Identification of Redundant Objective Functions in Multi-Objective Stochastic Fractional Programming Problems

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IDENTIFICATION OF REDUNDANT OBJECTIVE FUNCTIONS
IN MULTI-OBJECTIVE STOCHASTIC FRACTIONAL
PROGRAMMING PROBLEMS

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Redundancy in constraints and variables are usually studied in linear, integer and
non-linear programming problems. However, main emphasis has so far been given only
to linear programming problems. In this paper, an algorithm that identifies redundant
objective functions in multi-objective stochastic fractional programming problems is pro-
vided. A solution procedure is also illustrated. This reduces the number of objective
functions in cases where redundant objective functions exist.

Keywords: Stochastic programming; fractional programming; multi-objective program-
m ing; redundancy.

1. Introduction

Suppose that there is some kind of redundancy in a mathematical programming
problem. Obviously, the problem will be larger or contain more details than that of
without redundancy. There may be other effects caused by the inclusion of redun-
dancy. The sheer presence of redundancy in the problem may cause a false impres-
sion of having some influence. Since this need not be the real case, ones perception
of the system may be obscured because of redundancy. Something is redundant if
omitting the same does not affect the system of concern in any manner. Adopting
this definition, redundancy may be described as a phenomenon that permits reduc-
tion of a system to a simpler one having the same properties as the original system.
Though it may appear a little vague, for linear stochastic fractional programming (LSFP), this description is found sufficient.

Redundancy may occur in the formulation phase of a programming problem because of difficulties inherent in the formulation process, especially in large systems. In such circumstances, a problem may be formulated by several independent ways. Coordination and Communication also tend to introduce redundancy in a problem. Ease in formulation of problems is another cause of redundancy. It is rather convenient to use what are often referred to as summation, collection, or definitional equalities in, for example, summing the quantities of raw materials that go into a final product. That is, for example, the quantity of final product produced is the sum of certain variables like, the quantities of necessary raw materials. The equality added is conceptually redundant; it could be eliminated by substitution. Yet many individual would rather use the simpler formulation involving redundant constraints. Zionts and Wallenius (1976) points out that redundancy may arise in interactive multiple criteria programming.

Redundancy may have some favorable effects too both in the problem formulating stage and in the problem solving stage. Charnes and Cooper (1961) added constraints and variables to a problem to transform the problem from a linear programming problem to a transportation problem as it is much more easier to solve than the general linear programming problem. Furthermore, in case of numerical problems in ill-constrained system of linear equalities, redundant equality constraints could be used to overcome such numerical problems. However, the unfavorable effects of redundancy in general mathematical programming problem usually outnumber the favorable ones.

As stated earlier, mathematical programming problems that have been studied in relation to redundancy include linear, integer and non-linear programming, but the main effort has been in linear programming. Gal (1975) presented a note on redundancy and linear parametric programming. Gal and Laberling (1977) proposed an algorithm to identify redundant objective function in linear vector maximization problem. Mark et al. (1983) and Rhymend et al. (1999) proposed algorithms to identify redundant constraints a priori to the solution of linear programming problems. In this paper, an algorithm that identifies redundant objective functions in multi-objective stochastic fractional programming problems and that helps reduction in the number of objective functions in cases of existence of redundant objective functions is developed. LSFP problem is one of the optimization problems that can be solved using a number of different techniques within the constraint satisfaction paradigm. The solution presented in this paper falls along the lines of Charnes and Cooper (1954, 1961, and 1962). Further, an extension to more general form combining stochastic fractional programming with sequential linear programming (SLP) is also presented. Cheney and Goldstein (1952) and Kelly (1960) have originally presented the SLP method. The concept of solving a series of linear programming problem in order to obtain the solution of the original non-linear programming problem is known as SLP.
Stochastic programming deals with situations where uncertainty is present in the data or parameters of the optimization problem that are described by probabilistic variables rather than by deterministic. In various areas of real world, problems are modeled as stochastic programming. For example, modeling of an investment portfolio so as to meet random liabilities, modeling of strategic capacity investments, power systems, i.e., (modeling the operation of electrical power supply systems so as to meet consumers demand for electricity), cluster-based allocation of recruitment in manpower planning in Jeeva et al. (2002, 2004), etc. Literatures and applications of stochastic programming as well as fractional programming are available in Stancu-Minasian and Wets (1976), Stancu-Minasian (1977) and Stancu-Minasian and Tigan (1987). Nembou et al. (1996) presented a stochastic optimization model which has been applied to an existing hydro-thermal electricity generation planning problem in Port Moresby, Papua New Guinea. An application to a multiobjective fractional programming is discussed in Gulati et al. (1991). Duality for pseudolinear programming problems is studied and its application to certain multiobjective fractional programming is discussed in Bector et al. (1998). A direct approach to solve linear fractional programming (LFP) and to duality in LFP is given in Bajalinov (2003), in which the original linear fractional programming problem is considered as it is, without reducing it to a linear programming problem. Youhua Frank Chen (2005) proposed algorithms based on a fractional programming method, which is efficient and compatible to existing algorithms to determine optimal values for the two control parameters of stochastic inventory models.

