Periodic inventory model with unstable lead-time and setup cost with backorder discount

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Abstract: The main purpose of this study is to investigate the periodic inventory model with backorder price discounts, where shortages are partially backlogged. The application of just-in-time (JIT) philosophy i.e., crashing of the lead-time and setup cost has been carried component wise. Two cases have been discussed viz

1 protection interval demand distribution is known (normal distribution approach)
2 protection level demand distribution is unknown (minimax distribution approach).

An algorithm has been developed which jointly optimises the review period, the backorder price discount, the setup cost and lead-time for a known service level. Numerical results clearly indicate that significant savings could be achieved.

Keywords: inventory; setup cost; lead-time; crashing cost; backorder price discount.


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1 Introduction

In inventory management, the lead-time has been considered as a prescribed constant or a stochastic variable. In reality, lead-time usually consists of the following components as suggested by Tersine (1982): order preparation, order transit, supplier lead times, delivery lead time and setup time. To gain the competitive success in business, each firm knows the importance of time in the market place. Therefore, the objective of each firm is to reduce the lead-time in a practical situation to satisfy the customer’s demand on time. The application of just-in-time (JIT) philosophy includes the crashing of lead-time to run the system profitably. The lead-time can be decomposed into several crashing periods for making the working system effective. In many practical situations, lead-time is controllable or can be reduced at an extra-added cost. Generally, the extra cost of reducing lead-time involves administrative, transportation and supplier’s speed-up costs.

As a result, the client service and manufacturing schedule can be improved even with reduced safety stocks. Recently, several continuous review inventory models have been developed to consider lead-time as a decision variable (Liao and Shyu, 1991; Ben-daya and Raouf, 1994; Ouyang et al., 1999; Moon and Choi, 1998; Ouyang and Wu, 1997; Ouyang and Chuang, 1998; Wu and Tsai, 2001). One finds that these papers mainly lay emphasis on the benefits of reducing lead-time. So, we have considered supposition of reducing lead-time at an extra cost in this paper.

It is generally observed that while shortage occurs, demand can be captured partially. Some customers may prefer their demands to be backordered i.e., some customers whose needs are not urgent can wait for the item to be satisfied, while others who cannot wait have to fill their demand from another source which is lost sale case. But there are some factors that motivate the customer for the backorders out of which price discount from the supplier is the major factor. By offering sufficient price discounts, the supplier can secure more backorders through negotiation. With higher price discount, the supplier could fetch a large number of back order ratio. Kim and Park (1985) investigated an inventory model with a mixture of sales and time weighed backorders. Subsequently, Pan and Hsiao (2001) presented continuous inventory models with backorder discounts and variable lead-time. Later on, Pan and Hsiao (2005) expanded the model by considering the case where lead-time crashing cost is given as the function of reduced lead-time and ordered quantities. In this paper, the backorder discount has been taken as one of the decision variable. The consideration is as unsatisfied demand during the shortages can lead to optimal backorder ratio by controlling the price discount and the supplier is to minimise the relevant total inventory cost.

Further, Porteus (1985) examined the impact of capital investment in reducing setup costs in the classical EOQ model for the first time. Many researchers have reported several relationships between the amount of capital investment and setup cost level (Nori
and Sarker, 1996; Kim et al., 1992; Trevino et al., 1993; Hall, 1983, Sarker and Coates, 1997). Therefore, this article deals with two important aspects of JIT philosophy i.e., reduction of lead time and setup costs where the lead time and setup costs vary as a function of capital expense. Setup costs can be controlled and reduced through various efforts such as worker training, procedural changes and special equipment acquisition. Earlier, Chuang et al. (2004) presented a periodic review inventory model with variable lead-time and reduction of setup cost where the setup cost has been considered as the logarithmic function of investment without backorder discount. Alternatively, Cheng et al. (2004) have analysed the continuous inventory model with crashing of lead-time and setup cost component wise where the lead-time demand follows the normal distribution only. But, certain questions remain to be answered e.g., what will be the effect of backorder discount in periodic inventory model or whether the component wise reduction of lead-time and setup costs are beneficial in periodic inventory model or not? If yes, what is the intensity of benefits? If the demand during lead-time does not follow normal distribution, what will be the probable solution and so on? To answer these questions, this paper suggests a periodic review inventory model with backorder discount where lead time and setup costs have been reduced component wise, which can be taken as the mixture and extension of Cheng et al. (2004) and Chuang et al. (2004) work. In this article, periodic review inventory model has been considered where demand during the protection interval is partially backordered. Both cases of protection interval demand

1 known distribution

2 unknown distribution, where first and second finite moments are known, have been revealed.

The main purpose of our study is to optimise the review period, the backorder discount, the setup cost and lead-time with known service level. The lead-time and setup costs both are controllable and have shown that the significant saving could be obtained by backorder discount.

2 Notation and assumption

To develop the proposed model, we adopt the following notation and assumptions used in Cheng et al. (2004) in this paper.

2.1 Notation

\(D\) average demand per year

\(K\) fixed ordering cost per inventory cycle

\(h\) inventory holding cost per unit per year

\(R\) target level

\(\beta\) fraction of the demand back ordered during stock out period such as \(0 \leq \beta \leq 1\)
\( \beta_0 \) upper bound of the backorder ratio

\( \pi_0 \) marginal profit (i.e., cost of lost demand) per unit

\( \pi_c \) back order price discount offered by the supplier per unit

\( L \) length of lead-time

\( X \) protection interval demand which has a p.d.f. \( f_x \) with finite mean \( D(T+L) \) and standard deviation \( \sigma \sqrt{(T+L)}(> 0) \) for the protection interval \( (T+L) \) where \( \sigma \) denotes the standard deviation of the demand per unit

\( \Omega \) the class of p.d.f. \( f_x \) of the protection interval demand with finite mean \( D(T+L) \) and standard deviation \( \sigma \sqrt{T+L} \)

\( S \) fixed shortage cost, \$ per unit short

\( A \) safety factor

\( T \) length of a review period

\( E(.) \) mathematical expectation

\( x^+ \) maximum value of \( x \) and 0 i.e., \( x^+ = \max \{x, 0\} \)

\( EAC \) expected annual cost

\( EAC^* \) least upper bound of expected annual cost.

