A coefficient inequality for a subclass of the Carathéodory functions defined using conical domains

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\section*{A B S T R A C T}

For $0 \leq k < \infty$, let $\Omega_k$ be the conical domain in the complex plane $\mathbb{C}$ defined by

$$
\Omega_k = \{ w \in \mathbb{C} : w = u + iv, \ u^2 > k^2((u - 1)^2 + v^2), \ u > 0 \}.
$$

Let $q_k(z)$ be the Riemann map of $\mathcal{U} := \{ z \in \mathbb{C} : |z| < 1 \}$ onto $\Omega_k$ satisfying $q_k(0) = 1$, $q_k'(0) > 0$. Let $\mathcal{P}(q_k)$ be the class of analytic functions $h(z)$ subordinate in $\mathcal{U}$ to $q_k(z)$ and represented by $h(z) = 1 + b_1 z + b_2 z^2 + \cdots$, $(z \in \mathcal{U})$. Sharp estimates for $|b_2 - u b_1^2| (-\infty < u < \infty)$ are found in this note. This result improves upon an estimate of Kanas in terms of both bounds and ranges of the parameter $u$ [S. Kanas, Coefficient estimates in subclasses of the Carathéodory class related to conical domains, Acta Math. Univ. Comenianae LXIV 2 (2005) 149–161].

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\section*{1. Introduction}

Let $\mathcal{A}$ denote the class of functions analytic in the open unit disc

$$
\mathcal{U} := \{ z \in \mathbb{C} : |z| < 1 \},
$$

where $\mathbb{C}$ is, as usual, the complex plane. Given functions $f, g \in \mathcal{A}$, $f$ is said to be subordinate to $g$ in $\mathcal{U}$, denoted by

$$
f(z) \prec g(z) \quad (z \in \mathcal{U}),
$$

if there exists a function $w \in \mathcal{B}_0$ where

$$
\mathcal{B}_0 := \{ w \in \mathcal{A} : w(0) = 0, |w(z)| < 1, \ (z \in \mathcal{U}) \},
$$

such that

$$
f(z) = g(w(z)), \quad (z \in \mathcal{U}).
$$

It follows that

$$
f(z) < g(z) \quad (z \in \mathcal{U}) \Rightarrow f(0) = 0 \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).
$$

For fixed $k (0 \leq k < \infty)$ let $\Omega_k$ be the conic region given by

$$
\Omega_k = \{ w \in \mathbb{C} : w = u + iv, \ u^2 > k^2((u - 1)^2 + v^2), \ u > 0 \}.
$$

We note that $\Omega_k$ is a region in the right half-plane, symmetric with respect to real axis, and contains the point $(1, 0)$. More precisely: for $k = 0$, $\Omega_0$ is the right half-plane; for $0 < k < 1$, $\Omega_k$ is an unbounded region having boundary $\partial \Omega_k$, a rectangular hyperbola; for $k = 1$, $\Omega_1$ is still an unbounded region where $\partial \Omega_1$ is a parabola; and for $k > 1$, $\Omega_k$ is a bounded region enclosed by an ellipse.

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**Definition 1.1.** Let $q_k$ be the Riemann map of $U$ onto $\Omega_k$ satisfying $q_k(0) = 1$ and $q_k'(0) > 0$. We define the function class $\mathcal{P}(q_k)$ as follows:

$$\mathcal{P}(q_k) := \{ h \in \mathcal{A} : h(z) < q_k(z) \ (z \in U) \}.$$ 

In a recent paper, Kanas [1] discussed in detail the geometry of the region $\Omega_k$, the explicit form of the function $q_k$ and basic properties of functions in the class $\mathcal{P}(q_k)$. In the particular case $k = 0$, $\mathcal{P}(q_0)$ is the often discussed class of Carathéodory functions $h \in \mathcal{A}$ satisfying $h(0) = 1$ and $\Re(h(z)) > 0$ for $z \in U$.

