Full-rank representations of \(\{2,4\}\), \(\{2,3\}\)-inverses and successive matrix squaring algorithm

Predrag S. Stanimirović\(^1\), Dragana S. Cvetković-Ilić\(^2\), Sladjana Miljković\(^3\), Marko Miladinović\(^4\)

University of Niš, Department of Mathematics, Faculty of Science, P.O. Box 224, Višegradska 33, 18000 Niš, Serbia

E-mail: \(^1\)pecko@pmf.ni.ac.rs, \(^2\)gagamaka@ptt.rs, \(^3\)slagana256@yahoo.com, \(^4\)markomiladinovic@gmail.com

Abstract

We present the full-rank representations of \(\{2,4\}\) and \(\{2,3\}\)-inverses (with given rank as well as with prescribed range and null space) as particular cases of the full-rank representation of outer inverses. As a consequence, two applications of the successive matrix squaring (SMS) algorithm from [P.S. Stanimirović, D.S. Cvetković-Ilić, Successive matrix squaring algorithm for computing outer inverses, Appl. Math. Comput. 203 (2008), 19–29] are defined using the full-rank representations of \(\{2,4\}\) and \(\{2,3\}\)-inverses. The first application is used to approximate \(\{2,4\}\)-inverses. The second application, after appropriate modifications of the SMS iterative procedure, computes \(\{2,3\}\)-inverses of a given matrix. Presented numerical examples clarify the purpose of the introduced methods.

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1 Introduction and preliminaries

Let \(\mathbb{C}^{m \times n}\) and \(\mathbb{C}^{m \times n}_r\) denote the set of all complex \(m \times n\) matrices and all complex \(m \times n\) matrices of rank \(r\), respectively. \(I\) denotes the unit matrix of an appropriate order. By \(A^*\), \(\mathcal{R}(A)\), \(\text{rank}(A)\) and \(\mathcal{N}(A)\) we denote the conjugate transpose, the range, the rank and the null space of \(A \in \mathbb{C}^{m \times n}\).

The problem of pseudoinverses computation leads us to the four, so called, Penrose equations

\[
\begin{align*}
(1) & \quad AXA = A & (2) & \quad XAX = X & (3) & \quad (AX)^* = AX & (4) & \quad (XA)^* =XA.
\end{align*}
\]

For a subset \(S\) of the set \(\{1,2,3,4\}\), the set of all matrices obeying the conditions contained in \(S\) is denoted by \(A\{S\}\). Any matrix from \(A\{S\}\) is called \(S\)-inverse of \(A\) and is denoted by \(A^{(S)}\). By \(A\{S\}_s\) we denote the set of all \(S\)-inverses of \(A\) with rank \(s\). For any matrix \(A\) there exists a single element in the set \(A\{1,2,3,4\}\), called the Moore-Penrose inverse of \(A\) and denoted by \(A^\dagger\). For other important properties of generalized inverses see [1, 13].

\(^1\)Corresponding author

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Let the Drazin and the group inverse representation, restated in the next proposition, gave a new computational aspect for appropriate choice of matrices $M, N$ where $M, N$ are positive definite matrices of appropriate orders and $A$ is a subspace of $\mathbb{C}_n^{m \times n}$, $\text{rank}(GAF) = s$.