Section 2 of this paper deals with formulation of multi-objective stochastic fractional programming problem and Section 3 deals with conversion of stochastic constraints into deterministic constraints. Section 4 provides the conversion technique that helps us to convert stochastic fractional objective functions into deterministic constraints and Section 5 gives the basic definitions of redundancy. Section 6 provides the redundancy algorithm technique to find redundant objective functions that has been shown numerically with examples in Section 7 and conclusion has been drawn at the end.

2. Multi-Objective Stochastic Fractional Programming

The optimization of ratios of criteria gives more insight into the situation than the optimization of each criterion. Multi-objective fractional programming models for this reason have been of greater interest in recent time [Nykowski et al. (1985); Gulati et al. (1991); Weir et al. (1992); Bector et al. (1993); Dutta et al. (1993); Bector et al. (1998); Gulati et al. (1998); Arora et al. (2003); Lalitha et al. (2003); Arora et al. (2005); Lalitha et al. (2005)]. A general format of the multi-objective linear fractional programming problem (MOLFPP) with identical denominator could be seen in Dutta et al. (1993). It is also shown that the general method of solving MOLFPP by Nykowski and Zolkiewski (1985) is computationally more involved
than the method proposed by Dutta et al. (1993). Baba and Morimoto (1993) proposed a stochastic approximation method for solving the stochastic multi-objective programming problem (SMOPP) and Caballero et al. (2001) provided efficient solution concepts in SMOPP. Here the concepts of MOLFPP and SMOPP are combined.

A multi-objective stochastic fractional programming problem in a criterion space is defined as follows:

Max \( R(X) = [R_1(X), R_2(X), \ldots, R_k(X)] \), where

\[
R_y(X) = \frac{N_y(X) + \alpha_y}{D_y(X) + \beta_y}, \quad y = 1, 2, \ldots, k
\]  

Subject to

\[
\Pr \left[ \sum_{j=1}^{n} t_{ij} x_j \leq b_i \right] \geq 1 - p_i \quad i = 1, 2, \ldots, m
\]

\[
\sum_{j=1}^{n} t_{ij}^{(1)} x_j \leq b_i^{(1)} \quad i = m + 1, \ldots, h
\]

where, \( 0 \leq X_{n \times 1} = \|x_j\| \in \mathbb{R}^n \) is a feasible set, and \( R: \mathbb{R}^n \to \mathbb{R}^k \), \( T_{m \times n} = \|t_{ij}\| \), \( b_{m \times 1} = \|b_i\| \), \( i = 1, 2, \ldots, m \), \( j = 1, 2, \ldots, n \), \( \alpha_y, \beta_y \) are scalars.

\[
N_y(X) = \sum_{j=1}^{n} c_{yj} x_j \quad \text{and} \quad D_y(X) = \sum_{j=1}^{n} d_{yj} x_j.
\]

In this model, out of \( N_y(X), D_y(X), T \) and \( b \), at least one may be random variable. \( S = \{X | \text{Eqs. (2) (3), } X \geq 0, X \in \mathbb{R}^n \} \) is non-empty, convex and compact set in \( \mathbb{R}^n \).

3. Deterministic Equivalents of Probabilistic Constraints

Let \( T \) be a random variable in Eq. (2) and it follows \( N(u_{ij}, s_{ij}^2), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \), where \( u_{ij} \) is the mean and \( s_{ij}^2 \) is the variance. Let \( l_i = \sum_{j=1}^{n} t_{ij} x_j \), \( i = 1, 2, \ldots, m \).

\[
E(l_i) = \sum_{j=1}^{n} u_{ij} x_j; \quad V(l_i) = X'V_i X = \sum_{j=1}^{n} s_{ij}^2 x_j^2,
\]

where \( V_i \) — \( i \)th covariance matrix. When \( T \) is independent, the covariance terms become zero. The \( i \)th deterministic constraint for Eq. (2) is obtained from Charles and Dutta (2001, 2003) as follows:

\[
\Pr(l_i \leq b_i) \geq 1 - p_i \quad \text{(or)} \quad \Pr(Z_i \leq z_i) \geq 1 - p_i,
\]

where \( Z_i = (l_i - E(l_i))/\sqrt{V(l_i)} \) follows standard normal distribution and \( z_i = (b_i - E(l_i))/\sqrt{V(l_i)} \). Thus, \( \phi(z_i) \geq \phi(Kq_i) \), where \( 1 - p_i = q_i = \phi(Kq_i) \), is the cumulative distribution function of standard normal distribution. Clearly, \( \phi(\cdot) \) is a
non-decreasing continuous function, hence $z_i \geq K q_i$. Substituting in this the values of $E(l_i)$ and $V(l_i)$,

$$\sum_{j=1}^{n} u_{ij} x_j + K q_i \sqrt{\sum_{j=1}^{n} s_{ij}^2 x_j^2} \leq b_i$$

(4)

If $b_i$ is a random variable in Eq. (2), i.e., $b_i \sim N(u_{bi}, s_{bi}^2)$, $i = 1, 2, \ldots, m$, where $u_{bi}$, $s_{bi}^2$ are the mean and variance respectively. With the similar argument that led to the inequality in (4), one can obtain inequality (5), the $i$th deterministic constraint for Eq. (2) as follows:

$$\sum_{j=1}^{n} t_{ij} x_j \leq u_{bi} + K p_i s_{bi}$$

(5)