For the class of Carathéodory functions, i.e. $\mathcal{P}(q_0)$, the following result is well known:

**Theorem 1.2 (cf. [2]).** Let the function $h$ in $\mathcal{A}$ satisfy $h(0) = 1$ and $\Re(h(z)) > 0$ for $z \in U$. If

$$h(z) = 1 + b_1z + b_2z^2 + \cdots \ (z \in U),$$

then for $-\infty < u < \infty$,

$$|b_2 - ub_1^2| \leq \begin{cases} 
2 + (u - 1)|b_1|^2, & u > \frac{1}{2}; \\
2 - \frac{1}{2}|b_1|^2, & u = \frac{1}{2}; \\
2 - u|b_1|^2, & u \leq \frac{1}{2}.
\end{cases} \tag{1.3}$$

Kanas (cf. [1]) obtained similar estimates for the class $\mathcal{P}(q_k)$ ($0 \leq k < \infty$):

**Theorem 1.3 (cf. [1]).** Let $0 \leq k < \infty$ be fixed and the function $q_k$ of **Definition 1.1** be represented by the Taylor–Maclaurin series

$$q_k(z) = 1 + Q_1(k)z + Q_2(k)z^2 + \cdots, \ (z \in U). \tag{1.4}$$

If the function $h$ given by the Taylor–Maclaurin series (1.2) is a member of the class $\mathcal{P}(q_k)$, then for $-\infty < u < \infty$,

$$|b_2 - ub_1^2| \leq \begin{cases} 
Q_1(k) + (u - 1)Q_1(k)^2, & u \geq 1; \\
Q_1(k), & u \in (0, 1); \\
Q_1(k) - uQ_1(k)^2, & u \leq 0.
\end{cases} \tag{1.5}$$

The middle inequality in (1.5) is sharp for the function $h(z) = q_k(z^2)$.

Let $\mathcal{A}_0$ be the subclass of functions $f$ in $\mathcal{A}$ satisfying the normalization condition $f(0) = f'(0) - 1 = 0$. Thus, the functions in $\mathcal{A}_0$ are represented by the Taylor–Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_nz^n \ (z \in U). \tag{1.6}$$

Let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}_0$ (cf. [2]). Similarly, let $\mathcal{E} \mathcal{V}$ and $\mathcal{U} \mathcal{E} \mathcal{V}$ denote, respectively, the subclasses of $\mathcal{S}$ consisting of functions which are convex (cf. [2]) and uniformly convex (cf. [3]) in $U$. For fixed $k$ ($0 \leq k < \infty$) the function $f \in \mathcal{A}_0$ is said to be in $k \mathcal{U} \mathcal{E} \mathcal{V}$, the class of $k$-uniformly convex functions, if and only if the function $h \in \mathcal{A}$ defined by

$$h(z) = 1 + \frac{zf''(z)}{f'(z)} \ (z \in U), \tag{1.7}$$

is a member of the class $\mathcal{P}(q_k)$ (cf. [4]). Note that $0 \mathcal{U} \mathcal{E} \mathcal{V} = \mathcal{E} \mathcal{V}$ and $1 \mathcal{U} \mathcal{E} \mathcal{V} = \mathcal{U} \mathcal{E} \mathcal{V}$.

For functions $f \in k \mathcal{U} \mathcal{E} \mathcal{V}$ ($0 \leq k < \infty$) and represented by (1.6), the problem of finding sharp bounds for $|\mu a_2^2 - a_3|$ ($-\infty < \mu < \infty$) has been settled for $k = 0$ by Keogh and Merkes [5], for $k = 1$ by Ma and Minda [6] and for $0 < k < 1$ and $k > 1$ by the authors [7] (also see [8,9] for recent work). In the study of this problem an essential first step is to replace the function $h(z)$ of Eq. (1.7) by $q_k(w(z))$, ($w \in B_0$). Next by using the one-to-one correspondence between the class of the Schwarz functions $B_0$ and the class of the Carathéodory functions $\mathcal{P}(q_0)$ one writes

$$q_k(w(z)) = q_k \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right), \ (w \in B_0, p_1 \in \mathcal{P}(q_0)). \tag{1.8}$$

Upon substitution of the last part of Eq. (1.8) in (1.7) and by judicious applications of the estimates for $|ub_1^2 - b_2|$ from **Theorem 1.2** (above), for the functions $p_1 \in \mathcal{P}(q_0)$, the estimates for $|\mu a_2^2 - a_3|$ are usually found.