### 2.2 Assumptions

1. The inventory level is reviewed every \( T \) units of time. A sufficient quantity is ordered up to the target level \( R \), and the ordering quantity is arrived after \( L \) units of time.

2. The length of the lead-time \( L \) does not exceed an inventory cycle time \( T \) so that there is never more than a single order outstanding in any cycle.

3. The target level \( R = \text{Expected demand during the protection interval} + \text{ safety stock (SS) and SS} = A \times (\text{standard deviation of protection interval demand}) \), i.e.,

   \[
   R = D(T+L) + A \sigma \sqrt{T+L}
   \]

   where \( A \) is the safety factor and satisfies

   \[
   P[x > R] = q,
   \]

   \( q \) represents the allowable stock out probability (i.e., defined service level) during the protection interval and is given.

4. The lead-time \( L \) consists of \( n \) mutually independent components. The \( i \)th component has a minimum duration \( a_i \), and normal duration \( b_i \), and a crashing cost per unit time \( c_i \). Arranging \( c_i \) such that \( c_1 \leq c_2 \leq c_3 \ldots \leq c_n \) for the convenience. Since it is clear that the reduction of lead-time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.
Let $L_0 = \sum_{j=1}^{n} b_j$ and $L_i$ be the length of lead time with components 1, 2, …, $i$ crashed to their minimum duration, then $L_i$ can be expressed as $L_i = L_0 - \sum_{j=1}^{i} (b_j - a_j)$, $i = 1, 2, \ldots n$ and the lead time crashing cost per cycle $C(L)$ is given as

$$C(L) = c_i (L_{i-1} - L) + \sum_{j=1}^{i-1} c_j (b_j - a_j)$$

(Refer Chuang et al., 2004).

The setup cost $K$ consists of $m$ mutually independent components. The $j$th component has a normal cost $e_i$ and minimum cost $d_i$ and a crashing cost $f_i$ when the normal cost reduces to minimum cost. So, there is a discontinuous relationship between the crashing cost and setup cost reduction. We have arranged $f_i$ such that $f_1 \leq f_2 \leq f_3 \ldots \leq f_m$ for the sake of convenience. Since it is clear that the reduction of setup cost should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.

Let $K_0 = \sum_{i=1}^{m} e_i$ and $K_j$ be the setup cost with components 1, 2, …, $j$ crashed to their minimum cost, then $K_j$ can be expressed as $K_j = K_0 - \sum_{i=1}^{j} (e_i - d_i)$, $i = 1, 2, \ldots m$

and setup crashing cost per cycle $C(K_j)$ is given as $C(K_j) = \sum_{i=1}^{j} f_i$ (refer Cheng et al., 2004).

Assuming that a fraction $\beta (0 \leq \beta < 1)$ of the demand during the stock out period can be backordered so the remaining fraction $1 - \beta$ is lost. The backorder ratio $\beta$ is variable and is in proportion to the price discount $\pi$, offered by the supplier per unit. Thus $\beta = \frac{\beta_0 \pi}{\pi_0}$ where $0 \leq \beta_0 < 1$ and $0 \leq \pi \leq \pi_0$ (Pan and Hsiao, 2001).

### 3 Basic model

We have assumed that the protection interval demand $X$ has a p.d.f. $f_x$ with finite mean $D(T + L)$ and standard deviation $\sigma \sqrt{(T+L)}$ with the target level $R = D(T+L) + A\sigma \sqrt{T+L}$ where $A$ is already defined. As Montgomery et al. (1973) proposed the periodic review model, the expected net inventory at the beginning of the period is $R - DL + (1 - \beta)E(X - R)^+$, and the expected net inventory at the end of the
period is $R - DL - DT + (1 - \beta)E(X - R)^+$. So, the expected holding cost per year is approximately $h \left[ R - DL - DT + (1 - \beta)E(X - R)^+ \right]$ and the expected stock out cost per year is $rac{\pi_x \beta E(X - R)^+ + (S + \pi_0)(1 - \beta)E(X - R)^+}{T}$, where $E(X - R)^+$ is the expected demand shortage at the end of cycle i.e., $E(X - R)^+ = \int_{R}^{\infty} (x - R)f_x \, dx$ (refer Chuang et al., 2004).

When the lead-time $L$ is reduced to $L_i$ then the annual lead-time crashing cost is $\sum_{i=1}^{j} c_j$.

Similarly, the setup cost $K$ reduced to $K_j$, then the annual setup crashing cost is $\sum_{i=1}^{j} f_i$.

Now, the objective is to minimise the total expected annual cost $(EAC)$ which is the sum of the ordering cost, setup crashing cost, stock out cost, holding cost and lead time crashing cost. Symbolically, our problem is to minimise $EAC(T, \pi_x, L)$ where

$$EAC(T, \pi_x, L) = \frac{K}{T} + \sum_{i=1}^{j} f_i$$

$$+ \frac{\pi_x \beta E(X - R)^+ + (S + \pi_0)(1 - \beta)E(X - R)^+}{T}$$

$$+ h \left[ R - DL - DT + (1 - \beta)E(X - R)^+ \right] + \sum_{i=1}^{j} c_j$$

Also, we have assumed that the backorder ratio $\beta$ depends on the backorder price discount $\pi$, i.e.

$$\beta = \frac{\beta_{0, \pi}}{\pi_0}$$

and $R = D(T + L) + A \sigma \sqrt{T} + L$

where $A$ is safety factor. The equation (1) can be written as
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\[ EAC(T, \pi_x, L) = \frac{K + \sum_{j=1}^{J} c_j + \sum_{i=1}^{I} f_i}{T} \]

\[ E(X - R)^{+} \left[ \frac{\beta_0 \bar{\pi}_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] + \frac{\sigma \sqrt{T + L} \psi(A)}{T} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] \]

\[ + h \left[ \frac{DT}{2} + \Lambda \sigma \sqrt{T + L} + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \sigma \sqrt{T + L} \psi(A) \right] \] (2)

In the above model, the crashing of setup cost is independent of the lead-time, which means the reduction of setup costs can be carried out separately. Consider a case, where a firm conducts a special training course to its employees for implementing the better quality of the product instead of making new appointments, as quality implementation is one of the important concepts of JIT. However, the lead-time and ordering cost reductions may be related closely in some cases.