Suppose $T$ and $b_i$ are random variables in Eq. (2) i.e. $T \sim N(u_{ij}, s_{ij}^2)$ and $b_i \sim N(u_{bi}, s_{bi}^2)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, where $u_{ij}$ and $u_{bi}$ are means, and $s_{ij}^2$ and $s_{bi}^2$ are variances respectively. With the similar argument that led to the inequality in (4), one can obtain inequality (6), the $i$th deterministic constraint for Eq. (2) as follows:

$$\sum_{j=1}^{n} u_{ij} x_j - K p_i \sqrt{\sum_{j=1}^{n+1} s_{ij}^2 x_j^2} \leq u_{bi}$$

(6)

where $x_{n+1} = -1$.

4. Conversion of Objective Functions into Constraints

This section considers all the objective functions in the form of constraints (Charles and Dutta, 2003). The main feature of the model is that it takes into account the probability distribution of the objective functions by maximizing the lower allowable limit of the objective function under chance constraints where both numerator and denominator coefficients being random.

**Assumption.** $N_y(X) \sim N(\sum_{j=1}^{n} u_{cyj} x_j, \sum_{j=1}^{n} s_{cyj}^2 x_j^2)$ and $D_y(X) \sim N(\sum_{j=1}^{n} u_{dyj} x_j, \sum_{j=1}^{n} s_{dyj}^2 x_j^2)$, where $u_{cyj}$ and $u_{dyj}$ are means, and $s_{cyj}^2$ and $s_{dyj}^2$ are variances.

The unknown parameter $\lambda_y$, which is less than or equal to $R_y(X)$ is defined by,

$$R_y(X) \geq \lambda_y \quad \text{i.e.,} \quad \frac{N_y(X)}{D_y(X)} + \beta_y \geq \lambda_y \implies 0 \leq N_y(X) + \alpha_y - \lambda_y[D_y(X) + \beta_y]$$

There are two cases in this problem.

**Case 1.** $\alpha_y > 0$.

Let $f(X, \lambda_y; \alpha_y > 0) = \lambda_y[D_y(X) + \beta_y] - N_y(X) \leq \alpha_y$

$$E[f(X, \lambda_y; \alpha_y > 0)] = F^E(X, \lambda_y; \alpha_y > 0) = \lambda_y[D_y^E(X) + \beta_y] - N_y^E(X)$$
\[ V[f(X, \lambda_y; \alpha_y > 0)] = F^V(X, \lambda_y; \alpha_y > 0) = \lambda_y^2 D_y^V(X) + N_y^V(X) \]
\[ = \lambda_y^2 \sum_{j=1}^{n} s_{dyj}^2 x_j^2 + \sum_{j=1}^{n} s_{cyj}^2 x_j^2 = \sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2 \]  
\[ (7) \]

\[ \Pr[f(X, \lambda_y; \alpha_y > 0) \leq \alpha_y] \geq 1 - p_y^{(2)} \]
\[ \Rightarrow \lambda_y (D_y^E(X) + \beta_y) - N_y^E(X) + \phi^{-1}(q_y^{(2)}) \sqrt{\lambda_y^2 D_y^V(X) + N_y^V(X)} \leq \alpha_y \]
\[ \Rightarrow \lambda_y \left[ \sum_{j=1}^{n} u_{dyj} x_j + \beta_y \right] - \sum_{j=1}^{n} u_{cyj} x_j + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \leq \alpha_y \]
\[ (8) \]

\[ \Pr[f(X, \lambda_y; \alpha_y > 0) \leq \alpha_y] \geq 1 - p_y^{(2)} \]
\[ \Rightarrow \lambda_y (D_y^E(X) + \beta_y) - N_y^E(X) + \phi^{-1}(q_y^{(2)}) \sqrt{\lambda_y^2 D_y^V(X) + N_y^V(X)} \leq \alpha_y \]
\[ \Rightarrow \lambda_y \left[ \sum_{j=1}^{n} u_{dyj} x_j + \beta_y \right] - \sum_{j=1}^{n} u_{cyj} x_j + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \leq \alpha_y \]
\[ (9) \]

\[ \Pr[f(X, \lambda_y; \alpha_y > 0) \leq \alpha_y] \geq 1 - p_y^{(2)} \]
\[ \Rightarrow \lambda_y (D_y^E(X) + \beta_y) - N_y^E(X) + \phi^{-1}(q_y^{(2)}) \sqrt{\lambda_y^2 D_y^V(X) + N_y^V(X)} \leq \alpha_y \]
\[ \Rightarrow \lambda_y \left[ \sum_{j=1}^{n} u_{dyj} x_j + \beta_y \right] - \sum_{j=1}^{n} u_{cyj} x_j + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \leq \alpha_y \]
\[ (10) \]

**Case 2.** \( \alpha_y \leq 0. \)

Similar to Case 1 one can obtain the constraint given below:
\[ \sum_{j=1}^{n} u_{cyj} x_j - \lambda_y \left[ \sum_{j=1}^{n} u_{dyj} x_j + \beta_y \right] + \phi^{-1}(p_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \geq \alpha_y \]
\[ (11) \]

**5. Definitions**

The following definitions are defined in consent of Case 1 of Section 4. Similarly, one can define for Case 2.

