On the Design of a Class of Odd-Length Biorthogonal Wavelet Filter Banks for Signal and Image Processing

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Abstract

In this paper, we introduce an approach to the design of odd-length biorthogonal wavelet filter banks based on semidefinite programming employing Bernstein polynomials. The method is systematic and renders a simple optimization problem, yet it offers wavelet filters ranging from maximally flat to maximal passband/stopband width. The odd-length biorthogonal filter pairs are then used in multi-focus imaging to obtain a fully-focused image from a set of registered semi-focused input images at varying focus employing the distance transform and exponentially decaying function on the subbands in wavelet domain. Various images are tested and experimental results compare favorably to recent results in literature.

1. Introduction

Convex optimization and semidefinite programming (SDP) have been used frequently in wavelet filter design techniques, e.g., in [2] and [4]. In [10], a sum of squares based method was proposed to the design of halfband orthogonal wavelets considering nonnegativity of the frequency response of the product filter on the unit circle. Parametric Bernstein polynomials (PBP) have been used widely in wavelet design. An orthogonal class of wavelet filter banks was introduced in [5] based on the use of PBPs employing spectral factorization on product filters. In [9], a class of odd-length biorthogonal wavelet filters was introduced based on least squares optimization.

The most basic advantage of defining a problem via convex optimization is that the problem can then be solved reliably and efficiently. Although several state-of-the-art works can be found in literature, this paper is motivated by simplicity and flexibility of semidefinite programming to offer more control over the frequency response of the wavelet filters, number of zeros at $z = -1$, and passband/stopband ripples.

An application of the filters designed is shown in multi-focus imaging. The reconstruction of a geometric object and to retrieve spatial information from one or multiple observation is a challenging problem in computer vision. When a 3D scene is projected into a 2D image plane, the depth information is lost. The object points on the focus plane appear sharp and blurring is increased with the distance of imaging system from the focus plane. For the scene with considerably large depth, points on the focus plane have a sharp appearance while the rest of the scene points are blurred. Taking the advantage of discrete wavelet transform (DWT), the transformation provides detailed information in wavelet subbands of each frame in the sequence which provides more options to select the best fitting feature pixel among the input images. Experimental results and quantitative comparisons show that the proposed framework based on use of DWT employing the designed wavelet filters improves fused image extraction via multi-focus imaging.

2. Odd-length Biorthogonal Wavelet Design

Consider two PBPs, namely $B_1(x)$ and $B_2(x)$ as

$$B_\rho(x) = K_0^{B_\rho}(x) - \sum_{\ell = m_\rho}^{k_\rho} \alpha_\ell K_\ell^{B_\rho}(x) \tag{1}$$

where the terms $K_0^{B_\rho}(x)$ and $K_\ell^{B_\rho}(x)$ are defined by

$$\sum_{i=0}^{k_\rho} \binom{N_\rho}{i} x^i (1 - x)^{(N_\rho - i)}$$

and

$$\binom{N_\rho}{\ell} x^\ell (1 - x)^{(N_\rho - \ell)} - x^{N_\rho - \ell} (1 - x)^\ell$$

respectively, with $\ell = m_\rho, \ldots, k_\rho$, $0 < m_\rho \leq k_\rho = (N_\rho - 1)/2$, and $\rho = 1, 2$. Polynomials in (1) structurally satisfy halfband condition, i.e., $P(z) + P(-z) = 1$, $P(z) = B(-z(1 - z^{-1})^2/4)$, and $P(z) = -z^2(1 - z^{-2})^2 = \sin^2(\omega/2)$ [5]. Note that $P(z)$ has $2m$ zeros at $z = -1$, the so-called number of vanishing moments (NVM), equivalent to $m$ zeros at $z = 1$ in the variable $x$. Let $\Lambda(x)$ be the set of two kernel filters, $H(x)$ and $F(x)$, defined as follows to the design of an odd-length biorthogonal filter bank [9].
\[ H(x) \triangleq B_1(x) \]  
\[ F(x) \triangleq B_1(x) + 2B_2(x) - 2B_1(x)B_2(x). \]

