A Delay Decomposition Approach to $H_\infty$ Control of Networked Control Systems

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This paper is concerned with $H_\infty$ control of a networked control system (NCS). First, the NCS is modelled as a linear system with an interval time-varying delay. Second, a delay decomposition approach is developed to derive a less conservative bounded real lemma (BRL). Based on this BRL, delay-dependent conditions for the existence of a state feedback controller, which ensures internally asymptotic stability and a prescribed $H_\infty$ performance level of the closed-loop system, are derived in terms of a nonlinear matrix inequality. Third, the nonconvex feasibility problem is converted into a nonlinear minimization problem subject to a set of linear matrix inequalities (LMIs), from which the suitable controller can be designed by an iterative algorithm. Finally, two numerical examples are given to show the effectiveness of the obtained results.

Keywords: Networked control systems (NCSs), State feedback, $H_\infty$ control, A delay decomposition approach

1. Introduction

A networked control system (NCS) is a system in which the feedback control loop is closed through a network. In the last decade, NCSs have been attracting much attention due to some significant advantages, such as reduced installation and maintenance cost and increased system agility and so on, see for example [7] and references therein. However, network-induced delays and data dropouts, which make the analysis and synthesis of an NCS complicated, are usually inevitable in an NCS because of the message transmission and limit bandwidth. Therefore, there have many researchers to conduct research on this topic and in the existing literature there are some results such as results on stability analysis [10, 11, 13–15, 22], results on state feedback control [8, 16, 18, 20, 21], and results on $H_\infty$ stabilization and estimation [2, 3, 17, 19].

In this paper, we consider a typical NCS, shown in Fig. 1, where the physical plant is supposed to be a continuous-time system described by

\[
\begin{align*}
\dot{x}(t) &= A_0 x(t) + B_u u(t) + B_w w(t) \\
z(t) &= C_0 x(t) + D_u u(t) + D_w w(t) \\
x(t_0) &= x_0
\end{align*}
\]

where $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^l$ and $u(t) \in \mathbb{R}^m$ are the state, the controlled output and the control input, respectively; $w(t) \in \mathbb{R}^p$ is the disturbance input belonging to $L_2^2[t_0, \infty)$. The coefficient matrices $A_0, B_w, B_u, C_0, D_w$, and $D_u$ are known real constant matrices of appropriate dimensions.

Assume that the sampler is clock-driven, the controller and the zero-order hold (ZOH) are event-driven, and also assume that the data are transmitted with a signal packet under the Try-Once-Discard
(TOD) protocol [13]. With these assumptions, the continuous signal \( x(t) \) is sampled by the sampler, and then is transmitted with a single packet to the digital controller through the network to generate a control action. This control action is finally transmitted via the network again and is held by the ZOH to drive the physical plant. During the data transmission, data-packet dropouts and the network-induced delays are usually inevitable, which will possibly degrade the stability of the NCS. In the following, by employing a novel timing mechanism from [3], we will model the NCS as a linear system with time-varying delay, where the sampling intervals, the numbers of packet dropout, and the bounds of the network-induced delay are taken into account. More specifically, let the ZOH update at \( t_k (k = 1, 2, \cdots, \infty) \), then at each updating instant \( t_k \), the available state is \( x(t_k - \eta_k) \), where \( \eta_k \) denotes the sum of network-induced delays from the sampler to the controller and from the controller to the ZOH since the last updating instant \( t_{k-1} \). The network-induced delay \( \eta_k \) is unknown, possibly time-varying or even random variable, but known to be bounded by two scalars \( \eta_m \) and \( \eta_M \), that is

\[
0 \leq \eta_m \leq \eta_k \leq \eta_M < \infty
\]

(2)

In this situation, the state feedback controller can be taken as

\[
u(t) = u(t_k) = Kx(t_k - \eta_k), \quad t_k \leq t < t_{k+1}
\]

(3)

where the gain \( K \) is to be determined. Then, the resulting closed-loop system is

\[
\begin{aligned}
\dot{x}(t) &= A_0x(t) + B_ow(t) + B_uKx(t_k - \eta_k) \\
\end{aligned}
\]

\[
\begin{aligned}
z(t) &= C_0x(t) + D_ow(t) + D_uKx(t_k - \eta_k), \\
\end{aligned}
\]

\( t \in [t_k, t_{k+1}) \).

