θ-Euclidean k-Fuzzy Ideals of Semirings

D.R Prince Williams

Abstract—In this paper, we introduce the notion θ-Euclidean k-fuzzy ideal in semirings and to study the properties of the image and pre image of a θ-Euclidean k-fuzzy ideal in a semirings under epimorphism.

Keywords—semiring, fuzzy ideal, k-fuzzy ideal, θ-Euclidean L-fuzzy ideal, θ-Euclidean fuzzy k-ideal, θ-Euclidean k-fuzzy ideal.

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I. INTRODUCTION


Ayten Koç, Erol Balkanay [7, 8] introduced a concept of θ-Euclidean L-fuzzy ideals, θ-Euclidean left subset in rings and studied the properties of ideals θ-Euclidean L-fuzzy ideals, θ-Euclidean left subset in rings. C.B Kim et al [10] introduce the k-fuzzy ideal of semirings and studied the properties of the image and pre image of a k-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy k-ideals in k-semirings.

The purpose of this paper is to introduce θ-Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a θ-Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a θ-Euclidean k-fuzzy ideal.

II. PRELIMINARIES

An algebra (S, +, .) is said to be a semiring if (S, +) and (S, .) are semigroup satisfying a(b + c) = ab + ac and (b + c)a = ba + ca, for all a, b, c ∈ S. A semiring S may have an identity 1, defined by 1a = a = a1 and a zero 0, defined by 0 + a = a = a + 0 and 0a = 0 = a0 for all a ∈ S. A non-empty subset I of S is said to be left (resp., right) ideal if x, y ∈ I and r ∈ S imply that x + y ∈ I and rx ∈ I (resp.,xr ∈ I). If I is both left and right ideal of S, we say I is a two-sided ideal, or simply ideal, of S. A left ideal I of a semiring S is said to be a left k-ideal if a ∈ I and x ∈ S and if a + x ∈ I or x + a ∈ I then x ∈ I. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.

Definition 2.1 [10]: Let K and S be any sets and let f : K → S be a function. A fuzzy subset μ of K is called f-invariant if f(x) = f(y) implies μ(x) = μ(y), where x, y ∈ K.

Definition 2.2 [2]: A fuzzy subset μ of a semiring S is said to be fuzzy left (resp., right) ideal of S if

(i) μ(x + y) ≥ min{μ(x), μ(y)} and
(ii) μ(xy) ≥ μ(y) (resp., μ(xy) ≥ μ(x))

for all x, y ∈ S. If μ is a fuzzy ideal of S if it is both fuzzy left and a fuzzy right ideal of S.

Definition 2.3 [10]: A fuzzy ideal μ of a semiring S is said to be a k-fuzzy ideal of S μ(x + y) = μ(0) and μ(y) = μ(0) imply μ(x) = μ(0), for all x, y ∈ S.

Definition 2.4 [8]: Let θ : S → [0,1] and μ : S → [0,1] be a fuzzy subsets of S. For any, 0 ≠ y ∈ S the set
\( \mu_{x+y} = \left\{ \begin{array}{ll} x \in S & \text{there exists } q, r \in S \text{ such that } x = yq + r \\ \text{where either } r = 0 \text{ or else } \mu(r) \geq \max \{ \mu(y), \theta(y) \} & \end{array} \right. \)

is called a \( \theta \)-Euclidean level subset of \( \mu \).

II. \( \theta \)-EUCLIDEAN K-FUZZY IDEALS

**Definition 3.1:** Let \( S \) be a semiring and let \( \theta : S \rightarrow [0,1] \) be a non–constant fuzzy subset of \( S \). A fuzzy ideal \( \mu : S \rightarrow [0,1] \) of \( S \) is called a \( \theta \)-Euclidean k-fuzzy ideal if \( \mu \) satisfies the following axioms

(i) \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \) imply \( \mu(x) = \mu(0) \), for all \( x, y \) in \( R \).

(ii) For any \( x, y \in R \) with \( y \neq 0 \), there exists elements \( q, r \in R \) such that \( x = yq + r \) where either \( r = 0 \) or else \( \max \{ \mu(r), \theta(r) \} \geq \max \{ \mu(y), \theta(y) \} \).

**Example 3.2:** Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \rightarrow [0,1] \) be a fuzzy subset defined by

\[ \mu(a) = \left\{ \begin{array}{ll} 1 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even}, \\ 0 & \text{if } a \text{ is odd}. \end{array} \right. \]

Let \( \theta : S \rightarrow [0,1] \) be a fuzzy subset defined by

\[ \theta(a) = \left\{ \begin{array}{ll} 0 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a = 3, 5, 7, ..., \\ \frac{1}{2} & \text{otherwise}. \end{array} \right. \]

Clearly \( \mu \) is a \( k \)-fuzzy ideal of \( S \), also \( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S \).