Then the lowpass and highpass filters in the analysis and synthesis sides of the filter bank, \( H_0(z) \), \( F_0(z) \), \( H_1(z) = z^{-1}F_0(-z) \) and \( F_1(z) = zH_0(-z) \), respectively, are determined via variable transformation. For the obtained odd-length filters \( \text{NVM}_1 = 2m_1 \), \( \text{NVM}_2 = 2m_1 \cap \{m_p\} \), \( L_1 = 2N + 1 \), and \( L_2 = 4N + 1 \) where the notation \( L \) refers to filter length. Now let us assume \( B(x) \) be a polynomial of odd degree \( 2k + 1 \). The polynomial is said to be positive semidefinite (PSD) if \( B(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). Now we want to solve a semidefinite programming (SDP) problem with the coefficients of \( B(x) \) as variables, and the constraint that \( B(x) \) be PSD [2],[8]. Then \( B(x) \geq 0 \) for all \( x \in [a, b] \), iff there exist [8] two polynomials \( Q_p(x) \) and \( Q_r(x) \) of degree not more than \( k \) such that

\[ B(x) = (x-a)Q_p^2(x) + (b-x)Q_r^2(x). \]  

(4)

Now let us assume \( x_{cp} \) and \( x_{cr} \) be the passband and stopband edges of \( \Lambda(x) \), respectively. Using (4) and similar to [10], the following inequalities and equality representations are determined.

\[ 1 - \delta_{cp} \leq \Lambda(x) \leq 1 + \delta_{pc}, \ x \in [0, x_{cp}] \]  
(5)
\[ 1 + \delta_{pc} - \Lambda(x) = (x-0)U_{\mu_2}(x) + (x_{cp} - x)U_{\nu_2}(x) \]  
(6)
\[ \Lambda(x) + \delta_{pc} - 1 = (x-0)L_{\mu_1}(x) + (x_{cp} - x)L_{\nu_1}(x) \]  
(7)
\[ -\delta_{pc} \leq \Lambda(x) \leq \delta_{pc}, \ x \in [x_{cp}, 1] \]  
(8)
\[ \delta_{pc} - \Lambda(x) = (x-x_{cp})U_{\mu_2}(x) + (1-x)U_{\nu_2}(x) \]  
(9)
\[ \Lambda(x) + \delta_{pc} = (x-x_{cp})L_{\mu_1}(x) + (1-x)L_{\nu_1}(x) \]  
(10)

where polynomials \( L_{\mu_r}(x) \), \( L_{\nu_r}(x) \), \( U_{\mu_r}(x) \), \( U_{\nu_r}(x) \), \( r = 1, \ldots, 4 \), have degrees equal to or less than \( k \). Note that (5) to (10) show the derivations for passband/stopband region, and unlike \( B_1(x) \), the wavelet filters obtained via \( B_2(x) \) in a biorthogonal design need not necessarily an antisymmetric function which requires investigation in [0, 1]. Thus, half of derivations are used to find \( B_1(x) \) coefficients. Now consider the following design problem.

Let \( B_1(x) \) and \( B_2(x) \) be in the form of (1) with specified \( N_\rho, m_\rho, \delta_{p1,2}, \) and \( \delta_{s1,2} \), where \( \rho = 1, 2 \). We want to determine \( \alpha_{B1}^1 \) and \( \alpha_{B2}^1 \) such that the passband and stopband widths be maximized subject to (6), (7), (9), and (10). Note that (5) to (10) and the problem above indicate a general form, and therefore, in an equiripple design obviously \( \delta_{pc} = \delta_{p1} \).

Figure 1. Filter function characteristics of a low-pass kernel in the variable \( x \).

3. Design Examples and Discussions

Various combinations may be considered, depending on filter length, number of vanishing moments and ripples resulting in a class of biorthogonal wavelet filters, halfband pair filter banks in [7] and [9], with maximal passband/stopband widths. In terms of optimization, a two-stage cycle is performed, i.e., first \( \alpha_{B1}^1 \) are obtained and then \( \alpha_{B2}^1 \) are determined substituting the values obtained in the first cycle. In [1], we will show a SDP based design of biorthogonal filter banks in general regardless of filters being of odd- or even-length. We specify passband and stopband ripples of 1±0.02 in the variable \( x \). Table 1 shows two sample examples with \( N = 7 \) and 11. The obtained coefficients of polynomials in each design and the maximal passband and minimal stopband edges associated with each design are given in Table 1.