(4)

It is worth pointing out that the above timing mechanism has many advantages; for example, the sampler need not sample periodically, and the distance between two consecutive updating instants, i.e. \( t_{k+1} - t_k (k = 1, 2, \cdots) \), contains some useful information, which can be seen from the following

\[
t_{k+1} - t_k = T_{\text{Dropout}}^{k+1} + T_{\text{Sampler}}^{k+1} + \eta_{k+1}
\]

(5)

where \( T_{\text{Dropout}}^{k+1} \) denotes the cost time of finite data-packet dropouts and \( T_{\text{Sampler}}^{k+1} \) the corresponding sampling interval from the last updating instant \( t_k \) to the current updating instant \( t_{k+1} \). Clearly, when the network-induced delay \( \eta_{k+1} \) is sufficiently small and no packet dropouts occur, the distance \( t_{k+1} - t_k \) can be regarded as the two successive sampling interval; in the case that \( \eta_{k+1} \) and \( T_{\text{Sampler}} \) are sufficiently small, then \( t_{k+1} - t_k \) can reflect the numbers of packet dropouts; also, when \( T_{\text{Sampler}} \) is sufficiently small and no packet dropouts occur, \( t_{k+1} - t_k \) expresses the network-induced delay since the last updating instant \( t_k \).

For simplicity of presentation, denote \( \tau_{\text{max}} := \max\{t_{k+1} - t_k | k = 1, 2, \cdots\} \). From the above analysis, the admissible value of \( \tau_{\text{max}} \), with which the closed-loop system (4) is asymptotically stable with a prescribed \( H_\infty \) disturbance attenuation level, can be regarded as an important performance index of an NCS due to the fact that it takes into account information about the sampling interval, the number of packet dropouts, and the network-induced delay. The larger value of \( \tau_{\text{max}} \), the better performance of an NCS, which means a larger enduranceability on dropouts, and network-induced delays. Therefore, it is of significance in both theory and practice to seek a maximum admissible upper bound of \( \tau_{\text{max}} \) such that the closed-loop system (4) is asymptotically stable with a prescribed \( H_\infty \) disturbance attenuation level.
Let \( d(t) = t - t_k + \eta_k, \ t_k \leq t < t_{k+1} \). Then \( d(t) \) can be considered as a piecewise continuous function satisfying

\[
0 \leq \tau_m \leq d(t) \leq \tau_M < \infty, \ \forall t \geq t_0
\]  

where \( \tau_m := \eta_m \) and \( \tau_M := \tau_{\max} + \eta_M \). In this way, the resulting closed-loop system (4) is thus equivalently converted into a linear continuous-time system with an interval time-varying delay described by

\[
\begin{cases}
\dot{x}(t) = A_0 x(t) + B_u K x(t - d(t)) + B_u w(t) \\
z(t) = C_0 x(t) + D_u K x(t - d(t)) + D_u w(t),
\end{cases}
\]

\( t \in [t_k, t_{k+1}) \). 

(7)

The initial condition of the state \( x(t) \) on \([t_0 - \tau_M, t_0] \) is supplemented as

\[
x(\theta) = \phi(\theta), \ \ \theta \in [t_0 - \tau_M, t_0] \quad \text{with} \quad \phi(t_0) = x_0
\]

where \( \phi(\theta) \) is a continuous vector-valued function on \([t_0 - \tau_M, t_0] \).

In this paper, we address an \( H_\infty \) control problem: To determine the controller gain \( K \) such that the closed-loop system (7) is asymptotically stable with a prescribed \( H_\infty \) disturbance attenuation level \( \gamma \), i.e.

- the closed-loop system (7) with \( w(t) \equiv 0 \) is asymptotically stable; and
- the closed-loop system (7) is of a prescribed \( H_\infty \) disturbance attenuation level \( \gamma > 0 \), that is, \( z(t) \) satisfies \( \|z\|_2 < \gamma \|w\|_2 \) for all nonzero \( w(t) \in \mathcal{L}_2[t_0, \infty) \) under the initial condition \( x(\theta) \equiv 0 \), \( \forall \theta \in [t_0 - \tau_M, t_0] \).

As is well known, the Lyapunov-Krasovskii functional (LKF) approach is an effective method to deal with the above \( H_\infty \) control problem. One can have some less conservative results by choosing some proper LKFs. In [18], the authors chose an LKF as

\[
\hat{V}(t, x_t) = x^T(t)Px(t) + \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)R \dot{x}(s)dsd\theta
\]

and derived some results, which are conservative because the lower bound \( \tau_m \) of the time-varying delay was not taken into account. To reduce the conservatism of the results, different LKFs were proposed in [19, 12] and [9]. Compared with the LKFs in [19, 12], the LKF in [9] has some advantage due to the fact that the LKF was constructed by using both the upper bound \( \tau_M \) and the lower bound \( \tau_m \) of the time-varying delay. In order to further reduce conservatism of the results in [9], we now are analyzing the following LKF in [9]

\[
\hat{V}(t, x_t) = x^T(t)Px(t) + \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)Q_1 \dot{x}(s)dsd\theta
\]

\[
+ \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s)Q_2 \dot{x}(s)dsd\theta
\]

\[
+ \tau_m \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)R_1 \dot{x}(s)dsd\theta
\]

\[
+ \tau_M \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s)R_2 \dot{x}(s)dsd\theta
\]

\[
+ (\tau_M - \tau_m) \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s)S \dot{x}(s)dsd\theta
\]