Let scalar \( \lambda = \min\{\lambda_y \simeq R_y(X)|X \text{ be the unit vector and } y = 1, 2, \ldots, k\}. \)

Let the decision space be
\[ S^0 = \left\{ X \in \mathbb{R}^n | \lambda \sum_{j=1}^{n} u_{dyj} x_j - \sum_{j=1}^{n} u_{cyj} x_j + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \right. \]
\[ \left. \leq \alpha_y - \lambda \beta_y, \; y = 1, 2, \ldots, k, \; x_j \geq 0; \; j = 1, 2, \ldots, n \right\} \]

and
\[ S_w = \left\{ X \in \mathbb{R}^n | \lambda \sum_{j=1}^{n} u_{dyj} x_j - \sum_{j=1}^{n} u_{cyj} x_j + \phi^{-1}(q_y^{(2)}) \sqrt{\sum_{j=1}^{n} (\lambda_y^2 s_{dyj}^2 + s_{cyj}^2) x_j^2} \right. \]
\[ \left. \leq \alpha_y - \lambda \beta_y, \; x_j \geq 0; \; j = 1, 2, \ldots, n, \; y \neq w \right\} \]
Definition 5.1. The constraint form of objective function

\[ \lambda \sum_{j=1}^{n} u_{d w j} x_j - \sum_{j=1}^{n} u_{c w j} x_j + \phi^{-1}(q_{w}^{(2)}) \left\lfloor \sum_{j=1}^{n} (\lambda^2 s_{d w j}^2 + s_{c w j}^2) x_j^2 \right\rfloor \leq \alpha_w - \lambda \beta_w \]

is redundant in system (1) if and only if \( S^w = S_w \). The equivalent definition is \( S_w = S^w \) if and only if

\[ \lambda \sum_{j=1}^{n} u_{d w j} x_j - \sum_{j=1}^{n} u_{c w j} x_j + \phi^{-1}(q_{w}^{(2)}) \left\lfloor \sum_{j=1}^{n} (\lambda^2 s_{d w j}^2 + s_{c w j}^2) x_j^2 \right\rfloor \leq \alpha_w - \lambda \beta_w \] (13)

for all \( X \in S_w \) and hence,

\[ s_w(X) = \alpha_w - \lambda \beta_w - \lambda \sum_{j=1}^{n} u_{d w j} x_j - \sum_{j=1}^{n} u_{c w j} x_j - \phi^{-1}(q_{w}^{(2)}) \left\lfloor \sum_{j=1}^{n} (\lambda^2 s_{d w j}^2 + s_{c w j}^2) x_j^2 \right\rfloor \]

Definition 5.2. The constraint form of the \( w \)th objective function (13) is redundant in system (1) if and only if \( s_w = \min \{ s_w(X) | X \in S_w \} \geq 0 \).

Definition 5.3. The constraint form of the \( w \)th objective function (13) is strongly redundant in the system (1) if and only if \( s_w > 0 \). However, the constraint can be redundant without strongly redundant.

Definition 5.4. The constraint form of the \( w \)th objective function (13) is weakly redundant in the system (1) if and only if \( s_w = 0 \).

6. Identification of Redundant Constraints for LSFP

In this section, an algorithm is provided that helps to identify redundant fractional objective functions in multi-objective linear stochastic fractional programming problems. Charles and Dutta (2003) using sequential linear programming provided a method for linearizing the constraint version of fractional objective function defined in Sec. 4.

Consider linearizing the constraint form of fractional objective function

\[ \hat{R}_y(X) = \lambda \sum_{j=1}^{n} u_{d y j} x_j - \sum_{j=1}^{n} u_{c y j} x_j + \phi^{-1}(q_{y}^{(2)}) \left\lfloor \sum_{j=1}^{n} (\lambda^2 s_{d y j}^2 + s_{c y j}^2) x_j^2 \right\rfloor - \alpha_y + \lambda \beta_y \leq 0, \]

then,

\[ \hat{R}_y(X_{\text{int}}) + \nabla \hat{R}_y(X)^T (X - X_{\text{int}}) \leq 0, \quad y = 1, 2, \ldots, k, \quad X \geq 0. \] (14)

Inequality (14) can be viewed as \( \hat{R}_y^{(1)} X \leq \alpha_y - \lambda \beta_y, \quad X \geq 0, \quad y = 1, 2, \ldots, k \) for the following steps:

The matrix form of the above inequality can be viewed as

\[ \hat{R}_y^{(1)} X \leq \alpha - \lambda \beta, \quad X \geq 0, \quad \text{where } \hat{R}_y^{(1)} \in \mathbb{R}^{k \times n} \text{ and } (\alpha - \lambda \beta) \in \mathbb{R}^k. \]
Adding slack variables to the \( k \) constraints form of objective functions, pre-multiplying by the inverse of an appropriate basis and redefining the variables (both slacks and structural variables) as \( x_{j}^{NB} \) (or) \( x_{j}^{B} \) according to their status (NB for non-basic, and B for basic), yields an equivalence system

\[
\begin{bmatrix}
\hat{R}_{NB}^{(1)} - I \\
\end{bmatrix}
\begin{bmatrix}
x_{NB} \\
x_{B} \\
\end{bmatrix} = \eta, \quad x^{B}, x^{NB} \geq 0.
\]