It is known that the sets $A[2]_s = \{ F(GAF)^{-1}G \mid F \in \mathbb{C}_n^{m \times s}, G \in \mathbb{C}_s^{s \times m}, \text{rank}(GAF) = s \}$; $A[2] = \cup_{s=0}^r A[2]_s$; $A[2, 4]_s = \{(VA)^{\dagger}V \mid V \in \mathbb{C}_s^{s \times m}, VA \in \mathbb{C}_s^{s \times n} \}$; $A[2, 3]_s = \{ (U(A)^{\dagger}U \mid U \in \mathbb{C}_s^{m \times m}, AU \in \mathbb{C}_s^{m \times s} \}$; $A[1, 2] = \{ F(GAF)^{-1}G \mid F \in \mathbb{C}_n^{n \times r}, G \in \mathbb{C}_r^{r \times m}, \text{rank}(GAF) = r \} = A[2]_r$; $A[1, 2, 4] = \{ N^{\dagger}(VAN^{\dagger})^{-1}V \mid V \in \mathbb{C}_s^{s \times m}, \text{rank}(VAN^{\dagger}) = r \} = \{(VA)^{\dagger}V \mid VA \in \mathbb{C}_s^{s \times n} \}$; $A[1, 2, 3] = \{ (M^{\dagger}AU)^{-1}M^{\dagger} \mid U \in \mathbb{C}_s^{n \times r}, \text{rank}(M^{\dagger}AU) = r \} = \{(U(A)^{\dagger}U \mid AU \in \mathbb{C}_s^{n \times r} \}$; $A[1, 2, 3] = N^{\dagger}(M^{\dagger}AN^{\dagger})^{-1}M^{\dagger}$.

If $m = n$, $A[1, 2] = F_A(G_A^{-1}A^{-1}F_A^{\dagger})^{-1}G_A^{\dagger}$, $A[1, 2] = F_A^{\dagger}G_A^{\dagger}, l \geq \text{ind}(A), A[1, 2] = F_A^{\dagger}G_A^{\dagger}$.

It is known that the sets $A[2]_0, A[2, 3]_0, A[2, 4]_0$ and $A[2, 3, 4]_0$ are identical and contain a single element, the $n \times m$ zero matrix. For this purpose, it suffices to consider only positive $s$.

If $A \in \mathbb{C}_n^{m \times n}$, $T$ is a subspace of $\mathbb{C}_n^m$ of dimension $t \leq r$ and $S$ is a subspace of $\mathbb{C}_m$ of dimension $m - t$, then $A$ has a $(2)$-inverse $X$ such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if $AT \oplus S = \mathbb{C}_m$, in which case $X$ is unique and we denote it by $A[2]_{T \oplus S}$.

It is well known that the Moore-Penrose inverse and the weighted Moore-Penrose inverse $A[1]_{M,N}$, the Drazin and the group inverse $A[2]_D, A[2]_N$, as well as the Bott-Duffin inverse $A^\dagger(L)$ and the generalized Bott-Duffin inverse $A^\dagger(L)$ can be presented by a unified approach, as generalizations $A^\dagger(2)_{T \oplus S}$ for appropriate choice of matrices $T$ and $S$. For example, the next is valid for a rectangular matrix $A$ [1]:


where $M, N$ are positive definite matrices of appropriate orders and $A[2] = N^{-1}A^\dagger M$. For a given square matrix $A$ the next identities are satisfied [1, 3, 13]:

$$A[2]_{R(A^\dagger),N(A^\dagger)}; A[2]_{R(A^\dagger),N(A^\dagger)};$$

where $k = \text{ind}(A)$. If $A$ is a given square $L$-p.s.d. matrix and $L$ is a subspace of $\mathbb{C}_n$ which satisfies $AL \oplus L^\perp = \mathbb{C}_n$, $S = \mathcal{R}(P_L A)$, then the next identities are satisfied [3, 13, 18]:


A useful representation for $A^\dagger(2)_{T \oplus S}$ based on the usage of the group inverse is presented in [20]. This representation, restated in the next proposition, gave a new computational aspect for $A^\dagger(2)_{T \oplus S}$ inverse.
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**Proposition 1.2** [20] Let \( A \in \mathbb{C}^{m \times n} \) be of rank \( r \), let \( T \) be a subspace of \( \mathbb{C}^{n} \) of dimension \( s \leq r \), and let \( S \) be a subspace of \( \mathbb{C}^{m} \) of dimension \( m - s \). In addition, suppose that \( G \in \mathbb{C}^{n \times m} \) satisfies \( R(G) = T \) and \( N(G) = S \). If \( A \) has \( A_{T,S}^{\{2\}} \) then \( \text{ind}(AG) = \text{ind}(GA) = 1 \) and

\[
A_{T,S}^{\{2\}} = G(AG)^\# = (GA)^\# G.
\]

This result contains particular results corresponding to the Drazin inverse [19]. A general integral representation for \( A_{T,S}^{\{2\}} \) is introduced in [16]. The authors in [17] introduced and used the \( PQ \)-norm to derive some results on the condition number of the generalized inverse \( A_{T,S}^{\{2\}} \).