(10)

Notice that the second and third terms in (10) can be rewritten as

\[
\int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)Q_1 \dot{x}(s)dsd\theta
\]

\[
+ \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s)Q_2 \dot{x}(s)dsd\theta
\]

Notice also that the last three terms in (10) can be rearranged as

\[
\tau_m \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s) \left[ R_1 + \frac{\tau_M}{\tau_m} R_2 \right] \dot{x}(s)dsd\theta
\]

\[
+ (\tau_M - \tau_m) \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s) \left[ S + \frac{\tau_M}{\tau_M - \tau_m} R_2 \right] \dot{x}(s)dsd\theta
\]

Introducing some new matrix variables, one can simplify the LKF (10) as

\[
\hat{V}(t, x_t) = x^T(t)Px(t) + \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_1 \dot{x}(s)dsd\theta
\]

\[
+ \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s)Z_2 \dot{x}(s)dsd\theta
\]

\[
+ \tau_m \int_{-\tau_m}^0 \int_{t+\theta}^t \dot{x}^T(s)S_1 \dot{x}(s)dsd\theta
\]

\[
+ (\tau_M - \tau_m) \int_{-\tau_M}^0 \int_{t+\theta}^t \dot{x}^T(s)S_2 \dot{x}(s)dsd\theta
\]

(11)

From the above LKF, one can see that the matrices \( Z_2 \) and \( S_2 \) in (11) are defined on the whole delay interval \([-\tau_M, -\tau_m]\), which may lead to some conservative
result. Recently, Han [5,6] introduced a delay decomposition approach to study the stability and $H_{\infty}$ control of linear delay systems, where the time-delay is a constant delay. The idea of the approach is that the delay interval is uniformly divided into multiple segments, and a proper Lyapunov–Krasovskii functional is chosen with different weighted matrices corresponding to different segments in the Lyapunov–Krasovskii functional. Inspired by Han [5,6], we now extend the approach to the system (7), where the time-delay is an interval time-varying delay. Divide the delay interval $[-\tau_M, -\tau_m]$ into $N$ equidistant sub-intervals, i.e.,

$$[-\tau_M, -\tau_m] = \bigcup_{j=1}^{N} [-\tau_m - j\delta, -\tau_m - (j - 1)\delta]$$

where $\delta = (\tau_M - \tau_m)/N$ with $N$ a positive integer. Then, on each subinterval $[-\tau_m - j\delta, -\tau_m - (j - 1)\delta]$, choose a different weighting matrix to arrive at

$$V_j(t, x_t) := \int_{t-\tau_m - j\delta}^{t-\tau_m - (j-1)\delta} x^T(s)Q_jx(s)ds + \delta \int_{t-\tau_m - j\delta}^{t-\tau_m - (j-1)\delta} \dot{x}(s)R_j\dot{x}(s)ds + \theta_{j\delta}^{t-\tau_m - (j-1)\delta} \dot{x}(s)R_j\dot{x}(s)ds d\theta$$

where $j = 1, 2, \cdots, N$. Consequently, a new LKF candidate for system (7) is readily constructed as

$$V(t, x_t) = x^T(t)Px(t) + V_0(t, x_t) + \sum_{j=1}^{N} V_j(t, x_t)$$

(12)

where $V_0(t, x_t)$ is a functional defined on $[-\tau_m, 0]$ as

$$V_0(t, x_t) := \int_{t-\tau_m}^{t} x^T(s)Q_0x(s)ds + \tau_m \int_{t-\tau_m}^{t} \dot{x}(s)R_0\dot{x}(s)ds dB$$

Apparently, if $Q_j \to 0^+$ and $\delta R_j = \tau_m R_0 = R$, $(j = 1, 2, \cdots, N)$, then the LKF (12) immediately reduces to the one in (9). Also, if we set $Q_0 = Z_0$, $Q_j = Z_j$, $R_0 = S_1$, $R_j = NS_2$, $(j = 1, 2, \cdots, N)$, then the LKF (12) becomes (11). Therefore, the LKF (12) includes those in (9) and (11) as special cases.

In this paper, we will apply the proposed LKF (12) to deal with the $H_{\infty}$ control of the system (7). A new and less conservative bounded real lemma (BRL) will be first derived to ensure that the resulting closed-loop system (7) will be asymptotically stable with a prescribed $H_{\infty}$ disturbance attenuation level. By employing the cone complementary linearization technique, the nonconvex feasibility problem described by this BRL will be converted into a nonlinear minimization problem subject to a set of linear matrix inequalities (LMIs), which can be easily implemented by an iterative algorithm. Two examples will be finally given to demonstrate the effectiveness of the obtained results.

