Abstract

In this paper, we present several systematic techniques, based on the Voronoi diagram and its variants, to partition a one and two-dimensional simplex. The Fekete points are used as the input to generate the Voronoi diagram, as they are almost optimal for the polynomial interpolation in a simplex and concentrate near the edges.

Spectral (finite) volume reconstructions on the resulted partitions have small Lebesgue constants. When using the Dubiner basis, the reconstruction matrix is well conditioned. Moreover, the total number of the edges of the partitions (the total work when being used in the spectral volume methods) is shown to be at most twice the minimum number of edges of all partitions for reconstructions of the same order accuracy. These suggest that the obtained partitions are well suited for the spectral volume methods and other numerical methods which rely on reconstructions from cell averages.
1 Introduction

In [7], Wang proposed a new finite volume (FV) method, named spectral volume (SV) method, for hyperbolic conservation laws. The spectral volume method has several good properties: high-order accurate, conservative, geometrically flexible, and computationally efficient. (A comparison with the discontinuous Galerkin methods is given in [9].) In the spectral volume method, a volume or a cell (named spectral volume) is partitioned into non-overlapping sub-cells named control volumes (CVs). Then cell-averaged solutions on the control volumes are used to perform high order reconstructions, i.e., spectral volume reconstructions. The spectral volume reconstruction is different from the reconstruction procedure of previous finite volume methods, which employs cell-averaged solutions on the neighbour cells to perform the reconstruction.

The spectral volume reconstruction is basically an approximation problem. Given a smooth function and an approximation space, the accuracy of the spectral volume reconstructions only depends on the partition of the spectral volume. Not all partitions produce good results. For example, uniform partitions [8] yield bad results for high-order reconstructions because of the Runge phenomenon. As far as we know, no systematic technique has been developed to partition an $m$-dimensional simplex, $S^m \subset R^m$, except that in [8] Wang gave a few partitions for up to the fourth-order reconstruction on a standard equilateral triangle.

However, another approximation problem on a simplex, the interpolation based on node values, has been extensively studied in the past (see [4] and the references therein). Some almost optimal nodal sets for polynomial interpolations on a two-dimensional simplex are given in [4] and [6]. In this paper, we develop several systematic techniques, based on the well-known Voronoi diagram and its variants with those optimal nodal sets as the input, to generate partitions of a one and two-dimensional simplex. Using these techniques, we obtained partitions for up to the 14-th order polynomial reconstruction on an equilateral triangle.

The remaining of the paper is organized as follows. In Sec. 2, we restate the spectral volume reconstruction problem on a two-dimensional simplex. The Lebesgue constant is introduced as one measurement of the quality of the spectral volume reconstructions. Section 3 describes the
systematic techniques to partition a one and two dimensional-simplex. Finally, we summarize the paper and make some concluding remarks in Sec. 4.
2 Spectral Volume Reconstruction

In this section, we define the spectral volume reconstruction problem on an equilateral triangle, \( S^2 \). Some issues related to the quality of the reconstruction are also addressed.

The spectral volume reconstruction is a key element of the recently proposed spectral volume method [7, 8], in which a target cell is divided into non-overlapping sub-cells. The cell-averaged solutions on the sub-cells are then used to reconstruct an approximate solution on the target cell. The number of the sub-cells is the same as the dimension of the approximation space. In general, the approximation space can consist of any functions. Here, we focus on the space of polynomials of degree up to \( n \), denoted as \( P^n \). The dimension of this approximation space is

\[
N_n^2 = \dim P^n = \binom{2 + n}{2}.
\]

(1)

\( N = N_n^2 \) and \( P^n = P^n \) will be used to simplify the notations if there is no confusion.

Then the spectral volume reconstruction problem can be formally stated as follows.

**SV Reconstruction Problem on** \( S^2 \): Given any continuous function \( u(x, y) \) on \( S^2 \), i.e., \( u \in C(S^2) \), the spectral volume reconstruction is to

1. **Construct a partition** \( \Pi_n \) of \( S^2 \):

\[
S^2 = C_1 \cup \cdots \cup C_N,
\]

where \( \{C_1, \cdots, C_N\} \) are \( N \) non-overlapping sub-cells;

2. **Compute the projection** \( I_{\Pi_n} u \in P^n \) such that

\[
\int_{C_i} (I_{\Pi_n} u) \, dV = \int_{C_i} u(x, y) \, dV, \quad i = 1, \cdots, N,
\]

i.e., the projection \( I_{\Pi_n} u \) and \( u \) have the same average on all the sub-cells.