Full-rank representation of \( \{2\} \)-inverses with prescribed range and null space is determined in the next proposition, which originated in [7].

**Proposition 1.3** [7] Let \( A \in \mathbb{C}^{m \times n} \), \( T \) be a subspace of \( \mathbb{C}^{n} \) of dimension \( s \leq r \) and let \( S \) be a subspace of \( \mathbb{C}^{m} \) of dimensions \( m - s \). In addition, suppose that \( R \in \mathbb{C}^{n \times m} \) satisfies \( R(R) = T, N(R) = S \). Let \( R \) has an arbitrary full-rank decomposition, that is \( R = FG \). If \( A \) has a \( \{2\} \)-inverse \( A_{T,S}^{\{2\}} \), then:

1. \( GAF \) is an invertible matrix;
2. \( A_{T,S}^{\{2\}} = F(GAF)^{-1}G \).

Representation of outer inverses in the general form \( F(GAF)^{-1}G \), is applicable in numerical calculations. For example, such a representation has been exploited to define the determinantal representation of \( A_{T,S}^{\{2\}} \) inverse in [7] or the set \( A_{\{2\}} \) in [8]. Also, this representation has been used in the construction of the general successive matrix squaring algorithm for computing \( A_{T,S}^{\{2\}} \) [12] or in the block representation of the set \( A_{\{2\}} \) [9]. Moreover, algorithms for computing \( A_{T,S}^{\{2\}} \) inverse are immediately applicable to the most important generalized inverses.

On the other hand, the general representations of \( \{2,4\} \) and \( \{2,3\} \)-inverses of the form \((VA)^{\dagger}V \) and \( U(UA)^{\dagger} \), respectively, are not widely exploited in the literature. Several modifications of the hyper-power method are used in computation of \( \{2,3\} \) and \( \{2,4\} \)-inverses in [10]. Various representations of \( \{2,3\} \) and \( \{2,4\} \)-inverses with prescribed range and null space has been investigated [5, 7, 21, 22]. The expressions for \( \{2,3\} \) and \( \{2,4\} \)-inverses of a normal matrix by its Schur decomposition are discussed in [23]. But, these representations are not exploited in developing of some effective computational procedures.

For this reason, our main goal is to determine full-rank representations of \( \{2,4\} \) and \( \{2,3\} \)-inverses with prescribed range and null space as particular cases of the full-rank representation for generalized inverses \( A_{T,S}^{\{2\}} \). We also define full-rank representations of the sets \( A_{\{2,4\}} \) and \( A_{\{2,3\}} \) as particular cases of the full-rank representation of the set \( A_{\{2\}} \). Introduce full-rank representations enable adaptation of well-known algorithms for computing outer inverses with prescribed range and null space into corresponding algorithms for computing \( \{2,4\} \) and \( \{2,3\} \)-inverses. In this paper we derive an adaptation of the successive matrix squaring algorithm from [12].