Notation: Throughout this paper, the notations are standard. The superscripts ‘$-$’ and ‘$T$’ mean the inverse and transpose of a matrix, respectively; $\mathbb{R}^n$ denotes $n$-dimensional Euclidean space; $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices; $P > 0$ means that $P$ is positive definite; diag{\ldots} denotes a block-diagonal matrix; $(\psi_{ij})_{m \times m}$ means a big matrix of $m \times m$ block matrix elements $\psi_{ij}$, $(i, j = 1, 2, \cdots, m)$; $I$ and $0$ denote an identity matrix and a zero matrix of appropriate dimensions, respectively; $\lambda_{\min}(\mathbf{Q})$ stands for the minimum eigenvalue of a symmetric matrix $\mathbf{Q}$; $\mathcal{L}_2[0, \infty)$ denotes the space of square integrable vector functions on $[0, \infty)$ with norm $\| \cdot \|_2 = (\int_0^\infty \| \cdot \|^2 dt)^{1/2}$, where $\| \cdot \|$ denotes the Euclidean vector norm; and the symmetric term in a symmetric matrix is denoted by $\ast$, e.g., $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$.

2. $H_{\infty}$ Performance Analysis

In this section, we will present a new sufficient condition such that the system (7) is asymptotically stable with a prescribed $H_{\infty}$ disturbance attenuation level. To begin with, we introduce the following integral inequality.

Lemma 1 [4]: For any constant matrix $X \in \mathbb{R}^{n \times m}$, $X = X^T > 0$, a scalar function $h := h(t) \geq 0$ and a vector-value function $\dot{x} : [t-h, t] \to \mathbb{R}^m$ such that the following integration is well defined, then

$$-h \int_{t-h}^{t} \dot{x}^T(s)X\dot{x}(s)ds \leq \begin{bmatrix} \dot{x}(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -X & X \\ * & -X \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t-h) \end{bmatrix}$$

(13)

We now state and establish the following result.

Proposition 1: For some given scalars $\gamma > 0$, $\tau_m, \tau_M$, and a positive integer $N$, the system (7) is asymptotic-
ally stable with a prescribed $H_{\infty}$ disturbance attenuation level $\gamma$ if there exist $n \times n$ real matrices $P > 0$, $Q_j > 0, R_j > 0$ ($j = 0, 1, \cdots, N$) such that

\[
\dot{V}(t, x) = 2x^T(t)Px + x^T(t)Q_0x(t) - x^T(t - \tau_m)Q_0x(t - \tau_m) + \dot{x}^T(t)\left(\sum_{j=0}^{N} R_j\right)\dot{x}(t)
\]

\[
- \tau_m \int_{t-\tau_m}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds + \sum_{j=1}^{N} x^T(t - \tau_m - j\delta)Q_jx(t - \tau_m - j\delta)
\]

\[
(t - \tau_m - j\delta) - \sum_{j=1}^{N} x^T(t - \tau_m - j\delta)Q_jx(t - \tau_m - j\delta)
\]

\[
(t - \tau_m - j\delta) - \sum_{j=1}^{N} \delta \int_{t-\tau_m-j\delta}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds.
\]

For convenience, introduce a new vector

\[
\zeta(t) := [w^T(t)x^T(t - d(t))x^T(t - \tau_m)x^T(t - \tau_m - \delta)\cdots x^T(t - \tau_m - Nb)]^T
\]

Then, rewrite (7) as

\[
\begin{aligned}
\dot{x}(t) &= \Gamma_1\zeta(t) \\
\dot{z}(t) &= \Gamma_2\zeta(t).
\end{aligned}
\]

Substituting (23) into (22), after simple algebraic manipulation, we have

\[
\dot{V}(t, x) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)
\]

\[
= \zeta^T(t)\Gamma_3^T \Gamma_1 + \zeta^T(t)\Gamma_1^T \Gamma_3 + \Xi + \gamma^2 \sum_{j=0}^{N} \Gamma_1^T R_0 \Gamma_1
\]

\[
+ \delta^2 \zeta^T(t)\Gamma_1 + \Gamma_2^T \Gamma_1 \zeta(t) - \tau_m \int_{t-\tau_m}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds
\]

\[
- \sum_{j=1}^{N} \delta \int_{t-\tau_m-j\delta}^{t} \dot{x}^T(s)R_0\dot{x}(s)ds.
\]

where $\Upsilon, \Gamma_1$ and $\Gamma_2$ are defined in (17)–(19), respectively, and

\[
\Gamma_3 := [0 \ 0 \ I \ 0 \ \cdots \ 0]
\]

\[
\Xi := \text{diag}\{-\gamma^2 I, 0, Q_0, Q_1 - Q_0, \cdots, Q_N - Q_{N-1}, - Q_N\}
\]

We now establish the relationships between $x(t - d(t))$ and

\[
x(t - \tau_m), x(t - \tau_m - \delta), \cdots, x(t - \tau_m - N\delta)
\]

by using Lemma 1. Since $d(t)$ is a continuous function, for any $t \in [t_k, t_{k+1})$, there should exist a positive integer
Substituting (26), (27) and (28) into (24) yields

\[
\dot{V}(t, x_i) + z^T(t)z(t) - \gamma^2 w(t)
\]

where \(\Phi_0, \Phi_q\) are defined in (15) and (16), respectively.

