheta}, k_{\varphi}, \delta \}}{\max \{ 1, \lambda_{\max}(\Gamma_1^{-1}), \lambda_{\max}(\Gamma_2^{-1}), \vartheta_1^{-1}, \vartheta_2^{-1} \}}
\]

\( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) indicate the maximum and minimum eigenvalues of the matrix \( A \), respectively, and \( \delta \) and \( k_0 \) are positive values to be defined later.

**Proof.** The following positive definite function \( V \) is selected
\[
V = \frac{1}{2} \text{trace} \{ \Theta^T \Gamma_1^{-1} \Theta \} + \frac{1}{2} |\hat{q}|^2 + \frac{1}{2} |\hat{\omega}|^2 + |\hat{\vartheta}|^2
\]

Taking the derivative of \( V \) along the trajectory of the dynamic system (1) with the control law (15)-(22), and using (10) we have
\[
\dot{V} = 2 \frac{1}{2} \text{trace} \{ \Theta^T \Gamma_1^{-1} \Theta \} + \frac{1}{2} |\hat{q}|^2 + \frac{1}{2} |\hat{\omega}|^2 + |\hat{\vartheta}|^2
\]
In the above derivation the following property of trace is used:

\[
\text{trace}(Q^Tvw^T) = v^TQw
\]  
(35)

where \( v \in \mathbb{R}^{n \times 1} \), \( w \in \mathbb{R}^{m \times 1} \), and \( Q \in \mathbb{R}^{n \times m} \).

From Assumption A1, we have

\[
\partial V / \partial t + \partial V / \partial x \leq -\varphi_t(\|x\|).
\]

Following the Razumikhin-type theorem [5], we further assume that for a positive number \( \vartheta > 1 \), the following inequality holds

\[
V(x(t), \Theta(t), \bar{q}(\tau), \bar{\theta}(\tau), \tau) \leq \vartheta^2 V(x(t), \Theta(t), \bar{q}(t), \bar{\theta}(t), t)
\]
(36)

where \( t - h^* \leq \tau \leq t \). Then we can easily have (see Appendix)

\[
\|x(t - h(t))\| \leq \vartheta \| q + k \omega \|
\]  
(37)

with \( q \) and \( \omega \) defined in (26) and (27), respectively. It can be seen that this condition is more general than [1] and [2] because the upper bound in the right hand side of (37) may not be zero even when \( \varphi(x) = 0 \). \( \blacksquare \)

**Remark 4.** Note that the Razumikhin-type theorem only indicates the “existence” of \( \vartheta > 1 \). In [2], to guarantee the stability of the control system, it was required that \( \vartheta > 1 \) be “sufficiently small”. This requirement, however, is not quite reasonable as \( \vartheta \) is not a parameter to be arbitrarily designated. In this paper, an adaptive law is adopted to estimate the value of \( \vartheta \).

Using the fact that

\[
\alpha'(l + E) = \alpha' + \frac{E + E^T}{2}
\]

\[
\lambda\left(\frac{1}{r} + \frac{E}{2} + \frac{E^T}{2}\right) \geq 1,
\]

where \( \lambda(\cdot) \) indicates the eigenvalue of \( \cdot \), it follows that

\[
V \leq -\varphi_q(\|x\|) - \vartheta^2 \bar{q}_d + \gamma \rho \cdot \|q\| \cdot \|x\| + \|q\| \cdot \|d\|
\]

\[
+ \mu_t \text{trace}(\Theta^T\Theta') - \bar{q}^T \tilde{q} + \gamma_{t}^{0} \tilde{\omega} \cdot \tilde{\vartheta} + \gamma_{t}^{0} \tilde{\vartheta} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
\leq -\varphi_q(\|x\|) - \alpha' \bar{q}_d + \rho \gamma \cdot \|q\| \cdot \|x\| + \|q\| \cdot \|d\|
\]

\[
+ \mu_t \text{trace}(\Theta^T\Theta') - \bar{q}^T \tilde{q} + \gamma_{t}^{0} \tilde{\omega} \cdot \tilde{\vartheta} + \gamma_{t}^{0} \tilde{\vartheta} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
\leq -\varphi_q(\|x\|) - \alpha' \bar{q}_d + \frac{\kappa}{\kappa} \tilde{\omega} \cdot \tilde{\vartheta} + \frac{k}{k} \alpha' \cdot \alpha + \epsilon_2
\]