The successive matrix squaring algorithm, which is shown to be an equivalent to Shultz method [2], attracted the attention of many researchers and this technique was used in different papers for computing various generalized inverses [2, 14, 15]. Finally, this concept was generalized in a unified successive matrix squaring algorithm for computing \( \{2\} \)-inverses with prescribed range and null space [12]. Obviously this representation enables the computation of the most commonly used generalized inverses, such as the Moore-Penrose inverse, weighted Moore-Penrose inverse, the Drazin inverse etc. Chen et all in [2] observed that with minor modifications it is possible to adapt the SMS algorithm to find the solutions of the linear system \( Ax = b \).
Motivated by the SMS method for computing \( \{2\} \)-inverses with prescribed range and null space we further extend the capability of SMS method. Our intention in the present paper is focused on computing \( \{2,4\} \) and \( \{2,3\} \)-generalized inverses using this method. The algorithm for computing \( \{2,4\} \)-inverses can be reduced to a particular case of the classical SMS iterative scheme; on the other hand, the method for the construction of \( \{2,3\} \)-inverses requires a dual form of the successive matrix squaring method.

Two applications of the SMS algorithm are investigated in the third section. These results are established upon the full-rank representations of \( \{2,4\} \) and \( \{2,3\} \)-inverses, which are given in the second section. Finally, the numerical examples are presented in the fourth section.

## 2 Full-rank representation of \( \{2,4\} \) and \( \{2,3\} \)-inverses

In Lemma 2.1 we exactly distinguish sets \( A\{2,4\} \) and \( A\{2,3\} \) as subsets of the set \( A\{2\} \).

**Lemma 2.1.** Let \( A \in \mathbb{C}_{r \times n}^n \) be the given matrix and \( 0 < s \leq r \) a chosen integer. Assume that \( V \) and \( U \) are two arbitrary matrices satisfying \( \text{rank}(VA) = \text{rank}(V) \) and \( \text{rank}(AU) = \text{rank}(U) \). Then the following statements are valid:

- (a) \( A\{2,4\} = \{(VA)^*(VA(VA)^*)^{-1}V \mid V \in \mathbb{C}_{s \times m}^s \} \);
- (b) \( A\{2,3\} = \{U((AU)^*AU)^{-1}(AU)^* \mid U \in \mathbb{C}_{s \times s}^s \} \);
- (c) \( A\{1,2,4\} = \{(VA)^*(VA(VA)^*)^{-1}V \mid V \in \mathbb{C}_{r \times m}^r \} = A\{2,4\} \);
- (d) \( A\{1,2,3\} = \{U((AU)^*AU)^{-1}(AU)^* \mid U \in \mathbb{C}_{r \times r}^r \} = A\{2,3\} \).

**Proof.** The proof for the parts (a), (b) follows immediately from

\[
(VA)^*(VA(VA)^*)^{-1}V = (VA)^*((VA)^*)^{\dagger}(VA)^{\dagger}V = (VA)^{\dagger}V
\]

and

\[
U((AU)^*AU)^{-1}(AU)^* = U(AU)^{\dagger}((AU)^*)^{\dagger}(AU)^* = U(AU)^{\dagger}
\]

together with the general representations of the sets \( A\{2,4\} \) and \( A\{2,3\} \) from Proposition 1.1.

Representations (c) and (d) follows from well-known fact that \( X = F(GAF)^{(1)}G \in A\{1\} \) if and only if \( \text{rank}(GAF) = r \) (see, for example [13], Theorem 1.3.7). \( \square \)

Therefore, we conclude that \( \{2,4\} \)-inverses can be derived from the set of outer inverses in the particular case \( F = (VA)^* \) and \( G = V \). Similarly, \( \{2,3\} \)-inverses can be derived in the particular case \( F = U \) and \( G = (AU)^* \). Practically, the following general representations hold:

\[
A\{2,4\} = \{A(2) \in \mathbb{C}_{s \times m}^s \mid A(2) = F(GAF)^{-1}G, \ F = (VA)^*, G = V, \ V \in \mathbb{C}_{s \times m}^s, \ V \in \mathbb{C}_{s \times s}^s \};
\]

\[
A\{2,3\} = \{A(2) \in \mathbb{C}_{s \times m}^s \mid A(2) = F(GAF)^{-1}G, \ G = (AU)^*, F = U, \ U \in \mathbb{C}_{s \times s}^s, \ AU \in \mathbb{C}_{s \times s}^s \}.
\]