The matrix \( \hat{R}_{NB}^{(1)} \) is usually referred to as the Contracted Simplex Tableau (Dantzig, 1963). Let us refer to the elements of \( \hat{R}_{NB}^{(1)} \) as \( \gamma_{ij} \), \( \eta \) is the “updated right hand side” \( \hat{R}_{NB}^{(1)}(\alpha - \lambda \beta) \).

**Theorem 6.1.** A constraint form of objective function is redundant if and only if its associated slack variable \( s_{w} \) has the property \( s_{w} = x_{j}^{B} \) in a basic solution in which \( \gamma_{fj} \leq 0 \), \( j = 1, 2, \ldots, n \) and \( \eta_{f} \geq 0 \).

**Proof. If:** In a basic solution \( x_{j}^{B} = \eta_{f} - \sum_{j=1}^{n} \gamma_{fj} x_{j}^{B} \), since in any feasible solution the value of the \( x_{j}^{NB} \) will be at least zero, the sum is at least zero and hence, \( s_{w} = x_{j}^{B} \geq \eta_{f} \geq 0 \). Therefore \( s_{w} \geq 0 \).

**Only If:** Let us consider the \( f \)th row of tableau as the objective function for the sequential linear programming minimum \( \{s_{w}(X)|X \in S_{w}\} \); then if \( s_{w} \geq 0 \), it follows that in the optimal solution \( \gamma_{fj} \leq 0 \) for all \( j = 1, 2, \ldots, n \) with \( \eta_{f} \geq 0 \). Since this optimal solution is a feasible extreme point of \( S_{w} \), it is a basic feasible solution for the original set of constraint form of objective functions. \( \square \)

Since, in the theorem above \( s_{w} = \eta_{f} \), the constraint form of objective function is strongly redundant if \( \eta_{f} > 0 \) and weakly redundant if \( \eta_{f} = 0 \).

**Redundancy Algorithm**

1. A matrix of intercept is constructed with decision and slack variables as rows and columns respectively. This matrix is of order \( m \times n \).

   - if \( \alpha_{y} \leq 0 \), then \( \theta_{ji} = (\alpha_{y} - \lambda \beta_{y})/\hat{R}_{yij}^{(1)} \), \( \hat{R}_{yij}^{(1)} \geq 0 \), \( j = 1, 2, \ldots, n \)
   - else \( \theta_{ji} = (\alpha_{y} - \lambda \beta_{y})/\hat{R}_{yij}^{(1)} \), \( \hat{R}_{yij}^{(1)} < 0 \), \( j = 1, 2, \ldots, n \).

2. Identify the pivot element in each row if \( \alpha_{y} \leq 0 \), then \( \Psi_{j} = \max_{i}\{\theta_{ji}\} \) else \( \Psi_{j} = \min_{i}\{\theta_{ji}\} \), for all \( j \) while the objective is maximum, vice versa.

3. Score out the row and column corresponding to the entering and leaving variables. If a column has more than one maximum/minimum, score out those rows also.

4. The constraints corresponding to the slack variables in the unscored column, if any, ab initio are assumed and predicted to be redundant.
5. Remove these redundant constraint forms of fractional objective functions tenta-
tively from the original model.

6.1. General algorithm

1. Convert the stochastic fractional objective functions into constraint form using
Sec. 4.
(2003), linearize the constraint form of the objective functions.
3. Apply the algorithm under Sec. 6 to identify the redundant objective function
and ignore that objective function from the system.
4. Solve the reduced multi-objective stochastic fractional programming to get the
optimal solution as in Charles and Dutta (2001, 2003) or using any stochastic
programming solver.

7. Numerical Examples

Example 7.1.  

\[
\begin{align*}
\text{Max} & \quad R(X) = \left[ \frac{c_{11}x_1 + c_{12}x_2 + \alpha_1}{d_{11}x_1 + d_{12}x_2 + \beta_1}, \frac{c_{21}x_1 + c_{22}x_2 + \alpha_2}{d_{21}x_1 + d_{22}x_2 + \beta_2} \right] \\
\text{Subject to} & \quad 3x_1 + 5x_2 \leq 15, \ 5x_1 + 2x_2 \leq 10, \ a_1x_1 + a_2x_2 \leq 5,
\end{align*}
\]

where \( \alpha_1 = \alpha_2 = 0, \ \beta_1 = \beta_2 = 1, \ x_1, x_2 \geq 0. \)

Let the third constraint satisfy at least 90%. The mean and variance of the
random variables are given in Table 1.