The projection, \( I_{\Pi_n} u \), can be computed once the partition \( \Pi_n \) is known. Express \( I_{\Pi_n} u \) in a series sum of a complete basis of \( P^n \), \( \{p_1(x, y), \cdots, p_N(x, y)\} \),

\[
I_{\Pi_n} u = \sum_{i=1}^{N} a_i \ p_i(x, y).
\]

(3)
Denote $\bar{a}_i$ as the average of $u(x, y)$ over the sub-cells $C_i$, i.e.,
\[
\bar{a}_i = \frac{1}{V_i} \int_{C_i} u(x, y) \, dV, \quad i = 1, \cdots, N, \tag{4}
\]
where $V_i$ is the area of $C_i$. Plug (3) and (4) into (2), and rewrite the new equation into a matrix form
\[
A a = \bar{a}, \tag{5}
\]
where $\bar{a} = (\bar{a}_1, \cdots, \bar{a}_N)^T$, $a = (a_1, \cdots, a_N)^T$ and the reconstruction matrix $A$ takes the form
\[
A = \left( \begin{array}{ccc}
\frac{1}{V_1} \int_{C_1} p_1(x, y) dV & \cdots & \frac{1}{V_1} \int_{C_1} p_N(x, y) dV \\
\cdots & \cdots & \cdots \\
\frac{1}{V_N} \int_{C_N} p_1(x, y) dV & \cdots & \frac{1}{V_N} \int_{C_N} p_N(x, y) dV 
\end{array} \right). \tag{6}
\]

When the partition is non-singular, i.e., the matrix $A$ is non-singular, we solve Eq. (5) and substitute the solution $a$ back into the expression (3) to obtain the projection
\[
I_{\Pi_n} u = \sum_{i=1}^{N} \bar{a}_i \, L_i(x, y), \tag{7}
\]
where the cardinal basis functions $L = (L_1(x, y), \cdots, L_N(x, y))$ are given as
\[
L = (p_1(x, y), \cdots, p_N(x, y)) \, A^{-1}. \tag{8}
\]

Then we equip the space $P^n$ and $C(S^2)$ with an $L^\infty$ norm (supremum-norm, denoted as $\| \cdot \|$) and the induced functional norm
\[
\|I_{\Pi_n}\| = \sup_{\|u\| \neq 0} \frac{\|I_{\Pi_n} u\|}{\|u\|}.
\]

Since $|\bar{a}_i| \leq \|u\|$ for $i = 1, \cdots, N$, one can show that
\[
\|I_{\Pi_n}\| = \max_{(x, y) \in S^2} \sum_{i=1}^{N} |L_i(x, y)|. \tag{9}
\]

### 2.1 Error of Spectral Volume Reconstruction

One measurement of the quality of spectral volume reconstructions is the error. Similar to that of the polynomial interpolations, the error of the spectral volume reconstruction is bounded from below as
\[
\|u - u^*\| \leq \|u - I_{\Pi_n} u\|, \tag{10}
\]
where \( u^* \) is the optimal approximating polynomial whose existence is guaranteed by the continuity of \( u(x, y) \) [1]. Although it is difficult to determine such optimal approximation for general functions, it enables us to evaluate the quality of other approximations. From the linearity of the projection operator \( I_{\Pi_n} \) and the fact that \( I_{\Pi_n} f = f, \forall f \in P^n(S^2) \), one can verify that

\[
\|u - I_{\Pi_n} u\| \leq (1 + \Lambda(\Pi_n)) \|u - u^*\|, \tag{11}
\]

where

\[
\Lambda(\Pi_n) = \|I_{\Pi_n}\| = \max_{(x,y) \in S^2} \sum_{i=1}^{N} |L_i(x, y)|. \tag{12}
\]

\( \Lambda(\Pi_n) \) is called the Lebesgue constant [4] of the operator \( I_{\Pi_n} \).

From the way to compute \( I_{\Pi_n} u \) as previously described, one can show that the Lebesgue constant only depends on the partition \( \Pi_n \) when the approximation space is fixed.

**Lemma 1** For fixed approximation space \( P^n \), the partition \( \Pi_n \) determines the Lebesgue constant \( \Lambda(\Pi_n) \).

**Proof:** From the definition of the Lebesgue constant (c.f. (12)), it is enough to show that the same cardinal basis functions will be obtained for two different basis sets.