First, we consider the asymptotic stability of the system (7) with \(w(t) \equiv 0\). In this situation, we have

\[
\dot{V}(t, x_i) \leq \xi^T(t) \left[ \Phi_0 + \Phi_q + \gamma^2 \tau_m^2 I_{11} [R_0 \tilde{\Gamma}_1 + \delta \tilde{\Gamma}_2 \eta(t)] \right] \xi(t), t \in [t_k, t_{k+1})
\]

where

\[

\xi(t) := \begin{bmatrix} x^T(t-d(t)) & x^T(t) & x(t-\tau_m) & x^T(t-\tau_m-N\theta) \end{bmatrix}^T
\]

\[

\Phi_0 := \begin{bmatrix} 0 & (B_u K) P & 0 & \cdots & 0 & 0 \\ * & \varphi_0 & R_0 & \cdots & 0 & 0 \\ * & * & \varphi_1 & R_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \varphi_N & R_N \\ * & * & * & \cdots & * & \varphi_{N+1} \end{bmatrix}
\]

\[

\Phi_q := \begin{bmatrix} \tilde{\psi}_{ij} & \tilde{\psi}_{ij} \\ \tilde{\psi}_{ij} & \tilde{\psi}_{ij} \end{bmatrix}
\]

\[

\tilde{\psi}_{ij} := \begin{cases} -R_q, & i = j = 1, \\ R_q, & i = 1, j = q+2, q+3, \\ -R_q, & i = q+2, j = q+3, \\ 0, & \text{otherwise.} \end{cases}
\]

\[

\tilde{\Gamma}_1 := \begin{bmatrix} B_u K & A_0 & 0 & \cdots & 0 \end{bmatrix}
\]

Notice that (14) implies

\[

\begin{bmatrix} \Phi_0 + \Phi_q & \tau_m \tilde{\Gamma}_1 & \delta \tilde{\Gamma}_1 \eta \\ * & -R_0 & 0 \\ * & * & -\eta \end{bmatrix} < 0, \quad (q = 1, 2, \cdots, N)
\]

which leads to

\[

\Omega_q := \begin{bmatrix} \Phi_0 + \Phi_q & \tau_m \tilde{\Gamma}_1 & R_0 \tilde{\Gamma}_1 + \delta \tilde{\Gamma}_1 \eta \end{bmatrix} < 0, \quad (q = 1, 2, \cdots, N).
\]

Considering all the possibility of \(q\) in the set \(\{1, 2, \cdots, N\}\) and denoting

\[

\rho := \min \{\lambda_{\min}(-\Omega_q)|q = 1, 2, \cdots, N\}
\]
we have from (30) that $\dot{V}(t, x) \leq -\rho x^T(t) x(t) < 0$ for $x(t) \neq 0$, from which we can conclude that the system (7) with $w(t) \equiv 0$ is asymptotically stable.

Next, we consider the $H_\infty$ performance $\|z\|_2 < \gamma \|w\|_2$ is satisfied for all nonzero $w(t) \in L^2([t_0, \infty) \cup \{0\})$ under the zero initial condition $x(\theta) = 0$, $\forall \theta \in [t_0 - \tau_M, t_0]$. In fact, applying Schur complement to (14) yields

$$
\Phi_0 + \Phi_q + \tau_m^2 \Gamma_1^T R_0 \Gamma_1 + \delta^2 \Gamma_1^T Y \Gamma_1 + \Gamma_2^T \Gamma_2 < 0,
$$

$(q = 1, 2, \cdots, N)$

From (29), we have, for $t \in [t_k, t_{k+1}]

$$
\dot{V}(t, x) + z^T(t) z(t) - \gamma^2 w^T(t) w(t) < 0
$$

Integrating both sides of (31) on $t$ from $[t_k, t_{k+1}]$ yields

$$
\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} [\dot{V}(t, x) + z^T(t) z(t) - \gamma^2 w^T(t) w(t)] dt < 0
$$

that is

$$
\int_{t_0}^{\infty} [z^T(t) z(t) - \gamma^2 w^T(t) w(t)] dt < V(t, x)|_{t=t_0} - V(t, x)|_{t=\infty}
$$

Under zero initial conditions, $x(\theta) = 0$, $\forall \theta \in [t_0 - \tau_M, t_0]$, we have

$$
\int_{t_0}^{\infty} z^T(t) z(t) dt < \gamma^2 \int_{t_0}^{\infty} w^T(t) w(t) dt
$$

which means $\|z\|_2 < \gamma \|w\|_2$. This completes the proof.