Take \( p_1^{(2)} = 0.10 \) and \( p_2^{(2)} = 0.90. \) The deterministic equivalent of constraint
form of fractional objective functions is given below:

\[
\begin{align*}
6x_1 + 3x_2 - \lambda_1(5x_1 + 2x_2 + 1) + 1.28\sqrt{(2\lambda^2_1 + 2)x_1^2 + (\lambda^2_1 + 1)x_2^2} & \geq 0 \\
15x_1 + 10x_2 - \lambda_2(x_1 + x_2 + 1) - 1.28\sqrt{(\lambda^2_2 + 1)x_1^2 + (\lambda^2_2 + 1)x_2^2} & \geq 0
\end{align*}
\]

Let \( \lambda = \min\{1.125, 8.333\} = 1.125 \) at \( (x_1, x_2) = (1,1) \) from Eq. (12).

Therefore, inequalities (16)-(17) reduces to (18)-(19):

\[
0.3750x_1 + 0.7500x_2 + 1.28\sqrt{4.5313x_1^2 + 2.2656x_2^2} \geq 1.1250
\]

<table>
<thead>
<tr>
<th>Random Variables</th>
<th>( c_{11} )</th>
<th>( c_{12} )</th>
<th>( c_{21} )</th>
<th>( c_{22} )</th>
<th>( d_{11} )</th>
<th>( d_{12} )</th>
<th>( d_{21} )</th>
<th>( d_{22} )</th>
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<td>1</td>
<td>2</td>
<td>3</td>
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<tr>
<td>Variance</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 2. Matrix-of-intercept for Example 7.1.

<table>
<thead>
<tr>
<th>Slack/Decision Variables</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$\Psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.4527</td>
<td>0.0899</td>
<td>0.4527</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.6041</td>
<td>0.1497</td>
<td>0.6041</td>
</tr>
</tbody>
</table>

\[13.8750x_1 + 8.8750x_2 - 1.28\sqrt{2.2656x_1^2 + 2.2656x_2^2} \geq 1.1250\] \hspace{1cm} (19)

Using the SLP of Charles and Dutta (2003) the following linear constraints are obtained:

\[2.5997x_1 + 1.8623x_2 \geq 1.1250\] \hspace{1cm} (20)

\[12.5126x_1 + 7.5126x_2 \geq 1.1250\] \hspace{1cm} (21)

Adapt the redundancy algorithm given in Section 6.

From the Table 2, it can be inferred that the constraint due to second objective function is strongly redundant. Therefore, ignoring the second objective function the problem is solved. The bi-objective stochastic fractional programming problem reduces to SFP problem as follows:

\[
\begin{align*}
\text{Max} & \quad 0.90\lambda_1 \\
\text{Subject to} & \quad 6x_1 + 3x_2 - \lambda_1(5x_1 + 2x_2 + 1) \\
& \quad + 1.28\sqrt{(2\lambda_1^2 + 2)x_1^2 + (\lambda_1^2 + 1)x_2^2} \geq 0 \\
& \quad 3x_1 + 5x_2 \leq 15, 5x_1 + 2x_2 \leq 10, 2x_1 + 3x_2 + 1.28\sqrt{x_1^2 + x_2^2} \leq 5 \\
& \quad x_1, x_2, \lambda_1 \geq 0.
\end{align*}
\]

The solution is obtained as $x_1 = 1.5244, x_2 = 0.0000, \lambda_1 = 1.6890$. The corresponding value of objective functions in Eq. (15) is $[1.0608, 9.0580]$. Here, it is to be noted that 90% of importance is given to the first objective function.

Example 7.2. Let us consider Example 7.1 along with a third objective function as

\[
\begin{align*}
\text{Max} & \quad R(X) = \left[ \begin{array}{c} c_{11}x_1 + c_{12}x_2 + \alpha_1 \\
& c_{21}x_1 + c_{22}x_2 + \alpha_2 \\
& c_{31}x_1 + c_{32}x_2 + \alpha_3\end{array} \right] \\
& \left[ \begin{array}{c} d_{11}x_1 + d_{12}x_2 + \beta_1 \\
& d_{21}x_1 + d_{22}x_2 + \beta_2 \\
& d_{31}x_1 + d_{32}x_2 + \beta_3\end{array} \right]
\end{align*}
\]

\hspace{1cm} (23)

Subject to Example 7.1 constraints, where $\alpha_3 = -5, \beta_3 = 1, x_1, x_2 \geq 0$.

Let the third constraint satisfy at least 90%. The mean and variance of the random variables are given in Table 3.
Let us take $p_1^{(2)} = 0.70$, $p_2^{(2)} = 0.90$ and $p_3^{(2)} = 0.40$. The deterministic equivalents to constraint form of fractional objective functions are given below:

\begin{align*}
6x_1 + 3x_2 - \lambda_1(5x_1 + 2x_2 + 1) - 0.52\sqrt{(2\lambda_1^2 + 2)x_1^2 + (\lambda_1^2 + 1)x_2^2} & \geq 0 \quad (24) \\
15x_1 + 10x_2 - \lambda_2(x_1 + x_2 + 1) - 1.28\sqrt{(\lambda_2^2 + 1)x_1^2 + (\lambda_2^2 + 1)x_2^2} & \geq 0 \quad (25) \\
10x_1 + 12x_2 - \lambda_3(5x_1 + 2x_2 + 1) + 0.25\sqrt{(2\lambda_3^2 + 2)x_1^2 + (\lambda_3^2 + 1)x_2^2} & \geq 5 \quad (26)
\end{align*}