Here, two cases arise for distribution of lead-time demand i.e.,

a normal distribution
b unknown distribution

3.1 Lead time demand with normal distribution

In this section, we have assumed that the probability distribution of protection interval demand \( X \) has a normal distribution with mean \( D(T + L) \) and standard deviation \( \sigma \sqrt{T + L} \). So, the expected shortages occurring at the end of the cycle is given by

\[ E(X - R)^{+} = \int_{R}^{\infty} (x - R)f_x \, dx = \sigma \sqrt{T + L} \psi(A) > 0 \]

where \( \psi(A) = \phi(A) - A[1 - \Phi(A)] \), \( \phi \) and \( \Phi \) are the standard normal p.d.f. and d.f., respectively.

Therefore, equation (2) is reduced to

\[ EAC(T, \pi_x, L) = \frac{K + \sum_{j=1}^{J} c_j + \sum_{i=1}^{I} f_i}{T} \]

\[ \frac{\sigma \sqrt{T + L} \psi(A)}{T} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] \]

\[ + h \left[ \frac{DT}{2} + \Lambda \sigma \sqrt{T + L} + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \sigma \sqrt{T + L} \psi(A) \right] \] (3)
It can be checked that for fixed $T$ and $\pi$, $EAC(T, \pi, L)$ is a concave function of $L \in [L_{1}, L_{1-1}]$, because $\frac{\partial^2 EAC(T, \pi, L)}{\partial L^2} < 0$. So, for fixed $(T, \pi, L)$, the minimum total expected annual cost will occur at the end points of the interval $[L_{1}, L_{1-1}]$. On the other hand, for a given value of $L \in (L_{1-1}, L_{1})$, it can be shown that $EAC(T, \pi, L)$ is convex in $(T, \pi)$.

Thus for fixed $L \in (L_{1-1}, L_{1})$, the minimum value of $EAC(T, \pi, L)$ will occur at the point $(T, \pi)$ that satisfy $\frac{\partial EAC(T, \pi, L)}{\partial T} = 0$ and $\frac{\partial EAC(T, \pi, L)}{\partial \pi} = 0$. Now,

$$
\frac{\partial EAC(T, \pi, L)}{\partial T} = 0 \Rightarrow -\frac{\left\{K + \sum_{j=1}^{J} c_j + \sum_{i=1}^{I} f_i \right\}}{T^2} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] \times \sigma \sqrt{T + L} \psi(A) \times \sqrt{T} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] \times \sigma \frac{1}{2 \sqrt{T + L}} \psi(A) + \frac{\sigma}{2 \sqrt{T + L}} \psi(A) + h \left[ \frac{D}{2} + \frac{\Lambda \sigma}{2 \sqrt{T + L}} + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \frac{\sigma}{2 \sqrt{T + L}} \psi(A) \right]
$$

This can be written as

$$
\frac{\left\{K + \sum_{j=1}^{J} c_j + \sum_{i=1}^{I} f_i \right\}}{T^2} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] \times \sigma \sqrt{T + L} \psi(A) \times \sqrt{T} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right] \times \sigma \frac{1}{2 \sqrt{T + L}} \psi(A) + \frac{\sigma}{2 \sqrt{T + L}} \psi(A) + h \left[ \frac{D}{2} + \frac{\Lambda \sigma}{2 \sqrt{T + L}} + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \frac{\sigma}{2 \sqrt{T + L}} \psi(A) \right]
$$
where as

\[
\frac{\partial EAC(T, \pi, L)}{\partial \pi} = 0 \Rightarrow \pi = \frac{hT + (S + \pi_0)}{2} \tag{6}
\]

Since it is difficult to obtain the solutions for \(T\) and \(\pi\) explicitly as the evaluation of equation (5) and equation (6) need the value of each other. As a result, we must establish the following iterative algorithm to find the optimal \((T, \pi)\).

**Algorithm 1**

**Step 1**
For each \(L_i, i = 0, 1, 2 \ldots n\), execute (a)-(d).

(a) For each \(K_j, j = 0, 1, 2 \ldots m\), execute (b)-(d).

(b) Start with fixed service level = 0.8 that is, \(A_{ij} = 0.845\) and \(\psi(A_{ij})\) by checking the table from Silver and Peterson (1985, pp.699–708).

(c) By putting the value of \(\psi(A_{ij})\) into equation (5), using numerical search technique, evaluate \(T_{ij}\). If \(T_{ij} \geq L_i\) then go to (d) otherwise let \(T_{ij} = L_i\), go to (d).

(d) By using \(T_{ij}\), calculate the value of \(x_{ij}\) using the equation (6). Compare \(x_{ij}\) and \(\pi_0\). If \(x_{ij} \leq \pi_0\), then \(x_{ij}\) is feasible. Go to step (2)

Otherwise set \(x_{ij} = \pi_0\), go to step (2)

**Step 2**
For each \(\left(T_{ij}, x_{ij}, L_i\right)\), compute the corresponding expected annual cost

\[
EAC(T_{ij}, x_{ij}, L_i), j = 0, 1, 2 \ldots m
\]

from equation (3) and find minimum

\[
EAC(T_{ij}, x_{ij}, L_i)
\]

for \(j = 0, 1, 2 \ldots m\). Go to step 3.

**Step 3**
Find \(\min_{i=0,1,2,\ldots n} EAC(T_{ij}, x_{ij}, L_i)\). If \(EAC(T^*, x^*, L^*) = \min_{i=0,1,2,\ldots n} EAC(T_{ij}, x_{ij}, L_i)\). Hence \(\left(T^*, x^*, L^*\right)\) is the optimal solution. And hence, the optimal target level is

\[
R^* = D(T^* + L^*) + \pi_0 \sqrt{(T^* + L^*)}.
\]

Theoretically, for given \(K, D, h, \beta_0, \pi_0\) and each \(L_i(i = 0, 1, 2 \ldots n)\), from equations (5) and (6), we can obtain optimal values of \(T\) and \(\pi\), then the corresponding total expected annual cost can be found. Thus, the minimum total expected annual cost could be obtained when the lead-time demand is normally distributed.