Remark 1: Proposition 1 presents a new BRL for NCSs based on a new Lyapunov–Krasovskii functional. When controller gain $K$ is given, this proposition can be used to calculate the minimum $H_\infty$ performance $\gamma_{\min}$ for prescribed $\tau_m$ and $\tau_M$. Also, it can be employed to compute the upper bound of $\gamma$ for given $\tau_m$ and $\tau_M$. The numerical examples in Section 4 show that Proposition 1 can achieve much less conservative results than those in the literature. However, Proposition 1 depends on the integer $N$. The larger $N$, the less conservatism of Proposition 1, but the more CPU time. As a compromise, we can take some small values of $N$. For example, $N = 2$ or $N = 3$, to deal with the $H_\infty$ control of an NCS in a practical application.

3. $H_\infty$ Controller Design

In this section, we are in a position to solve the controller gain $K$ based on the matrix inequalities in (14). Since some nonlinear terms such as $(B_w K)^T P$ exist in the matrix inequalities in (14), it is not easy to directly derive the controller gain $K$. In the following, we first modify those matrix inequalities in (14) such that only two nonlinear terms are included. Then, we present an approach to solving the gain $K$ by employing the cone complementary linearization algorithm in [1]. To begin with, we have the following

Proposition 2: For some given scalars $\gamma > 0$, $\tau_m$, $\tau_M$, and a positive integer $N$, the closed-loop system (7) is asymptotically stable with a prescribed $H_\infty$ noise attenuation level $\gamma$ if there exist $n \times n$ real matrices $X > 0, Q > 0, R > 0 (j = 0, 1, \cdots, N)$ and $m \times n$ real matrix $Y$ such that

$$
\Xi_q := \begin{bmatrix}
\Sigma_0 + \Sigma_q & \tau_m A_1^T & 0 & \cdots & & 0 \\
\ast & -XR_0^T X & 0 & \cdots & & 0 \\
\ast & \ast & -XY^{-1} X & 0 & \cdots & 0 \\
\ast & \ast & \ast & -I & \cdots & 0 \\
\end{bmatrix}
$$

for $q \in \{1, 2, \cdots, N\}$

where

$$
\Lambda_1 := [B_w \ B_u Y \ A_0 X \ 0 \ \cdots \ 0] \\
\Lambda_2 := [D_w \ D_u Y \ C_0 X \ 0 \ \cdots \ 0] \\
Y := R_1 + R_2 + \cdots + R_N
$$

$$
\Sigma_0 := \begin{bmatrix}
-\gamma^2 I & 0 & B_w^T & 0 & 0 & \cdots & 0 & 0 \\
\ast & 0 & Y^T B_u & 0 & 0 & \cdots & 0 & 0 \\
\ast & \ast & \ast & \ast & \beta_0 & \cdots & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \beta_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \beta_{N+1} \\
\end{bmatrix}
$$

$$
\Sigma_q := (\Theta_q) (N+4) x (N+4) + (\Theta_q)^T (N+4) x (N+4)
$$

with

$$
\beta_j := \begin{cases}
A_0 X + X A_0^T + \tilde{Q}_0 - R_0, & j = 0; \\
\tilde{Q}_j - \tilde{Q}_{j-1} - R_j - R_{j-1}, & j = 1, \cdots, N; \\
-\tilde{Q}_N - R_N, & j = N + 1.
\end{cases}
$$
Moreover, the controller gain is given by \( K = YX^{-1} \).

**Proof:** Define a transformation matrix

\[
\mathcal{S} := \text{diag}(I, P_0^{-1}, \ldots, P_{N+3}^{-1}, R_0^{-1}, \gamma^{-1}, I).
\]

Pre- and post-multiplying \( \Psi_q(q = 1, 2, \cdots, N) \) defined in (14) by \( \mathcal{S}^T \) and \( \mathcal{S} \), respectively, together with the settings as \( X := P^{-1}, Y := KP^{-1} \) and

\[
R_i := P^{-1}R_i^{-1}, Q_i := P^{-1}Q_i^{-1}, (i = 0, 1, \cdots, N)
\]

one obtains

\[
\Xi_q = \mathcal{S}^T \Psi_q \mathcal{S}
\]

where \( \Xi_q \) is defined in (32). So, if (32) holds for \( q = 1, 2, \cdots, N \), then from (40), \( \Psi_q < 0 \), \( q = 1, 2, \cdots, N \). Therefore, the closed-loop system (7) is asymptotically stable with a prescribed \( H_\infty \) level \( \gamma \) from Proposition 1, which completes the proof. \( \square \)

It is clear that the matrix inequality \( \Xi_q < 0 \) has only two nonlinear terms, i.e. \( -XR_0^{-1}X \) and \( -X\gamma^{-1}X \). If we set \( R_0 = \epsilon_0X \) and \( \gamma = \epsilon_1X \), where \( \epsilon_0, \epsilon_1 > 0 \) are two scalars to be tuned, then the problem of solving the matrix inequalities in (32) become a standard convex optimization problem with parameter tuning. However, this linearization approach, as is well known, will lead to more conservative results.