Choose another basis set, \( (q_1(x, y), \ldots, q_N(x, y)) = (p_1(x, y), \ldots, p_N(x, y)) \cdot T \), where \( T \) is a constant non-singular matrix. According to Eq. (6), the new reconstruction matrix is

\[
\tilde{A} = A \cdot T.
\]

So the new cardinal basis is

\[
\tilde{L} = (q_1, \ldots, q_N) \cdot (\tilde{A})^{-1} = (p_1, \ldots, p_N) \cdot T \cdot T^{-1} A^{-1} = L,
\]

which proves the lemma.

According to (11) and (10), the magnitude of the Lebesgue constant reflects how close the spectral volume reconstruction is to the optimal polynomial approximation. Therefore, spectral volume reconstructions with small and slowly-increasing Lebesgue constants w.r.t. the order of the reconstruction are preferred. For simplicity we only consider partitions with sub-cells being
convex polygons with straight edges. However, this might keep us from obtaining spectral volume reconstructions with Lebesgue constants as small as those of the polynomial interpolations as shown in [6] and [4].

Another important issue is the work load when the spectral volume reconstruction procedure is used in solving partial differential equations. As shown in [8, 7], the work load is roughly proportional to the total number of the edges of the partition. Hence the optimal partitions should have minimum number of edges and lead to the smallest Lebesgue constant. If one wants to optimize the partition, one needs to minimize both the number of edges and the Lebesgue constant at the same time, which apparently is not an easy task. We made no such effort in this paper.

We also want to emphasize that it is necessary that the reconstruction matrix $A$ is well-conditioned for high-order spectral volume reconstructions because of the finite precision of computers. For polynomial interpolations on a triangle, this is usually achieved by choosing the Dubiner basis [3] instead of the notorious monomials, provided the nodal set is good.
3 Partitions from Voronoi Diagram and Its Variants

In this section, we describe a few systematic techniques to partition a one and two-dimensional simplex by using the Voronoi diagram and its variants.

3.1 The Voronoi Diagram and Its Variants

The following definition is a generalization of the two-dimensional Voronoi diagram [5, 2].

**Definition 1 (Voronoi Diagram)** Given distinct input points \( S = \{p_1, \cdots, p_N\} \) in \( \mathbb{R}^m \), the Voronoi diagram is a partition of \( \mathbb{R}^m \) into \( N \) non-overlapping polyhedral regions: \( \{V_1, \cdots, V_N\} \), such that \( p_i \in V_i, i = 1, \cdots, N \), and the Voronoi cell \( V_i \) is the set of points in \( \mathbb{R}^m \) which are closer to \( p_i \) than to any other points in \( S \), i.e.,

\[
V_i = \{x \in R^m : |x - p_i| \leq |x - q|, \forall q \in S - p_i\},
\]

where \( |x - y| \) represents the Euclidean distance between \( x \) and \( y \) (other distance functions can also be used).

All Voronoi cells and faces form a cell complex whose vertices and edges are called **Voronoi vertices** and **Voronoi edges**. The unbounded edges are also called **Voronoi rays**. When the input points, \( \{p_1, \cdots, p_N\} \), are in the plane (\( m=2 \)), we can bound the number of the Voronoi vertices and Voronoi edges by the following theorem [5, 2].

**Theorem 1** For \( N \geq 3 \), in the Voronoi diagram of \( N \) distinct points on the plane, the number of Voronoi vertices is at most \( 2N - 5 \); the number of Voronoi edges is at most \( 3N - 6 \).

When the input points are on a triangle, it is straightforward to generate a partition of the triangle from the Voronoi diagram, e.g., Fig. 1. This observation enables us to partition a triangle by choosing the input points and computing the Voronoi diagram. Furthermore, each Voronoi vertex (the circles in Fig. 1) is the circumcenter of one triangle with the vertices being three input points. Therefore, we can get a few variants of the Voronoi diagram, and thus different partitions,
Figure 1: Partition of a triangle from the Voronoi diagram of given points on the triangle. On the left graph, the dark dots are the input points; the circles are the Voronoi vertices; the thin solid lines are the Voronoi edges. The right graph displays the partition.

by replacing each Voronoi vertex (circumcenter) with the corresponding incenter, centroid or any other point related to that triangle. We recall that for a triangle, the circumcenter is the center its circumcircle, the incenter is the center its incircle, and the centroid is the intersection of the triangle’s three triangle medians.