### 3.2 Lead time demand with unknown distribution

If the lead-time demand does not follow normal distribution or the probability distribution is unknown with first two moments, then the solution can be obtained by minimax approach (see Ouyang et al., 1996). Since the probability distribution of \(X\) is unknown, we cannot find the exact value of \(E(X - R)^*\).
3.2.1 Solution by minimax approach

In this section, we relax the constraint over the form of the probability distribution of lead-time demand. Here, we assume that the lead-time demand has an unknown distribution with known finite mean $D(T + L)$ and standard deviation $\sigma \sqrt{T + L}$ where the target level $R = D(T + L) + A \sigma \sqrt{T + L}$. Now, we try to use a minimax distribution free procedure to solve this problem. So our problem is to solve:

$$\min_{T>0, \pi_x>0, T>0} \max_{F \in \Omega} EAC(T, \pi_x, L)$$

We use the following proposition (Gallego and Moon, 1993) to shorten the problem.

Proposition 1: For any $F \in \Omega$,

$$E(X - \tau)^+ \leq \frac{1}{2} \left\{ \sigma^2 (T + L) + (R - D(T + L))^2 - (R - D(T + L)) \right\}$$

(7)

Moreover, the upper bound (7) is tight. Then the equation (2) can be reduced to

$$EAC^w(T, \pi_x, L) = \frac{K + \sum_{j=1}^{l} c_j + \sum_{i=1}^{l} f_i}{T} + \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - S \beta_0 \pi_x - \beta_0 \pi_x \right] \sigma \sqrt{T + L} \left( \sqrt{1 + A^2} - A \right)$$

$$+ h \left[ \frac{DT}{2} + A \sigma \sqrt{T + L} + \frac{1}{2} \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \sigma \sqrt{T + L} \left( \sqrt{1 + A^2} - A \right) \right]$$

(8)

where $EAC^w(T, \pi_x, L)$ is the least upper bound of $EAC(T, \pi_x, L)$.

As notified in the preceding section, it can be shown that $EAC^w(T, \pi_x, L)$ is a concave function of $L \in [L_i, L_{i-1}]$ for fixed $T$ and $\pi_x$ (Appendix 1). Therefore, the minimum upper bound of the expected total annual cost will occur at the end point of the interval $L \in [L_i, L_{i-1}]$ for fixed value of $(T, \pi_x)$. Moreover, it can be shown that $EAC^w(T, \pi_x, L)$ is convex function of $T$ and $\pi_x$ for fixed $L$ (Appendix 2). Therefore, the first order conditions are necessary and sufficient conditions for optimality. Using the first condition of derivatives, we get
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\[
\left\{ \frac{K + \sum_{j=1}^{l} c_j + \sum_{i=1}^{l} f_i}{T^2} \right\} \\
+ \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S \frac{\beta_0 \pi_o - \beta_0 \pi_x}{\pi_0} \right] \frac{1}{2} \sqrt{(T + L) \left( \sqrt{1 + A^2} - A \right)} \\
+ \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S \frac{\beta_0 \pi_x - \beta_0 \pi_o}{\pi_0} \right] \frac{1}{4} \sigma \sqrt{(T + L) \left( \sqrt{1 + A^2} - A \right)} \\
+ h \left[ \frac{D}{2} + \frac{A \sigma}{2} \frac{1}{\sqrt{(T + L)}} + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \frac{1}{4} \frac{\sigma}{\sqrt{(T + L)}} \left( \sqrt{1 + A^2} - A \right) \right]
\]

and

\[
\pi_x = \frac{hT + (S + \pi_0)}{2}.
\]

Since, it is difficult to obtain the exact value of service factor \(A\) which depends upon the required service level on the basis of allowable stock out probability \(q\), because the p.d.f. \(f_x(x)\) is unknown. So, the following proposition has been used to find accurate value of \(A\).

Therefore, the algorithm to find the optimal review period, lead-time and backorder discount can be established by using the proposition given below:

Proposition: (Ouyang and Wu, 1997)

Let \(X\) represent the protection interval demand that has p.d.f. \(f_x(x)\) with finite mean \(D(T + L)\) and standard deviation \(\sigma \sqrt{T + L}\) then for any real number \(c > 0\), \(P[X > c] \leq \frac{\sigma^2 L}{\sigma^2 L + (c - RL)^2}\).

If we take \(R\) instead of \(c\), then \(P[X > R] \leq \frac{\sigma^2 (T + L)}{\sigma^2 (T + L) + (R - DL)^2}\) or

\(P[X > R] \leq \frac{1}{1 + A^2}\).

Furthermore, since it is assumed that the allowable stock out probability \(q\) during the lead time is known so \(q = P[X > R] \leq \frac{1}{1 + A^2}\). Hence \(q \leq \frac{1}{1 + A^2}\).

Therefore \(A \in \left[ 0, \frac{1}{\sqrt{q}} - 1 \right]\).
Algorithm 2

Step 1
For Each q, divide the interval \( \left[ 0, \frac{1}{\sqrt{q}} \right] \) into \( N \) equal subintervals. Let

\[
A_0 = 0, \quad A_N = \frac{1}{\sqrt{q}}
\]

\[
A_l = A_{l-1} + \frac{A_N - A_0}{N}, \quad l = 1, 2, \ldots, N - 1
\]

Step 2
For each \( L_i \) \((i = 0, 1, 2, \ldots, n)\) and \( K_j \) \((j = 0, 1, 2, \ldots, m)\) perform steps (3) and (4).

Step 3
For given \( A_l \in \{A_0, A_1, \ldots, A_N\}, l = 0, 1, 2, \ldots, N \), using numerical search technique, evaluate \( T_y \) from equation (9) simultaneously. If \( T_y \geq L_i \) then go to Step (4) otherwise set \( T_y = L_i \) and go to Step (4).

Step 4
By using \( T \), calculate the value of \( \pi_j \) using the equation (10). Compare \( \pi_j \) and \( \pi_0 \). If \( \pi_j \leq \pi_0 \) then \( \pi_j \) is feasible. Go to next step. Otherwise set \( \pi_j = \pi_0 \), go to Step (5)

Step 5
For each \( (T_y, \pi_y, L_i) \), compute the corresponding expected annual cost

\[
EAC^w(T_y, \pi_y, L_i), \quad j = 0, 1, 2, \ldots, m
\]

Step 6
Find \( \min_{A_l} EAC^w(T_y, \pi_y, L_i) \).

If \( EAC^w(T_{i,A}, \pi_{i,A}, L_{i,A}) = \min_{A_l} EAC^w(T_y, \pi_y, L_i) \).