**Example 3.3:** Let \( S \) be the set of Natural Numbers including zero and \( \mu : S \rightarrow [0,1] \) be a fuzzy set defined by

\[ \mu(a) = \left\{ \begin{array}{ll} 1 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even}, \\ 0 & \text{if } a \text{ is odd}. \end{array} \right. \]

Let \( \theta_1 : S \rightarrow [0,1] \) be a fuzzy subset defined by

\[ \theta_1(a) = \left\{ \begin{array}{ll} 0 & \text{if } a = 0 \\ \frac{1}{2} & \text{otherwise}. \end{array} \right. \]

So \( \mu \) is a \( k \)-fuzzy ideal but \( \mu \) is not a \( \theta_1 \)-Euclidean \( k \)-fuzzy ideal of \( S \).

**Theorem 3.4:** Let \( A \) be a non empty subset of \( S \). Let \( \mu \) be a fuzzy subset of a semiring \( S \) such that \( \mu \) is into \( [0,1] \), so that \( \mu \) is the characteristic function of \( A \). Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \) then \( A \) is a left ideal of \( S \).

Proof: The proof is easy and straightforward. \( \square \)

**Theorem 3.5:** Let \( \mu \) be a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \). Then for \( 0 \neq y \in S \), (i) \( \theta_0(y) \) is an ideal of \( S \) \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \) then \( A \) is a left ideal of \( S \).

Proof: The proof is similar to [8, Theorem 3.3]. \( \square \)

**Theorem 3.6:** Let \( \mu \) be a fuzzy ideal of a semiring \( S \). If \( \mu_{\theta_y} \) and \( \theta_{\mu_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \).

Proof: Suppose \( \mu \) is fuzzy ideal of semiring \( S \). For \( x, y \in S \), if \( \mu(x + y) = \mu(0) \) and \( \mu(y) = \mu(0) \), then \( \mu(x + y) \geq \min \{ \mu(x), \mu(y) \} \), since \( \mu \) is fuzzy ideal of \( S \).

Thus \( \mu \) is a \( k \)-fuzzy ideal of semiring \( S \).

We have \( \mu_{\theta_y} \) and \( \theta_{\mu_y} \) is the Euclidean level set of \( \mu \) and \( \theta \) respectively. Then, for \( x, y \in S \), with \( 0 \neq y \), there exists \( q, r \in S \) such that \( x = yq + r \) where either \( r = 0 \) or else \( \mu(r) \geq \max \{ \mu(y), \theta(y) \} \). Thus \( \mu(r) \geq \max \{ \mu(y), \theta(y) \} \).

Hence \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of a semiring \( S \). \( \square \)

**Definition 3.7** ([10]): Let \( f : S \rightarrow S' \) be a homomorphism of semirings. Let \( \mu \) be a fuzzy subset of \( S' \). We define a fuzzy subset \( f^{-1} \mu \) of \( S \) by \( f^{-1} \mu(x) = \mu(f(x)) \) for all \( x \in S \).

**Theorem 3.7:** Let \( f : S \rightarrow S' \) be an epimorphism of semirings and \( \mu \) be a fuzzy ideal of \( S' \). Then \( \mu \) is a \( \theta \)-Euclidean k-fuzzy ideal of \( S' \) if and only if \( f^{-1}(\mu) \)
is a $f^{-1}(\theta)$-Euclidean k-fuzzy ideal of fuzzy ideal of $S$.

**Proof:** Suppose $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$. 
(i) For all $x, y \in S'$

$$f^{-1}_\mu(x + y) = f^{-1}_\mu(f(x) + f(y)) \geq \min \{ f^{-1}_\mu(f(x)), f^{-1}_\mu(f(y)) \} = \min \{ f^{-1}_\mu(x), f^{-1}_\mu(y) \}$$

(ii) For all $x, y \in S'$

$$f^{-1}_\mu(xy) = f^{-1}_\mu(f(xy)) \geq \max \{ f^{-1}_\mu(f(x)), f^{-1}_\mu(f(y)) \} = \max \{ f^{-1}_\mu(x), f^{-1}_\mu(y) \}$$

(iii) For all $x, y \in S'$, if $f^{-1}_\mu(x + y) = f^{-1}_\mu(0)$

and $f^{-1}_\mu(y) = f^{-1}_\mu(0)$ then

$$f^{-1}_\mu(x) = f^{-1}_\mu(x) = \mu(x) = \mu(0) = f^{-1}_\mu(0).$$

(iv) We have $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$, then for any $x, y \in S'$, there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y) f(q) + f(r)$ where either $f(r) = 0$ or else

$$\max \{ \mu(f(q)), \mu(f(r)) \} \geq \max \{ \mu(f(r)), \mu(f(r)) \}.$$ 

Thus $f(x) = f(yq + r)$ where either $f(r) = 0$ or else

$$\max \{ f^{-1}_\mu(y), f^{-1}_\theta(r) \} \geq \max \{ f^{-1}_\mu(r), f^{-1}_\theta(r) \}.$$ 

Hence for any $x, y \in S'$ there exists elements $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else

$$\max \{ f^{-1}_\mu(y), f^{-1}_\theta(r) \} \geq \max \{ f^{-1}_\mu(r), f^{-1}_\theta(r) \}.$$ 

Conversely, suppose $f^{-1}_\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S$.