Let $\lambda = \min\{1.125, 8.333, 2.125\} = 1.125$ at $(x_1, x_2) = (1, 1)$ from Eq. (12). Therefore, inequalities (24)–(26) reduce to (27)–(29)

\begin{align*}
0.3750x_1 + 0.7500x_2 - 0.52\sqrt{4.5313x_1^2 + 2.2656x_2^2} & \geq 1.1250 \quad (27) \\
13.8750x_1 + 8.8750x_2 - 1.28\sqrt{2.2656x_1^2 + 2.2656x_2^2} & \geq 1.1250 \quad (28) \\
4.3750x_1 + 9.7500x_2 + 0.25\sqrt{4.5313x_1^2 + 2.2656x_2^2} & \geq 6.1250 \quad (29)
\end{align*}

Using the SLP of Charles and Dutta (2003) the following linear constraints are obtained:

\begin{align*}
-1.4765x_1 + 0.1033x_2 & \geq 1.1250 \quad (30) \\
12.5126x_1 + 7.5126x_2 & \geq 1.1250 \quad (31) \\
4.8095x_1 + 9.9673x_2 & \geq 6.1250 \quad (32)
\end{align*}

Let us take $p_1^{(2)} = 0.70$, $p_2^{(2)} = 0.90$ and $p_3^{(2)} = 0.40$. The deterministic equivalents to constraint form of fractional objective functions are given below:

\begin{align*}
6x_1 + 3x_2 - \lambda_1(5x_1 + 2x_2 + 1) - 0.52\sqrt{(2\lambda_1^2 + 2)x_1^2 + (\lambda_1^2 + 1)x_2^2} & \geq 0 \quad (24) \\
15x_1 + 10x_2 - \lambda_2(x_1 + x_2 + 1) - 1.28\sqrt{(\lambda_2^2 + 1)x_1^2 + (\lambda_2^2 + 1)x_2^2} & \geq 0 \quad (25) \\
10x_1 + 12x_2 - \lambda_3(5x_1 + 2x_2 + 1) + 0.25\sqrt{(2\lambda_3^2 + 2)x_1^2 + (\lambda_3^2 + 1)x_2^2} & \geq 5 \quad (26)
\end{align*}

Let $\lambda = \min\{1.125, 8.333, 2.125\} = 1.125$ at $(x_1, x_2) = (1, 1)$ from Eq. (12). Therefore, inequalities (24)–(26) reduce to (27)–(29)

\begin{align*}
0.3750x_1 + 0.7500x_2 - 0.52\sqrt{4.5313x_1^2 + 2.2656x_2^2} & \geq 1.1250 \quad (27) \\
13.8750x_1 + 8.8750x_2 - 1.28\sqrt{2.2656x_1^2 + 2.2656x_2^2} & \geq 1.1250 \quad (28) \\
4.3750x_1 + 9.7500x_2 + 0.25\sqrt{4.5313x_1^2 + 2.2656x_2^2} & \geq 6.1250 \quad (29)
\end{align*}

Using the SLP of Charles and Dutta (2003) the following linear constraints are obtained:

\begin{align*}
-1.4765x_1 + 0.1033x_2 & \geq 1.1250 \quad (30) \\
12.5126x_1 + 7.5126x_2 & \geq 1.1250 \quad (31) \\
4.8095x_1 + 9.9673x_2 & \geq 6.1250 \quad (32)
\end{align*}

Adapt the redundancy algorithm given in Sec. 6.

From the Table 4, it can be inferred that the constraint due to second objective function is strongly redundant. Therefore, ignoring the second objective function the problem is solved. Now, the Tri-objective SFP problem reduces to bi-objective

\begin{table}[h]
\centering
\caption{Matrix-of-intercept for Example 7.2.}
\begin{tabular}{c|c|c|c|c}
\hline
Slack/Decision Variables & $S_1$ & $S_2$ & $S_3$ & $\Psi$
\hline
$x_1$ & $\text{-ve}$ & 0.0899 & 1.2735 & 1.2735
\hline
$x_2$ & 10.8906 & 0.1497 & 0.6145 & 10.8906
\hline
\end{tabular}
\end{table}
SFP problem as follows:

\[
\begin{align*}
\text{Max} & \quad 0.30\lambda_1 + 0.60\lambda_3 \\
\text{Subject to} & \quad 6x_1 + 3x_2 - \lambda_1(5x_1 + 2x_2 + 1) \\
& \quad - 0.52\sqrt{(2\lambda_1^2 + 2)x_1^2 + (\lambda_1^2 + 1)x_2^2} \geq 0 \\
& \quad 10x_1 + 12x_2 - \lambda_3(5x_1 + 2x_2 + 1) \\
& \quad + 0.25\sqrt{(2\lambda_1^2 + 2)x_1^2 + (\lambda_1^2 + 1)x_2^2} \geq 5 \\
& \quad 3x_1 + 5x_2 \leq 15, 5x_1 + 2x_2 \leq 10, 2x_1 + 3x_2 + 1.28\sqrt{x_1^2 + x_2^2} \leq 5 \\
\end{align*}
\]

The solution is obtained as \( x_1 = 0.0000, x_2 = 1.1682, \lambda_1 = 0.8155 \) and \( \lambda_3 = 2.8160 \). The corresponding value of objective functions in Eq. (23) is \[1.05041, 5.3879, 4.1594\].