3.2 Partitions of $S^1$

$S^1$ is simply a line segment. Without loss of generality, let $S^1 = [-1, 1]$. One can generate a partition from any given input points set as follows. For example, suppose $-1 = x_0 < x_1 < \cdots < x_n = 1$ are the input points. Take $y_0 = x_0, y_{n+1} = x_n$ and $y_i = (x_i + x_{i-1})/2, i = 1, \cdots, n$. The points $\{y_i\}$ then define a partition: $V_i = [y_i, y_{i+1}], i = 0, \cdots, n$. A different points set will yield a different partition. Table 1 includes the Lebesgue constants corresponding to the Legendre Gauss-Lobatto (LGL) and Chebyshev Gauss-Lobatto (CGL) quadrature points. $\Lambda_{LGL}^i$ and $\Lambda_{CGL}^i$ represent the Lebesgue constants of the interpolation with the Legendre Gauss-Lobatto and Chebyshev Gauss-Lobatto points. $\Lambda_{LGL}^V$ and $\Lambda_{CGL}^V$ represent the Lebesgue constants for the spectral volume reconstruction on the partitions with LGL and CGL points being the input. $\Lambda_{Eq}^V$ denotes the Lebesgue constant for the spectral volume reconstruction on the uniform mesh, i.e., all sub-cells have the same size.
Note that the Lebesgue constants of the spectral volume reconstructions on the partitions from both LGL and CGL points are less than twice those of the polynomial interpolations, which are very close to the Lebesgue constants of the optimal nodal set [4]. In summary,

\[ \Lambda_{\text{Eq}} > \Lambda_{\text{LGL}} > \Lambda_{\text{CGL}} > \Lambda_{\text{CGL}} > \Lambda_{\text{LGL}}. \]  

(14)

Table 1: Lebesgue constants for one-dimensional spectral volume reconstructions and interpolations. \( n \) : the order of the spectral volume reconstruction or interpolation.

<table>
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<tr>
<th>( n )</th>
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3.3 Partitions of \( S^2 \)

We only study the standard equilateral triangle \( S^2 \) (c.f. Fig. 1) because any other triangle can be obtained from a linear transformation of this triangle. The linear transformation will not change the Lebesgue constant of the spectral volume reconstruction, which will be shown as below.

Lemma 2 Suppose the standard equilateral triangle \( S^2 \) has a partition \( \Pi_n = \{ C_1, \cdots, C_N \} \). Let \( \tilde{S} \) be a triangle defined on \( \xi - \eta \) plane, which can be obtained from a non-singular linear
transformation of $S^2$. If the partition of $\tilde{S}$, $\tilde{\Pi}_n = \{\tilde{C}_1, \cdots, \tilde{C}_N\}$, is obtained from $\Pi_n$ by the same linear transformation, then the spectral volume reconstructions on $\tilde{S}$ and $S^2$ have the same Lebesgue constant.

**Proof:** Denote $\tilde{V}_i$ as the area of the sub-cell $\tilde{C}_i$. Use $\tilde{A}$ and $\tilde{L}$ to represent the reconstruction matrix and cardinal basis set on $\tilde{S}$. For $S^2$, we use the same notations as those in Sec.2.

From the definition of the Lebesgue constant, it suffices to show that

$$\tilde{L}(\xi, \eta) = L(x, y), \quad \forall (\xi, \eta) \in \tilde{S} \text{ or } (x, y) \in S^2.$$ 

According to (8),

$$\begin{cases}
  L(x, y) = (p_1(x, y), \cdots, p_N(x, y)) A^{-1} \\
  \tilde{L}(\xi, \eta) = (p_1(\xi, \eta), \cdots, p_N(\xi, \eta)) \tilde{A}^{-1}
\end{cases}.$$

Since the transformation, $(\xi, \eta) \leftrightarrow (x, y)$, is linear, there exists a constant matrix $T$ such that

$$(p_1(\xi, \eta), \cdots, p_N(\xi, \eta)) = (p_1(x, y), \cdots, p_N(x, y)) \cdot T. \quad (15)$$