In the present paper we first sharpen the estimates in (1.5) for $|ub_1^2 - b_2|$ ($-\infty < u < \infty$) for the functions $h \in \mathcal{P}(q_k)$ ($0 \leq k < \infty$) and given by the series expansion (1.2). Our result improves upon the estimates in **Theorem 1.3**
Let the function \( f \in \mathcal{B}^1 \) be represented by (1.4). If the function \( h \) given by the Taylor–Maclaurin series (1.2) is a member of the function class \( \mathcal{P}(q_k) \), then for \( -\infty < u < \infty \),

\[
|b_2 - ub_1^2| \leq \begin{cases} 
    uQ_1(k)^2 - Q_2(k), & u > \alpha_1(k) \\
    Q_1(k), & \alpha_2(k) \leq u \leq \alpha_1(k) \\
    Q_2(k) - uQ_1(k)^2, & u < \alpha_2(k)
\end{cases}
\]  

(3.1)

where

\[
\alpha_1(k) = \frac{D(k) + 1}{Q_1(k)},
\]

(3.2)

\[
\alpha_2(k) = \frac{D(k) - 1}{Q_1(k)},
\]

(3.3)

and \( Q_1(k), Q_2(k) \) and \( D(k) \) are given by (2.2)–(2.4) respectively. All the inequalities in (3.1) are sharp.
Theorem 3.1

By the definition of subordination there exists a function \( w \in B_0 \) such that \( h(z) = q_k(w(z)), (z \in U) \). Suppose that \( w(z) \) is given by the series (2.6). A direct calculation gives

\[
\begin{align*}
    b_1 &= Q_1(k)d_1, \\
    b_2 &= Q_1(k)d_2 + Q_2(k)d_1^2 = Q_1(k)[d_2 + D(k)d_1^2].
\end{align*}
\]

Therefore

\[ b_2 - ub_1^2 = Q_1(k)[d_2 + (D(k) - uQ_1(k))d_1^2]. \quad (3.4) \]

This gives

\[ |b_2 - ub_1^2| = Q_1(k)|d_2 - d_1^2 + (1 + D(k) - uQ_1(k))d_1^2|. \]

Suppose that \( u > \alpha_1(k) \); then using the estimate \( |d_2 - d_1^2| \leq 1 \) from Lemma 2.2 and the well known estimate \( |d_1| \leq 1 \) of the Schwarz lemma, we get

\[ |b_2 - ub_1^2| \leq Q_1(k)[1 + |uQ_1(k) - D(k) - 1|] = uQ_1(k)^2 - Q_2(k). \quad (3.5) \]

This is precisely the first inequality in (3.1).

On the other hand if \( u < \alpha_2(k) \), then (3.4) gives

\[ |b_2 - ub_1^2| \leq Q_1(k)|d_2| + |D(k) - uQ_1(k)||d_1| \]

Applying the estimates \( |d_2| \leq 1 - |d_1|^2 \) of Lemma 2.2 and \( |d_1| \leq 1 \), we have

\[ |b_2 - ub_1^2| \leq Q_1(k)[1 + |D(k) - uQ_1(k) - 1||d_1|^2] \leq Q_2(k) - uQ_1(k)^2. \quad (3.6) \]

This is the last inequality in (3.1).

Lastly, if \( \alpha_2(k) \leq u \leq \alpha_1(k) \), then

\[ |D(k) - uQ_1(k)| \leq 1. \]

Therefore, (3.4) yields

\[ |b_2 - ub_1^2| \leq Q_1(k)[|d_2| + |d_1|^2] \leq Q_1(k)[1 - |d_1|^2 + |d_1|^2] = Q_1(k). \]

We get the middle inequality in (3.1).

We next discuss the sharpness of the inequalities in (3.1). Suppose \( u > \alpha_1(k) \). Then equality holds in (3.1), i.e. in (3.5) if \( d_1^2 = -1 \) and \( |d_2 - d_1^2| = 1 \). Therefore, \( w(z) = iz \) and the extremal function is \( q_k(iz) \).

Next, if \( u < \alpha_2(k) \), equality holds in (3.1), i.e. in (3.6) if \( |d_1| = 1 \) (and hence \( d_2 = 0 \)). Thus, \( w(z) \) is a rotation of \( z \) and the extremal function is a rotation of \( q_k(z) \).

Lastly, if \( \alpha_2(k) \leq u \leq \alpha_1(k) \), then equality holds in (3.1) if \( d_1 = 0 \) and \( |d_2| = 1 \). Therefore \( w(z) \) is a rotation of \( z^2 \) and \( h(z) = q_k(e^{i\theta}z^2) \). The proof of Theorem 3.1 is complete. \( \square \)

Remark 3.2. For \( 0 < k < \infty \) it is known (cf. [1]) that \( Q_1(k) - D(k) \leq 1 \). Therefore, \( \alpha_1(k) \geq 1 \) where \( \alpha_1(k) \) is defined by (3.2). Our Theorem 3.1, refines and sharpens the first and last estimates in (1.5) and determines the exact ranges of \( u \) for each case of (1.5).