Example 7.3.

\[
\begin{align*}
\text{Max} \quad R(X) &= \left[ \frac{c_{11}x_1 + c_{12}x_2 + a_1}{d_{11}x_1 + d_{12}x_2 + \beta_1}, \frac{c_{21}x_1 + c_{22}x_2 + a_2}{d_{21}x_1 + d_{22}x_2 + \beta_2} \right] \\
\text{Subject to} & \quad a_{11}x_1 + a_{12}x_2 \leq 20, a_{21}x_1 + a_{22}x_2 \leq b_2, 7x_1 + 2x_2 \leq b_3,
\end{align*}
\]

where \( a_1 = 5, a_2 = 7, \beta_1 = 3, \beta_2 = 4, x_1, x_2 \geq 0 \).

Let the first constraint be satisfied at least 90% and the second and third constraints be satisfied at least 80%. The mean and variance of the random variables are given in Table 5.

Let us take \( q_1^{(2)} = 0.10 \) and \( q_2^{(2)} = 0.90 \). The deterministic equivalents of constraint form of fractional objective functions are given below:

\[
\begin{align*}
\lambda_1(2x_1 + x_2 + 3) - 4x_1 - 3x_2 - 1.28\sqrt{(\lambda_1^2 + 1)x_1^2 + (\lambda_1^2 + 0.5)x_2^2} & \leq 5 \quad (35) \\
\lambda_2(5x_1 + 2x_2 + 4) - 8x_1 - 3x_2 + 1.28\sqrt{(2\lambda_2^2 + 1)x_1^2 + (2\lambda_2^2 + 0.5)x_2^2} & \leq 7 \quad (36)
\end{align*}
\]

Let \( \lambda = \min\{2.0000, 1.6364\} = 1.6364 \) at \((x_1, x_2) = (1, 1)\) from Eq. (12).

<table>
<thead>
<tr>
<th>Random Variables</th>
<th>c_{11}</th>
<th>c_{12}</th>
<th>c_{21}</th>
<th>c_{22}</th>
<th>d_{11}</th>
<th>d_{12}</th>
<th>d_{21}</th>
<th>d_{22}</th>
<th>a_{11}</th>
<th>a_{12}</th>
<th>a_{21}</th>
<th>a_{22}</th>
<th>b_2</th>
<th>b_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>4</td>
<td>3</td>
<td>8</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>6</td>
<td>7</td>
<td>18</td>
<td>13</td>
</tr>
<tr>
<td>Variance</td>
<td>1</td>
<td>0.5</td>
<td>2</td>
<td>0.5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
Therefore, inequalities (35) and (36) reduces to (37) and (38)

\[-0.7272x_1 - 1.3636x_2 - 1.28\sqrt{3.6778x_1^2 + 3.1778x_2^2} \leq 0.0908 \]  
\[0.1820x_1 + 0.2728x_2 + 1.28\sqrt{6.3556x_1^2 + 5.8556x_2^2} \leq 0.4544 \]  

Using the SLP of Charles and Dutta (2003) the following linear constraints are obtained:

\[-2.5251x_1 - 2.9171x_2 \leq 0.0908 \]  
\[2.5100x_1 + 2.4177x_2 \leq 0.4544 \]  

Adapt the redundancy algorithm given in Sec. 6.

From the Table 6, it can be inferred that the constraint due to second objective function is strongly redundant. Therefore, ignoring the second objective function the problem is solved. The bi-objective stochastic fractional programming problem reduces to SFP problem as follows:

\begin{equation}
\text{Max} \quad 0.90\lambda_1
\end{equation}

Subject to

\[\lambda_1(2x_1 + x_2 + 3) - 4x_1 - 3x_2 \]
\[-1.28\sqrt{(\lambda_1^2 + 1)x_1^2 + (\lambda_1^2 + 0.5)x_2^2} \leq 5 \]
\[4x_1 + 8x_2 + 1.28\sqrt{2x_1^2 + 3x_2^2} \leq 20 \]
\[6x_1 + 7x_2 + 0.84\sqrt{x_1^2 + x_2^2} + 3 \leq 18 \]
\[7x_1 + 2x_2 \leq 13 + 0.84\sqrt{6} = 15.0576 \]
\[x_1, x_2, \lambda_1 \geq 0. \]

The solution is obtained as \( x_1 = 1.9983, x_2 = 0.5349, \lambda_1 = 3.0723 \). The corresponding value of objective functions in Eq. (34) is \([1.9382, 1.6327]\). Here, it is to be noted that 90% of importance is given to the first objective function.

### 8. Conclusion

The method presented here identifies redundant stochastic fractional objective functions. The method has been developed with the intention of solving Multi-objective Stochastic Fractional Programming Problems and in process of identifying and
eliminating redundant stochastic fractional objective functions so as to reduce the number of objective functions, provided redundancy exist.

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