As we mentioned, the generalized inverses \( A^1, A^1_{M,N}, A^D, A^\# \) are identical with the generalized inverse \( A(2)_{T,S} \) for appropriate choices of the matrix \( R \) which is exploited in Proposition 1.3. More precisely, these pseudo-inverses can be derived in particular cases \( R = A^*, R = A^2 = N^{-1}A^*M, R = A^l, l \geq \text{ind}(A) \) and \( R = A, \) respectively (see, for example [1]). In the rest of this section we
give an answer to the problem of finding appropriate values for the matrix \( R \) which lead to \( \{2, 4\} \) and \( \{2, 3\} \)-inverses with prescribed range and null space. It follows from the following two statements.

Motivated by Proposition 1.3, we find alternative representations of \( \{2, 4\} \) and \( \{2, 3\} \)-inverses with prescribed range and null space in the general form which characterizes the set of outer inverses: \( F(GAF)^{-1}G \), where \( F \in \mathbb{C}^{n \times s} \), and \( G \in \mathbb{C}^{s \times m} \), \( s \leq \text{rank}(A) \).

**Theorem 2.1** For arbitrary matrix \( A \in \mathbb{C}_{r \times n}^m \) and arbitrary integer \( s \) satisfying \( 0 < s \leq r \) we have

\[
\begin{align*}
(a) \quad A[2, 4]_s &= \left\{ A_{N(VA)^{\perp}, N(V)}^{(2, 4)} \mid V \in \mathbb{C}^{s \times m}_s, \text{rank}(VA) = \text{rank}(V) \right\} \\
(b) \quad A[2, 3]_s &= \left\{ A_{R(U), R(\text{null})}^{(2, 3)} \mid U \in \mathbb{C}^{s \times m}_s, \text{rank}(AU) = \text{rank}(U) \right\}.
\end{align*}
\]

**Proof.** Let us choose an arbitrary element \( X \in A[2, 4]_s \). According to Proposition 1.1, \( X \) is of the form

\[
X = (VA)^\dagger V, \quad V \in \mathbb{C}^{s \times m}_s, \quad \text{rank}(VA) = \text{rank}(V).
\]

Since

\[
\text{rank}(X) = \text{rank}((VA)^\dagger) = \text{rank}(VA) = \text{rank}(V)
\]

we conclude

\[
R(X) = R((VA)^\dagger) = N(VA)\perp, \quad N(X) = N(V)
\]

and verify

\[
X \in \left\{ A_{N(VA)^{\perp}, N(V)}^{(2, 4)} \mid V \in \mathbb{C}^{s \times m}_s, \text{rank}(VA) = \text{rank}(V) \right\}.
\]

To verify the opposite inclusion, assume that

\[
X = A_{N(VA)^{\perp}, N(V)}^{(2, 4)}
\]

for the selected matrix \( V \in \mathbb{C}^{s \times m}_s \) satisfying \( \text{rank}(VA) = \text{rank}(V) \). According to Proposition 1.3, \( X \) is of the form

\[
X = F(GAF)^{-1}G, \quad R(F) = N(VA)\perp = R((VA)^\dagger), \quad N(G) = N(V).
\]

For example, it is possible to choose \( F = (VA)^\dagger \), \( G = V \). According to Lemma 2.1, we get \( X \in A[2, 4]_s \).

The part (a) of the proof is completed. The dual statement for \( \{2, 3\} \)-inverses can be verified in a similar way. \( \square \)

A correlation between outer inverses with prescribed range and null space with \( \{2, 4\} \) and \( \{2, 3\} \)-inverses with prescribed range and null space is determined in the next statement.