\[
+ \kappa \omega \rho \alpha \alpha - \kappa \kappa \tilde{\omega} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
+ \mu_t \text{trace}(\Theta^T\Theta') - \bar{q}^T \tilde{q} - \gamma_{t}^{0} \tilde{\omega} \cdot \tilde{\vartheta} - \gamma_{t}^{0} \tilde{\vartheta} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
- \frac{\partial^3 \rho^2 \beta}{\partial \rho \bar{q}} \alpha \alpha \left. \frac{\alpha}{\alpha} \right| + k \omega \rho \alpha \alpha - \kappa \kappa \tilde{\omega} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
+ \mu_t \text{trace}(\Theta^T\Theta') + \mu_t \bar{q}^T \bar{q}
\]

\[
+ \mu_t \tilde{\omega} \tilde{\vartheta} + \mu_t \tilde{\vartheta} \tilde{\vartheta} + \epsilon_1
\]

\[
(38)
\]

It can be easily verified that

\[
- \frac{\partial^2 \rho \bar{q}}{\partial \rho \bar{q}} \alpha \alpha \left. \frac{\alpha}{\alpha} \right| + k \omega \rho \alpha \alpha - \kappa \kappa \tilde{\omega} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
- \frac{\partial^2 \rho \bar{q}}{\partial \rho \bar{q}} \alpha \alpha \left. \frac{\alpha}{\alpha} \right| + k \omega \rho \alpha \alpha - \kappa \kappa \tilde{\omega} \cdot \tilde{\vartheta} + \epsilon_1
\]

\[
\leq \varphi_q(\|x\|) - \frac{1}{2} \mu_t \text{trace}(\Theta^T\Theta') - \frac{1}{2} \mu_t \bar{q}^T \bar{q} - \frac{1}{2} \mu_t \tilde{\omega}^2 - \frac{1}{2} \mu_t \tilde{\vartheta}^2
\]

\[
+ \frac{1}{2} \mu_t \text{trace}(\Theta^T\Theta') + \frac{1}{2} \mu_t \bar{q}^T \bar{q} + \frac{1}{2} \mu_t \tilde{\omega}^2 + \frac{1}{2} \mu_t \tilde{\vartheta}^2 + \epsilon_1
\]

(39)

where \( \epsilon = \epsilon_1 + \epsilon_2 + \epsilon_1 + \epsilon_2 \).

By choosing \( g(x, t) \) such that

\[
\varphi_q(\|x\|) \geq k \omega \varphi_q(x, t)
\]

(40)

\[
k_t = \frac{\epsilon + c_i}{\varphi_q(\|x\|)}
\]

(41)

and \( c_i \) is an arbitrary positive constant, then from (29) we have

\[
\dot{V} \leq -\varphi_q(\|x\|) + \epsilon \leq -c_1, \quad \forall x \in \mathbb{R}^n - E_0,
\]

(42)
which implies that there exists a constant $0 < \varepsilon'_0 < \varepsilon_0$ such that

$$\forall \mathbf{x} \in \mathcal{G}' - E'_0 \quad \dot{V} = 0$$

(43)

where $E'_0 = \{ \mathbf{x} : \| \mathbf{x} \| < \varepsilon'_0 \}$ is a subset of $E_0$. This indicates that the system will enter the set $E'_0$ in a finite time.

When $\mathbf{x} \in E'_0$, it is obvious that

$$k_i (\varepsilon_0 - \varepsilon'_0) \leq \mu_i \leq k_i E_0, \quad i = 1, 2, 3, 4.$$  

(44)

Define $\delta = \varepsilon_0 - \varepsilon'_0 > 0$, then from (29), (39) and (44) we obtain

$$
\dot{V} \leq - \varphi_i \| \mathbf{x} \| - \frac{1}{2} \mu_i \text{trace} (\Theta^T \Theta) - \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} \dot{\mathbf{q}} - \frac{1}{2} \mu_i \dot{\vartheta}^2 \\
+ \frac{1}{2} \mu_i \text{trace} (\Theta^T \Theta) + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 \\
\leq - \varphi_i \| \mathbf{x} \| - \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \text{trace} (\Theta^T \Theta) + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 \\
+ \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 + \varepsilon \leq - \kappa \dot{V} + \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 + \varepsilon \\
+ \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 + \varepsilon$$

(45)

where

$$k^* = \frac{2 \min \{ k_i, k_i \delta, k_i \delta, k_i \delta, k_i \delta \}}{max \{ 1, \lambda_{\min} (\Gamma_1^{-1}), \lambda_{\min} (\Gamma_2^{-1}), \gamma_3, \gamma_4 \}}.$$  