(i) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$.

$$f^{-1}_\mu(ab) = f^{-1}_\mu(f(x)f(y)) \geq \max \{ f^{-1}_\mu(f(x)), f^{-1}_\mu(f(y)) \} = \max \{ \mu(a), \mu(b) \}.$$ 

(ii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$.

$$\mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}_\mu(xy) \geq \max \{ f^{-1}_\mu(x), f^{-1}_\mu(y) \} = \max \{ \mu(x), \mu(y) \} = \max \{ \mu(a), \mu(b) \}.$$ 

(iii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$, if $\mu(a + b) = \mu(0)$ and $\mu(b) = \mu(0)$ imply

$$\mu(a) = \mu(f(x)) = f^{-1}_\mu(f(x)) = f^{-1}_\mu(0) = \mu(0)$$

(iv) For any $x, y, q, r \in S$ then

$a = f(x), b = f(y), c = f(q), d = f(r) \in S'$.

We have $f^{-1}_\mu$ is a $\theta$-Euclidean k-fuzzy ideal of fuzzy ideal of $S$, then there exists $q, r \in S$ such that $x = yq + r$ either $r = 0$ or else

$$\max \{ f^{-1}_\mu(y), f^{-1}_\theta(r) \} \geq \max \{ f^{-1}_\mu(r), f^{-1}_\theta(r) \}.$$ 

That is $f(x) = f(yq + r)$ either $f(r) = 0$ or else $x = f(yq) + f(r)$ where either $f(r) = 0$ or else

$$\max \{ f^{-1}_\mu(y), f^{-1}_\theta(r) \} \geq \max \{ f^{-1}_\mu(r), f^{-1}_\theta(r) \}.$$ 

Thus there exists $c, d \in S'$ such that $a = bc + d$ either $r = 0$ or else

$$\max \{ f^{-1}_\mu(y), f^{-1}_\theta(r) \} \geq \max \{ f^{-1}_\mu(r), f^{-1}_\theta(r) \}.$$ 

**Definition 3.8:** Let $f: S \rightarrow S'$ be an homomorphism of the semirings. Let $\mu$ be a fuzzy subset of $S$, we define a fuzzy subset $f(\mu)$ of $S'$ by

$$f(\mu)(y) = \begin{cases} \sup \{ \mu(t) \mid t \in R, f(t) = y \} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

**Theorem 3.9:** Let $f: S \rightarrow S'$ epimorphism of semirings. Let $\mu$ be a $f$-invariant $\theta$-Euclidean k-fuzzy ideal of $S$. Then $f(\mu)$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$.

**Proof:** Suppose $x, y \in S'$ such that $x = f(a), y = f(b)$, for all $a, b \in S$. Then $x + y = f(a) + f(b) = f(a + b)$ and $xy = f(ab)$ hold. Since $\mu$ is $f$-invariant

Thus

(i) $f(\mu)(x + y) = f(\mu)f(a + b) = \sup \{ \mu(t) \mid t \in S, f(t) = f(a + b) \}$
is a \( k \)-fuzzy ideal of \( S \).

\( \mu \) is a \( \theta \)-Euclidean \( k \)-fuzzy ideal of \( S' \).

\[ \mu \circ f : S \to [0,1] \]

\[ \mu' \circ f : S \to [0,1] \]

\[ S \]

\[ S' \]

\[ \mu' \]

\[ \mu \]

It was proved that \( \mu \) is a fuzzy ideal of \( S \) [5] and \( \mu \) is a \( \theta \)-Euclidean fuzzy ideal of \( S \) [7].

Hence, \( \mu' \circ f : S \to [0,1] \) is a \( (\theta' \circ f) \)-Euclidean \( k \)-fuzzy ideal of \( S \).

REFERENCES


D.R. Prince Williams, received his Masters degree in Mathematics (1991) and Master of Philosophy in Mathematics (1992) from Pachiyappa’s College, University of Madras, Chennai, India. He received his PhD (1999-2004) from Department of Mathematics, Anna University, and Chennai, India. From 1992 to 2004 he worked as Mathematics Faculty in various Engineering Colleges in Chennai, India. From 2005, he is working as Mathematics Faculty in Department Information Technology, Salalah College of Technology, Salalah, Sultanate of Oman. He published many papers in national and international journals. His research interests are in the areas of Fuzzy Algebraic Structures, Software Reliability and Mathematical Modeling.