**Corollary 2.1** Let \( A \in \mathbb{C}^{m, n} \) be given matrix and \( R \in \mathbb{C}^{m, n} \), \( s \leq r \), be arbitrary but fixed matrix. Assume that \( F \in \mathbb{C}^{s \times s} \) and \( G \in \mathbb{C}^{s \times m} \) form the full-rank factorization \( R = FG \). Let \( R(R) = T \) be a subspace of \( \mathbb{C}^n \) of dimension \( s \leq r \) and \( N(R) = S \) be a subspace of \( \mathbb{C}^m \) of dimensions \( m - s \). Then the following statements are satisfied:

1. In the case \( R = (VA)^\dagger V \), (or \( F = (VA)^\dagger \), \( G = V \), \( V \in \mathbb{C}^{s \times m} \) is an arbitrary matrix), the outer inverse \( A_{T,S}^{(2)} \) \( F(GAF)^{-1}G \) reduces to the \( \{2, 4\} \)-inverse

\[
A_{R(((VA)^\dagger)\perp, N(V)}^{(2, 4)} = A_{N(VA)^{\perp}, N(V)}^{(2, 4)}(VA)((VA)^\dagger)^{-1}V.
\]
(2) In the case $R = U(AU)^*$, (or $F = U$, $G = (AU)^*$, $U \in \mathbb{C}^{n \times s}$ is an arbitrary matrix), the outer inverse $A^{(2)}_{T,S} = F(GAF)^{-1}G$ reduces to the $\{2,3\}$-inverse

$$A^{(2,3)}_{T,N((AU)^*)} = A^{(2,3)}_{R(U),R((AU)^*)} = U((AU)^*AU)^{-1}(AU)^*. $$

3 Computing $\{2,4\}$ and $\{2,3\}$-inverses by SMS iterative schemes

In view of the previous general representations of $\{2,4\}$ and $\{2,3\}$-inverses, in what follows, we analyze two particular cases of the SMS algorithm in order to obtain $\{2,4\}$ and $\{2,3\}$-inverses of the matrix $A$.

Let $A \in \mathbb{C}^{m \times n}$ is given and $R \in \mathbb{C}^{n \times m}$, $0 < s \leq r$ be an arbitrary but fixed matrix.

We consider the general iterative scheme used in [2, 12, 14, 15]

$$X_1 = Q, \quad X_{k+1} = PX_k + Q, \quad k \in \mathbb{N}, \quad (3.1)$$

where $P = I - \beta RA$, $Q = \beta R$ and $\beta$ is a relaxation parameter. The classical SMS iterative scheme is appropriate for computing $\{2,4\}$-inverses. We find that the dual iterative scheme of the form

$$Y_1 = Q, \quad Y_{k+1} = Y_k P Y + Q, \quad k \in \mathbb{N}, \quad (3.2)$$

where $P = I - \beta AR$, $Q = \beta R$ is more appropriate in computation of $\{2,3\}$-inverses.

It is well known that the iterative process (3.1) can be accelerated by means of $(m + n) \times (m + n)$ composite matrix $T$, partitioned in the block form

$$T = \begin{bmatrix} P_X & Q \\ 0 & I \end{bmatrix}. \quad (3.3)$$

The improvement of (3.1) can be done by computing the matrix powers

$$T^k = \begin{bmatrix} P_X^k & \sum_{i=0}^{k-1} P_X^i Q \\ 0 & I \end{bmatrix}. $$

It is not difficult to see that the iterative scheme (3.1) produces $X_k = \sum_{i=0}^{k-1} P_X^i Q$. Hence, the matrix $X_k$ is equal to the right upper block in $T^k$. In turn, $T^{2k}$ can be computed by the matrix squaring repeated $k$ times, that is

$$T_0 = T, \quad T_{i+1} = T^2, \quad i = 0, 1, \ldots, k - 1. \quad (3.4)$$

It is clear that

$$T_k = T^{2k} = \begin{bmatrix} P_X^{2k} & \sum_{i=0}^{2k-1} P_X^i Q \\ 0 & I \end{bmatrix}. \quad (3.5)$$