Step 7
Find \( \min_{i=0,1,2,…,n} EAC^w(T_{i,A}, \pi_{i,A}, L_{i,A}) \).

Then, \( (T', \pi', L') \) is the required optimal solution.

4 Numerical example

In order to illustrate the solution algorithms, we have considered an inventory system having data used by Chuang et al. (2004) and Cheng et al. (2004): \( D = 600 \) units per year, \( K = \$200 \) per order, \( S = \$50 \) per short out, \( \sigma = 7 \) units per week, \( \pi_0 = \$150 \) per unit, \( h = \$20 \) per unit per year, \( q = 0.2 \) where \( A_0 = 0 \) and \( A_N = 2, \ N = 200 \).

The lead-time and setup cost has three components, which have been shown in Table 1 and Table 2.

**Table 1** Lead time data

<table>
<thead>
<tr>
<th>Lead time component ( i )</th>
<th>Normal duration (days) ( b_i )</th>
<th>Minimum duration (days) ( a_i )</th>
<th>Unit Crashing cost, ( c_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>6</td>
<td>0.4</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>6</td>
<td>1.2</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>9</td>
<td>5.0</td>
</tr>
</tbody>
</table>
Table 2  Setup cost data

<table>
<thead>
<tr>
<th>Setup cost component j</th>
<th>Normal cost, e_i</th>
<th>Minimum cost, d_i</th>
<th>Total crashing cost as reduced to minimum cost, f_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>20</td>
<td>56</td>
</tr>
<tr>
<td>2</td>
<td>80</td>
<td>20</td>
<td>168</td>
</tr>
<tr>
<td>3</td>
<td>40</td>
<td>10</td>
<td>350</td>
</tr>
</tbody>
</table>

We have solved the cases for different upper bounds of the backorder ratio $\beta = 0, 0.5$ and 1 which represents the cases of no backorder, partially backorder and no lost sale. At first, Table 3 provides the solution without the crashing of lead-time and setup cost with normal distribution.

Table 3  Optimal solution of numerical example ($T^*, L^*$ in weeks) without crashing of lead-time and setup cost ($K = 200$) for normal distribution

<table>
<thead>
<tr>
<th>$\beta_0$</th>
<th>$L^*$</th>
<th>$T^*$</th>
<th>$R^*$</th>
<th>$\pi^*$</th>
<th>EAC($T, \pi, L$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8</td>
<td>17.00</td>
<td>305.50</td>
<td>103.27</td>
<td>4343.94</td>
</tr>
<tr>
<td>0.5</td>
<td>8</td>
<td>15.99</td>
<td>293.90</td>
<td>102.85</td>
<td>4091.05</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>14.92</td>
<td>281.51</td>
<td>102.87</td>
<td>3822.54</td>
</tr>
</tbody>
</table>

Then, by applying the algorithm, crashing has been carried out for lead-time and setup costs for different backorder ratio and illustrated in Table 4(a) and Table 4(b). It is observed that by reducing the lead time and setup cost, the total expected cost decreases.

Table 4(a)  Crashing of lead-time and setup costs for different backorder ratio.

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>$L_i$</th>
<th>$C(L_i)$</th>
<th>$C(K_i)$</th>
<th>$\beta_0 = 0$</th>
<th>$\beta_0 = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$T$ $EAC$ $R$ $\pi^*$</td>
<td>$T$ $EAC$ $R$ $\pi^*$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0</td>
<td>17.00 3433.94 305 103.27</td>
<td>15.99 4091.05 293 103.08</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>0</td>
<td>56</td>
<td>16.94</td>
<td>4331.69 304 103.26</td>
<td>15.94 4078.02 293 103.06</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>0</td>
<td>224</td>
<td>18.32</td>
<td>4650.19 320 103.52</td>
<td>17.39 4415.01 310 103.34</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>0</td>
<td>574</td>
<td>21.89</td>
<td>5477.78 362 104.21</td>
<td>21.12 5279.04 353 104.06</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>5.6</td>
<td>0</td>
<td>16.27</td>
<td>4130.19 271 103.13</td>
<td>15.36 3901.12 261 102.95</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>5.6</td>
<td>56</td>
<td>16.21</td>
<td>4117.39 271 103.12</td>
<td>15.30 3887.50 260 102.94</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>5.6</td>
<td>224</td>
<td>17.64</td>
<td>4449.15 287 103.39</td>
<td>16.81 4237.30 278 103.23</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>5.6</td>
<td>574</td>
<td>21.33</td>
<td>5303.08 330 104.10</td>
<td>20.65 5125.74 322 103.97</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>22.4</td>
<td>0</td>
<td>15.50</td>
<td>3902.92 237 102.98</td>
<td>14.73 3705.72 228 102.83</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>22.4</td>
<td>56</td>
<td>15.45</td>
<td>3889.47 236 102.97</td>
<td>14.67 3691.57 227 102.82</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>22.4</td>
<td>224</td>
<td>16.94</td>
<td>4236.29 253 103.26</td>
<td>16.24 4055.00 245 103.12</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>22.4</td>
<td>574</td>
<td>20.75</td>
<td>5119.29 297 103.99</td>
<td>20.18 4968.92 291 103.88</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>57.4</td>
<td>0</td>
<td>15.40</td>
<td>3850.55 223 102.96</td>
<td>14.73 3677.29 215 102.83</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
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<td>56</td>
<td>15.34</td>
<td>3837.02 222 102.95</td>
<td>14.66 3663.13 214 102.82</td>
</tr>
<tr>
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<td>3</td>
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<td>224</td>
<td>16.84</td>
<td>4186.03 239 103.24</td>
<td>16.23 4026.68 232 103.12</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>57.4</td>
<td>574</td>
<td>20.67</td>
<td>5073.26 283 103.97</td>
<td>20.17 4940.86 278 103.88</td>
</tr>
</tbody>
</table>
Table 4(b)  Crashing of lead-time and setup costs for different backorder ratio