In the sequel, utilizing the cone complementary idea [1], we are about to convert the nonconvex feasibility problem of (32) into a nonlinear minimization problem subject to a set of LMIs, which can be effectively solved by employing an modified iterative algorithm. For this goal, introduce two variables \( Z_1 > 0 \) and \( Z_2 > 0 \) satisfying \( XR_0^{-1}X \geq Z_1 \) and \( X\gamma^{-1}X \geq Z_2 \), which are, respectively, equivalent to

\[
\begin{bmatrix}
R_0^{-1}X \\
X^{-1}Z_1
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
\gamma^{-1}X \\
X^{-1}Z_2
\end{bmatrix} \geq 0.
\]

Let \( \mathcal{P} = X^{-1}, \mathcal{Q} = \gamma^{-1}, \mathcal{R} = R_0^{-1} \), and \( Z_1 = Z_1^{-1}, Z_2 = Z_2^{-1} \). Then, similar to [1], the nonconvex feasibility problem of (32) can be converted into a nonlinear minimization problem subject to LMIs as follows.

**Nonlinear Minimization Problem**

Minimize \( \text{Tr}(\mathcal{P}X + \mathcal{Q}\gamma + \mathcal{R}R_0 + \mathcal{Z}_1Z_1 + \mathcal{Z}_2Z_2) \)

Subject to

\[
\begin{bmatrix}
\Sigma_0 + \Sigma_q & \tau_m A_m^T & \delta A_1^T A_2^T \\
* & -Z_1 & 0 \\
* & * & -Z_2 \\
* & * & * & -I
\end{bmatrix} \begin{bmatrix}
\mathcal{P} & \mathcal{Q} \\
\mathcal{P} & \mathcal{Z}_1 \\
\mathcal{Z}_1 & \mathcal{Z}_1 & \mathcal{Z}_2 \\
\mathcal{Z}_2 & \mathcal{Z}_2 & \mathcal{Z}_2 \\
\end{bmatrix} < 0, \quad (q = 1, 2, \cdots, N)
\]

An iterative algorithm can be used to solve the above nonlinear minimization problem, which is stated in the following.

**Algorithm 1:** Maximize \( \tau_M \) for given \( \tau_m \) and \( \gamma \).

**Step 1** Choose a sufficiently small initial \( \tau_M \) such that (42)–(44) are feasible. Set \( \tau_M^{i+1} = \tau_M^{ini} \).

**Step 2** Find a feasible set \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{Z}_1, \mathcal{Z}_2, X, Z_1, Z_2, Y, R_j, Q_j \) satisfying (42)–(44). Set \( i = 0 \).

**Step 3** Solve the following LMI problem for the variables \( (\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{Z}_1, \mathcal{Z}_2, X, Z_1, Z_2, Y, R_j, Q_j) \) \( (j = 0, 1, \cdots, N) \):

Minimize \( \text{Tr}

\[
\begin{bmatrix}
\mathcal{P}X + X^T\mathcal{P} + \mathcal{Q}\gamma + \gamma^T\mathcal{Q} + \mathcal{R}R_0 + \mathcal{Z}_1Z_1 + \mathcal{Z}_2Z_2 \\
\end{bmatrix} \begin{bmatrix}
\mathcal{P}^{-1} & \mathcal{Q}^{-1} \\
\mathcal{P}^{-1} & \mathcal{Z}_1 \\
\mathcal{Z}_1 & \mathcal{Z}_1 & \mathcal{Z}_2 \\
\mathcal{Z}_2 & \mathcal{Z}_2 & \mathcal{Z}_2 \\
\end{bmatrix} \begin{bmatrix}
\mathcal{P}^{-1} & \mathcal{Q}^{-1} \\
\mathcal{P}^{-1} & \mathcal{Z}_1 \\
\mathcal{Z}_1 & \mathcal{Z}_1 & \mathcal{Z}_2 \\
\mathcal{Z}_2 & \mathcal{Z}_2 & \mathcal{Z}_2 \\
\end{bmatrix} \begin{bmatrix}
\mathcal{P}^{-1} & \mathcal{Q}^{-1} \\
\mathcal{P}^{-1} & \mathcal{Z}_1 \\
\mathcal{Z}_1 & \mathcal{Z}_1 & \mathcal{Z}_2 \\
\mathcal{Z}_2 & \mathcal{Z}_2 & \mathcal{Z}_2 \\
\end{bmatrix} < 0,
\]

\( (q = 1, 2, \cdots, N) \).