$T$ is also non-singular as the transformation is non-singular. (The non-singularity of $T$ can be proved by evaluating Eq. (15) at $N$ distinct points.) Use Eq. (15) in the computation of the reconstruction matrix $\tilde{A}$, i.e.,

$$\tilde{A} = \left(\begin{array}{ccc}
\frac{1}{V_1} \int_{\tilde{C}_1} \tilde{p}_1(\xi, \eta) d\tilde{V} & \cdots & \frac{1}{V_1} \int_{\tilde{C}_1} \tilde{p}_N(\xi, \eta) d\tilde{V} \\
\cdots & \cdots & \cdots \\
\frac{1}{V_N} \int_{\tilde{C}_N} \tilde{p}_1(\xi, \eta) d\tilde{V} & \cdots & \frac{1}{V_N} \int_{\tilde{C}_N} \tilde{p}_N(\xi, \eta) d\tilde{V}
\end{array}\right)$$

to obtain

$$\tilde{A} = A \cdot T.$$

Hence,

$$\tilde{L}(\xi, \eta) = (p_1(\xi, \eta), \cdots, p_n(\xi, \eta)) \tilde{A}^{-1} = (p_1(x, y), \cdots, p_N(x, y)) \cdot T \cdot (A \cdot T)^{-1} = L(x, y).$$

The first kind of partitions, denoted as $\Pi^F_{\text{out}}$, are from the Voronoi diagram$^1$ of the two-dimensional Fekete points [6] on the triangle. As previously mentioned, each Voronoi vertex is

$^1$Fortune’s code is used to compute the 2-D Voronoi Diagram
Figure 2: The 7-th order partition from the Voronoi diagram of given Fekete points. On the left graph, the thin solid lines are the Voronoi rays; the circles are the Voronoi vertices. The right graph is the partition. The outside Voronoi vertices are also plotted for clarity.

the circumcenter of one triangle with vertices being three input points. So there might be some Voronoi vertices which are outside the big equilateral triangle. When this happens, the partition is not so easy to get as shown in Fig. 1. However, as demonstrated in Fig. 2, we can still obtain a partition by replacing each outside Voronoi vertex with the intersection point of the corresponding Voronoi ray and the edge of the big equilateral triangle. With this strategy, we generate partitions up to the 14-th order (Fig. 3). The Lebesgue constants of the spectral volume reconstructions are listed in Tab. 2. The number of distinct edges is bounded as follows.

**Theorem 2** The total number of edges in the partition, $\Pi^F_{\text{out}}$, is less than twice the minimum number of edges of any partition leading to the same order spectral volume reconstruction.

**Proof:** For the $n$-th order partition $\Pi^F_{\text{out}}$, according to Theorem 1, the Voronoi diagram contributes at most $(3N - 6)$ to the total number of edges, where $N = \binom{2+n}{2}$ (c.f. Eq. (1)) is the total number of input points on the triangle. Besides that, we should also count the $3(n+1)$ edges which lie on the edges of the big equilateral triangle. So the total number of edges of the $n$-th partition $\Pi^F_{\text{out}}$ is at most

$$3N - 6 + 3(n+1) = \frac{3}{2}(n^2 + 5n).$$  \hfill (16)
Consider any partition leading to an $n$-th order spectral volume reconstruction. Denote $X$ as the number of edges of the partition which are on the edges of the big equilateral triangle, and $Y$ as the number of remaining edges (called inside edges since they are inside the big equilateral triangle). Clearly, $X = 3(n + 1)$. Since an $n$-th order reconstruction needs $N$ polygons, each of which has at least three edges, one can show that

$$X + 2Y \geq 3N,$$

where the coefficient before $Y$ is due to the fact that each inside edge belongs to two polygons. Thus, the minimum number of distinct edges for any $n$-th order partition satisfies

$$X + Y \geq X + \frac{3N - X}{2} = 3(n + 1) + \frac{3N - 3(n + 1)}{2} = \frac{3}{4}(n^2 + 5n + 4).$$

Comparing the above equation with (16) proves the theorem.

However, this simple usage of Voronoi diagram does not yield very small Lebesgue constants, as shown in Fig 7, for high order spectral volume reconstructions.

We derive the second kind of partitions (denoted as $\Pi_{in}^F$) from one variant of the Voronoi diagram in which each Voronoi vertex is replaced by the corresponding incenter. By doing that, the structure of the partitions is more similar to the structure of the input points in the sense of layered structures and concentration near the edges (see Fig. 3 and 4). This is due to the fact that the incenter of a triangle is always inside the triangle. We believe that the layered structure and being concentrated near the edges of the sub-cells are crucial for the partition to produce small Lebesgue constants. As expected, the partition $\Pi_{in}^F$ produces smaller Lebesgue constants than $\Pi_{out}^F$ for most cases of up to the 14-th order spectral volume reconstructions (Tab. 2).

Unfortunately, as shown in Fig. 7 there is a sudden increase in the Lebesgue constants of the 8-th or higher order $\Pi_{in}^F$ partitions. By examining Fig. 4 more carefully, we notice that the layered structure is a bit “distorted” in the place close to the edges of the big triangle. The “distortion” is responsible for the sudden increase of the Lebesgue constants.