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>L_i</th>
<th>C(L_i)</th>
<th>C(K_j)</th>
<th>T</th>
<th>EAC</th>
<th>R</th>
<th>π_0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>0</td>
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<td>3822.54</td>
<td>281</td>
<td>102.87</td>
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<td>8</td>
<td>0</td>
<td>56</td>
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<td>3808.57</td>
<td>280</td>
<td>102.86</td>
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<tr>
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<td>0</td>
<td>8</td>
<td>0</td>
<td>224</td>
<td>16.41</td>
<td>4167.71</td>
<td>298</td>
<td>103.16</td>
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<tr>
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<td>8</td>
<td>0</td>
<td>574</td>
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<td>5073.57</td>
<td>343</td>
<td>103.91</td>
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<tr>
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<td>1</td>
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<td>5.6</td>
<td>0</td>
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<td>3658.84</td>
<td>250</td>
<td>102.77</td>
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<td>5.6</td>
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<td>249</td>
<td>102.76</td>
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<td>224</td>
<td>15.94</td>
<td>4015.40</td>
<td>267</td>
<td>103.06</td>
</tr>
<tr>
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<td>3</td>
<td>6</td>
<td>5.6</td>
<td>574</td>
<td>19.94</td>
<td>4942.99</td>
<td>314</td>
<td>103.83</td>
</tr>
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<td>0</td>
<td>13.91</td>
<td>3498.56</td>
<td>218</td>
<td>102.68</td>
</tr>
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<td>22.4</td>
<td>56</td>
<td>13.85</td>
<td>3483.58</td>
<td>218</td>
<td>102.66</td>
</tr>
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<td>22.4</td>
<td>224</td>
<td>15.50</td>
<td>3866.30</td>
<td>237</td>
<td>102.98</td>
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<td>103.77</td>
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<td>57.4</td>
<td>0</td>
<td>14.02</td>
<td>3496.56</td>
<td>207</td>
<td>102.70</td>
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<td>1</td>
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<td>56</td>
<td>13.96</td>
<td>3481.69</td>
<td>206</td>
<td>102.68</td>
</tr>
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<td>57.4</td>
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<td>15.60</td>
<td>3861.76</td>
<td>225</td>
<td>103.00</td>
<td></td>
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<td>3</td>
<td>57.4</td>
<td>574</td>
<td>19.67</td>
<td>4805.58</td>
<td>272</td>
<td>103.78</td>
<td></td>
</tr>
</tbody>
</table>

The optimal inventory results with relevant savings where lead-time and setup cost have been crashed given in Table 5. From Table 5, we also observe that the total annual expected cost decreases as the backorder ratio increases since supplier can fetch a large number of backorders by offering the price discount with no loss although with less cost.

Table 5  Optimal solution of numerical example (T^*, L^* in weeks) with crashing of lead-time and setup cost K for normal distribution

<table>
<thead>
<tr>
<th>β_0</th>
<th>L^*</th>
<th>T^*</th>
<th>R^*</th>
<th>K^*</th>
<th>π_s^*</th>
<th>EAC(T^<em>, π_s^</em>, L^*)</th>
<th>Saving (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>15.34</td>
<td>222.34</td>
<td>140</td>
<td>102.95</td>
<td>3837.02</td>
<td>11.67</td>
</tr>
<tr>
<td>0.5</td>
<td>3</td>
<td>14.66</td>
<td>214.53</td>
<td>140</td>
<td>102.82</td>
<td>3663.13</td>
<td>10.46</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>13.96</td>
<td>206.35</td>
<td>140</td>
<td>102.68</td>
<td>3481.69</td>
<td>8.92</td>
</tr>
</tbody>
</table>

Note: saving % = \left(\frac{EAC(T, \pi_s, L) - EAC(T^*, K^*, L^*)}{EAC(T, \pi_s, L)}\right) \times 100%

Furthermore, Table 6 listed the optimal results for controllable lead-time and setup cost with unknown distribution.

Table 6  Optimal solution of numerical example [T^*, π_s^*, L^* in weeks] with crashing of lead-time and setup cost for unknown distribution

<table>
<thead>
<tr>
<th>β_0</th>
<th>L^*</th>
<th>T^*</th>
<th>R^*</th>
<th>K^*</th>
<th>π_s^*</th>
<th>EAC(T^<em>, π_s^</em>, L^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>15.49</td>
<td>238.82</td>
<td>140</td>
<td>102.98</td>
<td>4183.87</td>
</tr>
<tr>
<td>0.5</td>
<td>3</td>
<td>14.79</td>
<td>230.70</td>
<td>140</td>
<td>102.85</td>
<td>4002.17</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>14.06</td>
<td>222.17</td>
<td>140</td>
<td>102.70</td>
<td>3812.36</td>
</tr>
</tbody>
</table>
Table 7 provides the comparison of unknown distribution model with that of normal distribution model to obtain expected value of additional information (EVAI), which is the largest amount that the supplier would be willing to pay for knowing the distribution of protection interval demand. It is the difference of total expected annual cost that is obtained by substituting the optimal solution of unknown distribution and normal distribution cases in equation (3), i.e., \( \text{EVAI} = EAC^N(T, \pi, L) - EAC^N(T^N, \pi^N, L^N) \), which could be referred to as the extra cost for utilising the unknown distribution instead of normal.

<table>
<thead>
<tr>
<th>( \beta_0 )</th>
<th>( EAC^N(T, \pi, L) )</th>
<th>( EAC^N(T^N, \pi^N, L^N) )</th>
<th>EVAI</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3889.49</td>
<td>3837.02</td>
<td>52.47</td>
</tr>
<tr>
<td>0.5</td>
<td>3691.68</td>
<td>3663.13</td>
<td>28.55</td>
</tr>
<tr>
<td>1</td>
<td>3483.95</td>
<td>3481.69</td>
<td>2.26</td>
</tr>
</tbody>
</table>

It is interesting to observe that an increment in backorder ratio based on price discount offered by the supplier reduces the total expected annual cost by 9.26% in normal distribution case and 8.63% in unknown distribution. Moreover, the extra cost for utilising the unknown distribution instead of normal, that is EVAI has been given above for different value of backorder ratio.