**Step 4** If matrix inequality (32) is satisfied and
where \( \varepsilon \) is a prescribed sufficiently small positive number, then increase \( \tau_m^i \) to some extent and set \( \tau_m^i = \tau_m^i \) and go back to Step 2. If one of the conditions (32) and (45) is not satisfied within a specified number of iterations, then exit, otherwise, set \( l = l + 1 \) and go to Step 3.

Remark 2: In Algorithm 1, (32) and (45) are used as the stopping criteria since it is difficult to exactly obtain the minimum value of the related nonlinear minimization problem. If (32) and (45) are satisfied, we can conclude from Proposition 2 that system (7) is asymptotically stable with a prescribed \( H_\infty \) performance value \( \gamma_{10} \) for given \( \tau_m \) and \( \tau_M \).

4. Numerical Examples

In this section, we present two examples to show the effectiveness of the obtained results.

Example 1: Consider the system described by

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t)
\]  \hspace{1cm} (46)

In [22], the network-based controller is designed as \( u(t) = [-3.75, -11.5] x(t) \). Using this controller and employing the criteria in [22, 10, 18, 19, 12, 9], the maximum allowable transfer intervals (MATIs) are 2.7 \times 10^{-4}, 0.7805, 0.8695, 0.8871, 0.9410 and 1.0081, respectively. We are now applying Proposition 1 with \( \tau_m = 0 \) and different \( N \) in this paper to this example. Fig. 2a plots MATIs for different \( N \). Clearly, with \( N \) increasing, the MATI grows fast at the beginning, but slowly when \( N \geq 4 \). On the other hand, the cost of CPU computation time in deriving the MATIs for various \( N \) is calculated and the result is depicted in Fig. 2b. From these two figures, the relationships among MATI, the cost of CPU computation time and \( N \) are clearly disclosed: the larger \( N \), the larger MATI, but the more CPU computation time. In particular, when \( N \geq 4 \), the MATI grows slowly, but the cost of CPU computation time sharply increases. For the tradeoff between the MATI and the CPU computation time, taking \( N = 4 \) is a good choice to derive a less conservative result which is sufficient for this example.

In order to consider the effect of the external disturbance on the system, we reexpress the system (46) as

\[
\begin{cases}
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} w(t) \\
z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + 0.1 u(t).
\end{cases}
\]  \hspace{1cm} (47)

With the same controller, for \( \tau_m = 0 \) and \( \tau_M = 0.8695 \), the minimum allowed \( H_\infty \) disturbance attenuation level \( \gamma_{\text{min}} \) is calculated as 6.82, 1.0005, and 0.7813, by using the criteria in [19, 9] and Proposition 1 with \( N = 3 \) in this paper. It is clear that the criterion in this paper can

![Fig. 2. The achieved MATIs and the cost of CPU time for different values of \( N \).](image-url)
Table 1. The achieved controller gain $K$ for different values of $\tau_m$

<table>
<thead>
<tr>
<th>$\tau_m$</th>
<th>Iterations</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>$[-0.0371 - 0.1305]$</td>
</tr>
<tr>
<td>0.5</td>
<td>50</td>
<td>$[-0.0370 - 0.1310]$</td>
</tr>
<tr>
<td>0.2</td>
<td>110</td>
<td>$[-0.0365 - 0.1309]$</td>
</tr>
</tbody>
</table>

provide a smaller $H_\infty$ disturbance attenuation level $\gamma_{\min}$ than those in [19] and [9].

Example 2: Consider the system (1) with

$$A_0 = \begin{bmatrix} -0.8 & 0.1 \\ 0.2 & 0.05 \end{bmatrix}, \quad B_u = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

$C_0 = [0.3 \ 0.1]$, $D_u = 0.2$, $D_v = 0.1$

Since two eigenvalues of the matrix $A_0$ are $-0.8229$ and $0.0729$, the open-loop system $\dot{x}(t) = A_0 x(t)$ is unstable. In the sequel, we design a state feedback controller of form (3) such that the resulting closed-loop system is asymptotically stable with a prescribed $H_\infty$ disturbance attenuation level $\gamma$. By applying Algorithm 1 with $N = 2$, for $\gamma = 0.8$ and $\tau_m = 7.35$, the obtained controller gain $K$ is listed in Table 1 for different lower bound $\tau_m$. From this table, one can see that for $\tau_m = 0.2$, after 110 iterations, a state feedback controller with gain $K = [-0.0365 - 0.1309]$ is obtained. Under the controller, supposing that the sample period is $0.6s$ and the network-induced delay bounds are $\nu_m = 0.2s$ and $\eta_m = 1.75s$, the admissible maximum number of data dropouts is allowed to $5$ between any two consecutive updating instants.

5. Conclusion

The $H_\infty$ control problem for NCSs has been addressed. A delay decomposition approach has been developed to derive a less conservative delay-dependent BRL. The desired $H_\infty$ controller has been designed by solving a nonlinear minimization problem using an iterative algorithm. The effectiveness of the proposed method has been verified by two numerical examples.

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References