Analogously, one can easily verify that the iterative scheme (3.2) can be accelerated by means of the $(n + m) \times (n + m)$ composite matrix $S$, partitioned in the following block form

$$S = \begin{bmatrix} P_Y & 0 \\ Q & I \end{bmatrix}. \quad (3.6)$$
The improvement of (3.2) can be done by computing the matrix powers

\[ S^k = \begin{bmatrix} P Y^k & 0 \\ \sum_{i=0}^{k-1} Q P Y^i & I \end{bmatrix}. \]

It is not difficult to see that the iterative scheme (3.2) gives \( Y_k = \sum_{i=0}^{k-1} Q P Y^i \). Hence, the matrix \( Y_k \) is equal to the left lower block in \( S^k \). In turn, \( S^{2k} \) can be computed by \( k \) repeated squaring, that is

\[ S_0 = S \]
\[ S_{i+1} = S_i^2, \quad i = 0, 1, \ldots, k - 1. \] (3.7)

Obviously,

\[ S_k = S^{2k} = \begin{bmatrix} P Y^{2k} & 0 \\ \sum_{i=0}^{2k-1} Q P Y^i & I \end{bmatrix}. \] (3.8)

It is not difficult to verify that \( X_k \equiv Y_k, \ k \in \mathbb{N} \). This implies that the upper right block in (3.5) is equal to the lower left block in (3.8).

As it is expected all known results that hold for the iterative scheme (3.1), analogously, can be proven for the iterative scheme (3.2). For convenience we will state the corresponding results, that can be applicable to the iterative scheme (3.2) and the SMS scheme (3.7). This result is analogous to Theorem 2.1 from [12].

**Theorem 3.1** Let \( A \in \mathbb{C}^{m \times n} \) and \( R \in \mathbb{C}^{n \times m} \), \( 0 < s \leq r \) be given such that

\[ AR(R) \oplus N(R) = \mathbb{C}^m. \] (3.9)

Let \( \beta \) be a fixed real number satisfying

\[ \max_{1 \leq i \leq t} |1 - \beta \lambda_i| < 1, \] (3.10)

where \( \text{rank}(AR) = t \) and \( \lambda_i, i = 1, \ldots, t \) are eigenvalues of \( AR \). Then sequence of approximations

\[ Y_{2^k} = \sum_{i=0}^{2^k-1} Q P Y^i, \quad k \geq 0 \]

determined by (3.2) converges to the outer inverse \( Y = A^{(2)}_{R(R), N(R)} \) of \( A \). In the case of convergence we have the following estimate

\[ \frac{\| Y - Y_{2^k} \|}{\| Y \|} \leq \max_{1 \leq i \leq t} |1 - \beta \lambda_i|^{2^k} + O(\varepsilon), \quad k \geq 0, \] (3.11)

where \( \| \cdot \| \) satisfies condition

\[ \rho(M) \leq \| M \| \leq \rho(M) + \epsilon \] (3.12)

for \( M = I - \beta RA \).

The remaining statements from [12] can be proved analogously.

We now define two particular cases of the SMS algorithm which generate the classes of \( \{2, 4\} \) and \( \{2, 3\} \)-inverses of the matrix \( A \).
1. Let \( V \in \mathbb{C}^{s \times m} \) is an arbitrary matrix such that \( \text{rank}(VA) = \text{rank}(V) = s \), where \( 0 < s \leq r \). For the iterative scheme given by (3.1), we take \( F = (VA)^*, G = V \) and \( R = FG = (VA)^*V \), which implies
\[
P_X = I - \beta (VA)^*VA, \quad Q_X = \beta (VA)^*V, \tag{3.13}
\]
where \( \beta \) is a relaxation parameter.