5 Conclusions

The main purpose of this paper is to investigate the effect of controlling the lead-time and setup cost in periodic inventory model with backorder price discount as a decision variable, and to solve the cases of known (normal) as well as of unknown distribution of protection interval demand. When unsatisfied demand occurs, a supplier offers price discount to the customers for the stock out items to secure more backorders. That is why we have considered the dependency of backorder ratio on the amount of price discount. The crashing of lead-time and setup cost is a very natural phenomenon in realistic inventory models, which have been carried out independently, under this study. Usually, in crashing one tries to reduce the time, to achieve this one requires additional resources, which costs money. Therefore, one looks for a trade off between the time and the cost. Moreover, safety stock protection is required for lead time plus the order interval i.e., \((T + L)\), it is higher for the larger lead time and lower for the smaller lead time. With this very fact, crashing of lead time has been carried out in this paper. It has been shown from the results that the review period \((T)\) and the maximum inventory level \((R)\) decrease with the reduction of lead time. Further, when the reduction of lead-time accompanies a decrease of setup cost with backorder discount to customers then the total expected cost decreases significantly.
Acknowledgements

The authors are grateful to the anonymous referees for their useful suggestions and comments. The first author would like to acknowledge the support of Research Grant No. Dean (R)/R&D/2009/487, provided by University of Delhi (India) for conducting this research. The second author would like to thank University Grant Commission (UGC) for providing the fellowship to accomplish the research.

References


Periodic inventory model with unstable lead-time and setup cost


Appendix 1

Proof of $EAC^w(T, \pi_x, L)$ is concave in $L \in (L_i, L_{i-1})$ for fixed $T, \pi_x$.

\[
\frac{\partial EAC^w(T, \pi_x, L)}{\partial L} = \frac{\left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x - \beta_0 \pi_x}{\pi_0} \right] \times \frac{\sigma}{4} (T + L)^{-1} \left( \sqrt{1 + A^2} - A \right)}{(T + L)^{3/2}} + \frac{h \lambda \sigma}{2} \left( T + L \right)^{-1} \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \frac{\sigma}{4} (T + L)^{-1} \left( \sqrt{1 + A^2} - A \right)
\]

\[
\frac{\partial^2 EAC^w(T, \pi_x, L)}{\partial L^2} = \frac{-1}{8} \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_x - \beta_0 \pi_x}{\pi_0} \right] \times \frac{\sigma}{4} (T + L)^{-3/2} \left( \sqrt{1 + A^2} - A \right) - \frac{h \lambda \sigma}{4} \left( T + L \right)^{-3/2} \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \frac{\sigma}{8} (T + L)^{-3/2} \left( \sqrt{1 + A^2} - A \right) < 0
\]

Therefore EAC is concave in $L \in (L_i, L_{i-1})$ for fixed $(T, \pi_x)$. 
Appendix 2

Proof of $EAC^w(T, \pi_s, L)$ is a convex function of $(T, \pi_s)$.

For a given value of $L \in (L_i, L_{i-1})$, we first obtain the Hessian Matrix $H$ as follows:

$$H = \begin{bmatrix} \frac{\partial^2 EAC^w(T, \pi_s, L)}{\partial T^2} & \frac{\partial^2 EAC^w(T, \pi_s, L)}{\partial T \partial \pi_s} \\ \frac{\partial^2 EAC^w(T, \pi_s, L)}{\partial \pi_s \partial T} & \frac{\partial^2 EAC^w(T, \pi_s, L)}{\partial \pi_s^2} \end{bmatrix}$$ (A2)

Then, evaluating principal minor of $H$, the first principal minor of $H$ that is $|H_{11}| > 0$, where

$$|H_{11}| = \frac{\partial^2 EAC^w(T, \pi_s, L)}{\partial T^2} \Rightarrow - \left( \frac{K + \sum_{j=1}^{i} c_j + \sum_{i=1}^{f} f_i}{T^2} \right)$$

$$- \left[ \frac{\beta_0 \pi_s^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_s}{\pi_0} - \beta_0 \pi_s \right] \times \frac{\sigma}{T \sqrt{T + L}} \left( \sqrt{1 + A^2} - A \right)$$

$$+ \left[ \frac{\beta_0 \pi_s^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_s}{\pi_0} - \beta_0 \pi_s \right] \times \frac{\sigma}{4 \sqrt{T(L + L)}} \left( \sqrt{1 + A^2} - A \right)$$

This can be written as

$$\left( \frac{K + \sum_{j=1}^{i} c_j + \sum_{i=1}^{f} f_i}{2T^2(T + L)} \right) =$$

$$\frac{hD}{4(T + L)} \left[ A + \left( 1 - \frac{\beta_0 \pi_s}{\pi_0} \right) \left( \sqrt{1 + A^2} - A \right) \right] \left( T + L \right)^{-3}$$

$$\left[ \frac{\beta_0 \pi_s^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_s}{\pi_0} - \beta_0 \pi_s \right] \sigma \left( T + L \right)^{-3} \left( \sqrt{1 + A^2} - A \right)$$

$$+ \left[ \frac{\beta_0 \pi_s^2}{\pi_0} + S + \pi_0 - \frac{S \beta_0 \pi_s}{\pi_0} - \beta_0 \pi_s \right] \sigma \left( T + L \right)^{-3} \left( \sqrt{1 + A^2} - A \right)$$

$$- \frac{4T}{2T^2}$$
Periodic inventory model with unstable lead-time and setup cost

or

\[
\frac{K + \sum_{j=1}^{i} C_j + \sum_{i=1}^{f_i}}{2T^2 (T + L)} = \frac{hD}{4(T + L)} + \frac{h\sigma}{4} \left[ A + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \left( \sqrt{1 + A^2} - A \right) \right] (T + L)^{-3} \\
+ \frac{N(\pi_x) \sigma (T + L)^{-3}}{4T} \left( \sqrt{1 + A^2} - A \right) - \frac{N(\pi_x) \sigma (T + L)^{-1}}{2T^2} \left( \sqrt{1 + A^2} - A \right)
\]

where

\[
N(\pi_x) = \left[ \frac{\beta_0 \pi_x^2}{\pi_0} + S + \pi_0 - S \frac{\beta_0 \pi_x}{\pi_0} - \beta_0 \pi_x \right], \tag{A4}
\]