Let the functions \( F(z) \) and \( G(z) \) in \( k-\mathcal{UCV} \) be defined, for \( z \in U \), respectively by

\[
1 + \frac{zF''(z)}{F'(z)} = q_k(z), \quad F'(0) = 1, \quad F(0) = 0
\]

and

\[
1 + \frac{zG''(z)}{G'(z)} = q_k(z^2), \quad G'(0) = 1, \quad G(0) = 0.
\]

It is easily seen that for \( z \in U \),

\[
F(z) = z + \frac{Q_1(k)}{2}z^2 + \frac{1}{6}(Q_1^2(k) + Q_2(k))z^3 + \cdots = z + A_2(k)z^2 + A_3(k)z^3 + \cdots \quad (3.7)
\]

and

\[
G(z) = z + \frac{Q_1(k)}{6}z^2 + \cdots = z + B_3(k)z^3 + \cdots. \quad (3.8)
\]

In the following theorem we present a shortened and direct proof for the Fekete-Szegö inequalities for the class \( k-\mathcal{UCV} \) found recently by the authors [7,8].
Theorem 3.3. Let the function \( f \) given by (1.6) be in the class \( k - \mathcal{U}C^V \) \((0 \leq k < \infty)\). Then

\[
|\mu a_2^2 - a_3| \leq \begin{cases} 
A_1(k) - \mu A_2(k), & \mu > \delta_1(k); \\
B_1(k), & \delta_2(k) \leq \mu \leq \delta_1(k); \\
\mu A_2(k) - A_3(k), & \mu < \delta_2(k);
\end{cases}
\]

(3.9)

where

\[
\delta_1(k) = \frac{2}{3} \left[ 1 + \frac{D(k) + 1}{Q_1(k)} \right],
\]

\[
\delta_2(k) = \frac{2}{3} \left[ 1 + \frac{D(k) - 1}{Q_1(k)} \right],
\]

and \( A_2(k), A_3(k), B_3(k) \) and \( D(k) \) are given by (3.7), (3.8) and (2.5). All the estimates in (3.9) are sharp.

Proof. By definition there exists a function \( h \in \mathcal{A} \), represented by (1.2) and subordinate to \( q_k \), such that

\[
1 + \frac{zf''(z)}{f'(z)} = h(z), \quad (z \in \mathcal{U}).
\]

Substituting the corresponding series expansions and by equating coefficients we get

\[
a_2 = \frac{b_1}{2}, \quad a_3 = \frac{1}{6} (b_1^2 + b_2).
\]

Therefore

\[
|\mu a_2^2 - a_3| = \frac{1}{6} \left| b_2 - \left( \frac{3\mu}{2} - 1 \right) b_1^2 \right|.
\]

An application of Theorem 3.1, with \( u = \left( \frac{3}{2} \mu - 1 \right) \), gives

\[
|\mu a_2^2 - a_3| \leq \begin{cases} 
\frac{1}{6} \left( \frac{3}{2} \mu - 1 \right) Q_1(k)^2 - Q_2(k), & \mu > \delta_1(k); \\
\frac{1}{6} Q_1(k), & \delta_2(k) \leq \mu \leq \delta_1(k); \\
\frac{1}{6} Q_2(k) - \left( \frac{3}{2} \mu - 1 \right) Q_1(k)^2, & \mu < \delta_2(k).
\end{cases}
\]

(3.10)

By using (3.7) and (3.8), it can be easily verified that the estimate (3.10) is equivalent to (3.9). The proof of Theorem 3.3 is complete. \( \square \)

Remark 3.4. Upon substitution of the values of \( Q_1(k), Q_2(k) \) from Lemma 2.1 for the ranges \( k = 1, 0 < k < 1, k > 1 \) into the expressions for \( A_2(k), A_3(k) \) and \( B_3(k) \), our Theorem 3.3 yields the Fekete–Szegö inequalities for \( \mathcal{U}C^V \) and \( k - \mathcal{U}C^V \) \((0 < k < 1, k > 1)\) found in [6–8].

References