By solving (45) we can establish that

$$
\dot{V}(t) \leq e^{-\kappa t} V(0) + \frac{1}{k^*} \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 + \varepsilon \\
+ \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 + \varepsilon
$$

which implies that the parameter estimation errors $\Theta, \mathbf{q}, \dot{\vartheta}$ and $\mathbf{q}$ converge exponentially to the residual set

$$
S = \{ \Theta, \mathbf{q}, \dot{\vartheta} : \text{trace} (\Theta^T \Theta) + \mathbf{q}^T \mathbf{q} + \dot{\vartheta}^2 + \dot{\vartheta}^2 \\
+ \frac{1}{2} k_i \delta \dot{\vartheta}^2 + \frac{1}{2} \mu_i \dot{\mathbf{q}}^T \mathbf{q} + \frac{1}{2} \mu_i \dot{\vartheta}^2 + \varepsilon \}
$$

where

$$k^* = \frac{1}{k^*} \min \{ \lambda_{\min} (\Gamma_1^{-1}), \lambda_{\min} (\Gamma_2^{-1}), \gamma_3, \gamma_4 \}.$$  

Remark 5. It is easy to find an appropriate scalar function $V_i$ and a vector $g$ to meet the condition (40). For instance, we can simply choose $V_i = \frac{1}{2} \mathbf{x}^T \mathbf{x}$ and $g = \frac{\varepsilon + c}{2 \varphi_i (\varepsilon_0)} \mathbf{x}$, then

$$
\varphi_i (\| \mathbf{x} \|) = \frac{\varepsilon + c}{\varphi_i (\varepsilon_0)} V_i.
$$

V. SIMULATION STUDY

In this section, a simulation study is given to demonstrate the effectiveness of the proposed control scheme.

Consider a nonlinear uncertain time-varying system described by (1) associated with

$$
\mathbf{f} = \begin{bmatrix}
-1.5 x_1 (1 + x_2^2) \\
\sin (2 \pi t) x_1 + 0.3 x_1 x_2
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
0.0 \\
1.0
\end{bmatrix},
$$

$$
H_0 = [\cos (x_1), \sin (x_1), \sin (x_1), x_1 x_2], \quad f_0 = 0.9 (t - h(t)),
$$

$$
d_1 = \begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix},
$$

$$
d_2 = \sqrt{0.7 \sin^2 (8x_1) + 0.8 \cos^2 (5t) x_2^2}.
$$

The real values of the unknown parameter $\theta_1$ and $\theta_2$ are 0.3 and $-0.4$, respectively. The delay function $h(t)$ is given in Fig. 1. The upper bound of $d_2$ is $\rho_0 = \sqrt{q_1 + q_2 x_2^2}$ with unknown constants $q_1$ and $q_2$. We choose the function vector $\mathbf{g}(\mathbf{x}, t)$ as

$$
\mathbf{g}(\mathbf{x}, t) = \begin{bmatrix}
h(t) \\
0
\end{bmatrix}
$$

Fig. 1. Time-varying delay $h(t)$.  

\hfill
\[ g = \begin{bmatrix} -1.5x_1(1 + x_4^2) \\ -5x_2 \end{bmatrix} . \]

then obviously \( v = [\sin(2t) x_1 + 0.3x_3 x_2 + 5x_2]. \)

The control and adapting parameters are chosen to be
\[ \epsilon_1 = \epsilon_2 = \epsilon_3 = 0.1, \quad k_{s_1} = k_{s_2} = k_{s_3} = k_{s_4} = 1.0, \]
\[ \Gamma_1 = \Gamma_2 = \text{diag}(1, 1), \quad \gamma_1 = \gamma_4 = 1. \]

The initial state condition is \( x(t) = [-0.7 \cos(t) -0.4 \cos(t)]^T, t \in [-h^*, 0], \) and the initial values of the parameter estimates \( \Theta, \Phi, \omega \) and \( \vartheta \) are zeros.

The simulation results are given in Fig. 2. It can be seen that the control scheme has achieved a good performance, and the control signal is quite smooth.

VI. CONCLUSION

In this paper, the control problem for a class of dynamical systems with time-varying state delay, unknown parameters and disturbances is considered. By combining adaptive control and robust control methods, a new continuous adaptive robust controller is developed which guarantees the uniform boundedness of the system and assure that the regulating error enters the arbitrarily designated zone in a finite time. The effectiveness of the control scheme is verified by both theoretical analysis and simulation studies.