We have

\[
\frac{\partial^2 EAC^w (T, \pi_x, L)}{\partial T^2} = \psi(T) \\
- \frac{h\sigma}{4} \left[ A + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \left( \sqrt{1 + A^2} - A \right) \right] (T + L)^{-3} \tag{A5}
\]

where

\[
\psi(T) = \frac{2 \left( K + \sum_{j=1}^{i} C_j + \sum_{i=1}^{f_i} \right)}{T^3} + \frac{2N(\pi_x) \sigma (T + L)^{-3}}{T^3} \left( \sqrt{1 + A^2} - A \right) \\
- \frac{N(\pi_x) \sigma (T + L)^{-1}}{T^2} \left( \sqrt{1 + A^2} - A \right) - \frac{N(\pi_x) \sigma (T + L)^{-3}}{4T} \left( \sqrt{1 + A^2} - A \right)
\]

Now,

\[
| H_{11} | = \frac{\partial^2 EAC^w (T, \pi_x, L)}{\partial T^2} = \psi(T) \\
- \frac{h\sigma}{4} \left[ A + \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) \left( \sqrt{1 + A^2} - A \right) \right] (T + L)^{-3}
\]

Let
\[
\delta(T) = \frac{K + \sum_{j=1}^{l} c_j + \sum_{i=1}^{l} f_i}{2T^2(T+L)} - \frac{N(\pi_x)\sigma(T+L)^\frac{3}{2}}{4T} \left(\sqrt{1 + A^2} - A\right)
\]
\[
+ \frac{N(\pi_x)\sigma(T+L)^{-\frac{3}{2}}}{2T^2} \left(\sqrt{1 + A^2} - A\right)
\]

Then \(\delta(T) > \frac{h\sigma}{4} \left[A + \left(1 - \frac{\beta_0\pi_x}{\pi_0}\right)\left(\sqrt{1 + A^2} - A\right)\right] (T + L)^{-\frac{3}{2}}\)

So, from we have

\[|H_{11}| > \psi(T) - \delta(T) > 0\]

Since

\[
\psi(T) - \delta(T) = \frac{\left(K + \sum_{j=1}^{l} c_j + \sum_{i=1}^{l} f_i\right)(3T + 4L)}{2T^3(T+L)} + \frac{N(\pi_x)(T + 4L)\sigma(T+L)^\frac{1}{2}}{2T^3(T + L)} > 0
\]

The second principal minor of \(H\) is

\[
|H_{22}| = \begin{bmatrix}
\frac{\partial^2 EAC''(T, \pi_x, L)}{\partial T^2} & \frac{\partial^2 EAC''(T, \pi_x, L)}{\partial T \partial \pi_x} \\
\frac{\partial^2 EAC''(T, \pi_x, L)}{\partial \pi_x \partial T} & \frac{\partial^2 EAC''(T, \pi_x, L)}{\partial \pi_x^2}
\end{bmatrix}
\]

Therefore,

\[
|H_{22}| = \left\{\frac{\partial^2 EAC''(T, \pi_x, L)}{\partial T^2} \times \frac{\partial^2 EAC''(T, \pi_x, L)}{\partial \pi_x^2} - \left(\frac{\partial^2 EAC''(T, \pi_x, L)}{\partial T \partial \pi_x}\right)^2\right\}
\]

where
\[ \frac{\partial^2 EAC^w(T, \pi_x, L)}{\partial \pi_x^2} = \frac{\beta_0 \sigma(T + L)^{\frac{1}{2}}(\sqrt{1 + A^2} - A)}{\pi_0 T} \quad \text{and} \]
\[ \frac{\partial^2 EAC^w(T, \pi_x, L)}{\partial \pi_x \partial T} = \frac{2T}{\pi_0} \left( \frac{2\beta_0 \pi_x}{\pi_0} - \frac{S \beta_0}{\pi_0} - \beta_0 \right) \sigma(T + L)^{\frac{1}{2}}(\sqrt{1 + A^2} - A) \]
\[ - \frac{2}{\pi_0} \frac{S \beta_0}{\pi_0} \left( \frac{2\beta_0 \pi_x}{\pi_0} - \beta_0 \right) \sigma(T + L)^{\frac{1}{2}}(\sqrt{1 + A^2} - A) \]

Also,
\[ |H_{22}| = \left| \frac{\partial^2 EAC^w(T, \pi_x, L)}{\partial \pi_x^2} \right| - \left( \frac{\partial^2 EAC^w(T, \pi_x, L)}{\partial T \partial \pi_x} \right)^2 \]
\[ = \left( \frac{K + \sum_{j=1}^l c_j + \sum_{i=1}^j f_i}{2T^3(T + L)} \right) \left( \frac{3(T + 4L)}{2T^3(T + L)} \right) + \frac{N(\pi_x)(T + 4L)\sigma(T + L)^{\frac{1}{2}}(\sqrt{1 + A^2} - A)}{2T^3(T + L)} \]
\[ \times \left( \frac{\partial^2 EAC^w(T, \pi_x, L)}{\partial \pi_x^2} \right)^2 \]

After simplification, we have
\[ |H_{22}| = \frac{\sigma \beta_0}{\pi_0} \left( \frac{K + \sum_{j=1}^l c_j + \sum_{i=1}^j f_i}{4T^4(T + L)} \right) \left( \frac{(T + L)^{\frac{1}{2}} - \frac{h^2 \sigma \beta_0}{\pi_0} \left( \sqrt{1 + A^2} - A \right) (3T + 4L)\sqrt{1 + A^2} - A}{4T^4(T + L)} \right) \]
\[ + \frac{4N(\pi_x) - \left( \frac{2\beta_0 \pi_x}{\pi_0} - \frac{S \beta_0}{\pi_0} - \beta_0 \right)^2}{4T^4(T + L)} + \frac{4N(\pi_x) - \left( \frac{2\beta_0 \pi_x}{\pi_0} - \frac{S \beta_0}{\pi_0} - \beta_0 \right)^2}{4T^4(T + L)} \]
\[ + \frac{4N(\pi_x) - \left( \frac{2\beta_0 \pi_x}{\pi_0} - \frac{S \beta_0}{\pi_0} - \beta_0 \right)^2}{4T^4(T + L)} > 0 \]

Because \( 0 < B_0 < 1 \Rightarrow \pi_0 (4 - B_0) - B_0 S > 3\pi_0 - S > 0 \) where \( \pi_0 \) is the marginal profit and \( S \) is the shortage cost.

\[ \therefore \ EAC^w \] is a convex function in \((T, \pi_x)\) for a given value of \( L \in [L, L_{i-1}] \).

This completes the proof.