However, the Fekete points set itself has a very nice structure (Fig. 5), based on which we come up with the third kind of partitions, denoted as $\Pi_{mass}^F$. There are three steps to build the
Figure 3: Partition $\mathcal{P}_{\text{out}}^F$. $n$: Order of SV reconstruction; $N$: Number of sub-cells; ‘.’: input Fekete points on the triangle.

Figure 4: Partition $\mathcal{P}_{\text{in}}^F$. $n$: Order of SV reconstruction; $N$: Number of sub-cells; ‘.’: input Fekete points on the triangle.
partition, which are demonstrated in Fig. 5. At first, we concatenate the input points layer by layer (see the left graph of Fig. 5). Then we construct a triangular mesh as shown in the middle graph of Fig. 5. Finally, for each input point inside the big triangle, we construct a polygon containing this input point by connecting the centroids (or any other point) of the small triangles those share the input point (see the right graph of Fig. 5). When an input point is on the edge of the big triangle, one can construct a polygon containing the point by using two more points on the edges of the big equilateral triangle. Some examples of the partitions are displayed in Fig. 6. We also tried to use the incenter instead of centroid of those small triangles. It yields almost the same Lebesgue constants as the centroid does.

The same upper bound holds for the number of the edges of the partitions $\Pi^F_{in}$ and $\Pi^F_{mass}$.

**Theorem 3** *The total number of edges in the partition $\Pi^F_{in}$ or $\Pi^F_{mass}$, is less than twice the minimum number of edges for any partition leading to the same order SV reconstruction.*

The proof is omitted as it is basically the same as that of **Theorem 2**.

For comparison, we also compute the Lebesgue constants of the partition $\Pi^E_{v}$, which is from the Voronoi diagram of equispaced (in the area coordinate system) points on the equilateral triangle. The Lebesgue constants for all partitions are listed in Tab. 2. And Table 3 contains the 2-norm condition numbers of the reconstruction matrix when the Dubiner basis is used. Figure 7 displays the ratios of the Lebesgue constants from the above partitions to those of the polynomial interpolations based on the Fekete points.
Figure 5: Three steps to construct the partition $\Pi^F_{\text{out}}$. '•': input Fekete points. Left: the layered structure of Fekete points; Middle: the triangular mesh; Right: the method to build the polygon (thick line) for one input point. 'o': vertices of the polygon.

Figure 6: Partition $\Pi^F_{\text{mass}}$. n: Order of SV reconstruction; N: Number of sub-cells; '•': input Fekete points on the triangle.
Table 2: Lebesgue constants for two-dimensional spectral volume reconstructions and interpolations. $\Lambda^F$ stands for the Lebesgue constant for the interpolations with Fekete points. $\Lambda^F_{\text{out}}, \Lambda^F_{\text{in}}, \Lambda^F_{\text{mass}}$, and $\Lambda^F_{\text{V}}$ represent the Lebesgue constants for the SV reconstruction on the partition $\Pi^F_{\text{out}}, \Pi^F_{\text{in}}, \Pi^F_{\text{mass}}$ and $\Pi^F_{\text{V}}$ respectively.

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Table 3: The 2-norm condition numbers of the spectral volume reconstruction matrix on the two-dimensional partitions.

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Figure 7: The ratio of the Lebesgue constants of two-dimensional spectral volume reconstructions to those of the interpolation on Fekete points. "-\triangle": $\Pi^F_{\text{out}}$ partition; "-o": $\Pi^F_{\text{in}}$ partition; "-x": $\Pi^F_{\text{mass}}$ partition;
4 Conclusions

We have developed several systematic techniques to partition a one and two-dimensional simplex by using the well-known Voronoi diagram and its variants. The resulted partitions have layered structure and the sub-cells are concentrated near the edges. These two properties are found to be crucial for the partitions leading to small Lebesgue constants.

The spectral volume reconstructions on those partitions have small Lebesgue constants, one of which is roughly twice the Lebesgue constant of the same order interpolation based on the almost optimal nodal sets. The total number of edges (the total work when being used in the SV method) of the partitions is showed to be at most twice the minimum number of edges of all partitions for the reconstructions of the same order accuracy. When using the Dubiner basis, the spectral volume reconstruction matrix is very well-conditioned. All of these suggest that the partitions are a good choice for the spectral volume methods and other numerical methods which rely on reconstructions from cell averages.

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References