APPENDIX

The Bound of \( x(t - h(t)) \)

Denote \( \Sigma = \text{trace} \{ \Theta \Gamma_1^{-1} \Theta \} + \Phi \Gamma_2^{-1} \Phi + \Theta \Gamma_2^{-1} \Theta^2 + \Theta \Gamma_1^{-1} \Theta \}

\[ \gamma_4^{-1} \vartheta^2, \]
then from (36) we have
\[ \frac{1}{2} x^T (\tau \Theta(t) + \frac{1}{2} \Gamma_1^{-1} \Theta(t) + \Theta(t)) \leq \vartheta^2, \]
\[ t - h^* \leq t \to t . \]

It follows that
\[ x^T (\tau \Theta(t) + \frac{1}{2} \Gamma_1^{-1} \Theta(t) + \Theta(t)) \]
\[ \leq \vartheta^2 \{ x^T \Gamma_1^{-1} \Theta(t) + \frac{1}{2} \Gamma_1^{-1} \Theta(t) + \Theta(t) \}
\[ + \gamma_4^{-1} \Phi^2 + \gamma_4^{-1} \vartheta^2 \}
\[ \leq \vartheta^2 \{ x^T + \sqrt{\lambda_{\max}(\Gamma_1^{-1})} \Theta(t) + \sqrt{\lambda_{\max}(\Gamma_2^{-1})} \Phi(t) \}
\[ + \gamma_4^{-1} \Phi^2 + \gamma_4^{-1} \vartheta^2 \}
\[ \leq \vartheta^2 \{ x^T + \sqrt{\lambda_{\max}(\Gamma_1^{-1})} \Theta(t) + \sqrt{\lambda_{\max}(\Gamma_2^{-1})} \Phi(t) \}
\[ + \gamma_4^{-1} \Phi^2 + \gamma_4^{-1} \vartheta^2 \}
\[ \leq \vartheta^2 (1 + \gamma_4^{-1} \vartheta) \}
\[ \leq \vartheta^2 k \}
\[ (47) \]

where \( k \) is a positive value which is sufficiently large such that \( k > 1 + \gamma_4^{-1} \vartheta \), \( \vartheta \) and \( \vartheta \) are defined in (26) and (27).

Fig. 2. (a) \( x_1(t) \); \( x_2(t) \); (b) Evolution of input voltage \( u(t) \).

REFERENCES

Jian-Xin Xu received his Bachelor degree from Zhejiang University, China in 1982. He attended the University of Tokyo, Japan, where he received his Master’s and Ph.D. degrees in 1986 and 1989 respectively. All his degrees are in Electrical Engineering.

He worked for one year in the Hitachi Research Laboratory, Japan and for more than one year in Ohio State University, U.S.A. as a Visiting Scholar.

In 1991 he joined the National University of Singapore, and is currently an associate professor in the Department of Electrical Engineering. His research interests lie in the fields of learning control, variable structure control, fuzzy logic control, discontinuous signal processing, and applications to motion control and process control problems. He is a senior member of IEEE.

Qingwei Jia received the B.Sc. and M.Sc. degrees from Tsinghua University, P.R.China, and the Ph.D. degree from the National University of Singapore in 1988, 1993 and 1999, respectively.

In 1997, he joined the Coding and Signal Processing Division of Data Storage Institute of Singapore, where he is currently a senior research engineer. His research interests include adaptive and robust control, automatic gain control and timing-recovery in digital communications, servo control technology and new read channel design for disk drives.

T.H. Lee received the B.A. degree with First Class honours in the Engineering Tripos from Cambridge University, England, in 1980, and the Ph.D. degree from Yale University in 1987. He is a professor, and Head of the Control Division at the Department of Electrical Engineering of the National University of Singapore. He is also the Vice-Dean (Research) in the Faculty of Engineering there.

Dr. Lee’s research interests are in the areas of adaptive systems, knowledge-based control and intelligent mechatronics. He has published 3 books, and over 100 technical papers in international journals and conference proceedings in these areas, and he currently holds Associate Editor appointments in Automatica (an IFAC Journal); the IEEE Transactions in Systems, Man and Cybernetics; Control Engineering Practice (an IFAC journal); and the International Journal of Systems Science. He is also the Regional Editor (for the Far East) of Mechatronics journal (Oxford, Pergamon Press).

Dr. Lee was a recipient of the Cambridge University Charles Baker Prize in Engineering.