Fig. 2 illustrates magnitude responses of \( H_0(z) \) and \( H_1(z) \) respectively, for the case \( N = 7 \) and \( m = 2 \) where the corresponding maximally flat pair is also shown for reference with dotted curve. The wavelet filter coefficients of this example and the corresponding smooth and symmetric scaling function and wavelets are given in Figs. 3 (a) to (d). In the following section, we show the application of the designed wavelet filter pairs in image fusion. The method offers a design tool to obtain a class of biorthogonal filter pairs based on some desired characteristics. Therefore one can simply customize a filter pair for different applications such as image/video denoising, detection and segmentation, classification and feature extraction.

4. Application to Multi-focus Imaging

The optical lenses of imaging sensor especially with long focal lengths only have a limited depth of field so a complete picture may not be always feasible. Multi-focus image fusion is a process of obtaining a fully-
Table 1. Coefficients to implement the wavelet filters for two sample cases, \( N = 7 \) and \( N = 11 \). The notation \( \bar{\omega} \) refers to normalized frequency (\( \bar{\omega}(\cdot) = \omega(\cdot)/\pi \)).

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<th>( N )</th>
<th>( m )</th>
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Figure 2. Magnitude responses of the filter pair for the case \( N = 7 \), \( m = 2 \) (solid curve). Maximally flat (dotted curve).

Figure 3. Filter bank coefficients, (a) lowpass; (b) highpass. Scaling function (solid curve) and wavelets (dotted curve), (c) analysis; (d) synthesis.

focused image from a set of registered semi-focused input images. The main goal of multi-focus image fusion is to transfer the most relevant information found in source images into a fused image.

Fig. 4 shows the entire process of the proposed image fusion technique employing the filters designed in Section 3 with \( N = 7 \) and \( m = 2 \). Our earlier algorithm in [6] is modified in terms of replacing the SU-SAN operator with discrete wavelet transform (DWT). Taking the advantage of DWT, the transformation provides detailed information in wavelet subbands of each frame in the sequence; leading to more options to select the best fitting feature among the entire input images. Firstly, the pre-registered multi-focus image sequence is decomposed by DWT, and then the maximum selection rule is applied on the wavelet subbands of each image at the last level of the DWT. Finally, a decision map is determined employing the distance transform with the exponentially decaying function, proposed in [6].

The exponentially decaying function uses neighborhood information in each subband assuming that the intensity far from a pixel is equal to 1 and it approaches this limiting value exponentially. Decision map for fusion is constructed by comparing the value of subbands; frame number with higher value is mapped onto the decision map. Using the decision map, pixels are extracted from corresponding subbands from the image set and then an inverse DWT yields final fused image. For simulation we used the chess, cone, book, and clock image datasets. The results obtained for the chess dataset of 29 images of size 800 × 600 are shown in Fig. 5. The resultant fused images using well documented focus measure (FM) methods, namely, sum of modified Laplacian (FM\(_{SML}\)), Tenengrade (FM\(_T\)), gray level variance (FM\(_{GLV}\)) are shown in Figs. 5 (a) to (c), respectively. It is seen that the fusion in Fig. 5 (d), employing designed wavelet filters, is significantly clearer than previously proposed methods. For the quantitative comparison we have used mutual information (MI) and structural similarity information matrices (SSIM). The MI and SSIM comparison without noise and in presence of Gaussian noise with a variance of 0.001 are given in Tables 2 and 3 respectively, confirming the experimental results and visual improvements.
5. Conclusions

In this paper, we have introduced a simple yet efficient technique to the design of odd-length biorthogonal wavelet filters. The method is focused on formulating a semidefinite programming optimization problem for a biorthogonal structure employing two kernel functions, where design parameters are more controllable. It provides tuning opportunity to impose desired properties depending on what one may need in an application. An application of a sample wavelet filter is shown in multi-focus imaging. Experiments and results show the performance improvement of DWT based technique employing the proposed filters in terms of fusion assessment factors and visual perception. In view of success of the presented approach in fully-focused image fusion, promising results are expected in relevant applications such as 3D shape reconstruction and object identification if a similar approach is followed.

Acknowledgment

This work is supported in part by Canada Research Chair program and NSERC Discovery Grant.

References