2. Let \( U \in \mathbb{C}^{n \times s} \), is an arbitrary matrix satisfying \( \text{rank}(AU) = \text{rank}(U) = s \), where \( 0 < s \leq r \). For the iterative scheme given by (3.2), we use the particular case \( F = U, G = (AU)^* \) and \( R = FG = U(AU)^* \), which gives
\[
P_Y = I - \beta AU(AU)^*, \quad Q_Y = \beta U(AU)^*, \tag{3.14}
\]
where \( \beta \) is a relaxation parameter.

The next theorem presents the main result of this section.

**Theorem 3.2** Let \( A \in \mathbb{C}^{m \times n}_r \), \( 0 < s \leq r \) be chosen integer.

1. If \( V \in \mathbb{C}^{s \times m}_r \) is chosen matrix such that \( \text{rank}(VA) = \text{rank}(V) = s \), then the sequence of approximations
\[
X_{2^k} = \sum_{i=0}^{2^k-1} (I - \beta (VA)^*VA)^i \beta (VA)^*V, \quad k \geq 0
\tag{3.15}
\]
determined by the SMS iterative scheme (3.5), where \( P_X \) and \( Q_X \) are defined in (3.13), converges in the matrix norm \( \| \cdot \| \) to \( \{2, 4\}\)-inverse of \( A \), which is equal to \( A^{(2, 4)}_{\mathcal{N}(UAU)^* - \mathcal{N}(V)} \) if \( \beta \) is a fixed real number such that
\[
\max_{1 \leq i \leq s} |1 - \beta \lambda_i| < 1,
\tag{3.16}
\]
where \( \lambda_i, i = 1, \ldots, s \) are nonzero eigenvalues of \( (VA)^*VA \). In the case of convergence, the the sequence \( X_k \) satisfies the error estimation
\[
\frac{\|X - X_{2^k}\|}{\|X\|} \leq \max_{1 \leq i \leq r} |1 - \beta \lambda_i|^{2^k} + O(\varepsilon), \quad k \geq 0,
\tag{3.17}
\]
where \( \| \cdot \| \) satisfies condition (3.12) for \( M = I - \beta (VA)^*V \).

2. If \( U \in \mathbb{C}^{n \times s}_r \) is chosen matrix such that \( \text{rank}(AU) = \text{rank}(U) = s \), then the sequence of approximations
\[
Y_{2^k} = \sum_{i=0}^{2^k-1} (I - \beta AU(AU)^*)^i \beta U(AU)^*, \quad k \geq 0
\tag{3.18}
\]
determined by the SMS iterative process (3.8), where \( P_Y \) and \( Q_Y \) are defined in (3.14), converges in the matrix norm \( \| \cdot \| \) to \( \{2, 3\}\)-inverse of \( A \), which is equal to \( A^{(2, 3)}_{\mathcal{R}(U), \mathcal{R}(AU)} \) in the case when \( \beta \) is a fixed real number satisfying
\[
\max_{1 \leq i \leq r} |1 - \beta \lambda_i| < 1,
\tag{3.19}
\]
where \( \lambda_i, i = 1, \ldots, s \) are nonzero eigenvalues of \( AU(AU)^* \).

In the case of convergence, the next error estimation holds for the sequence \( Y_k \)
\[
\frac{\|Y - Y_{2^k}\|}{\|Y\|} \leq \max_{1 \leq i \leq r} |1 - \beta \lambda_i|^{2^k} + O(\varepsilon), \quad k \geq 0,
\tag{3.20}
\]
where \( \| \cdot \| \) satisfies condition (3.12) for \( M = I - \beta U(AU)^*A \).
Proof. 1. According to main result from [2], we have

\[
\lim_{k \to \infty} X_{2^k} = X = \lim_{k \to \infty} \left( \sum_{i=0}^{2^k-1} (I - \beta (VA)^i (VA)^*) \right) \cdot V
\]

= \((VA)^\dagger V\)

From the general representation of \(\{2,4\}\)-inverses from Proposition 1.1, we obtain \(X \in A\{2,4\}\).

Moreover, the conditions imposed in this case are equivalent with Theorem 2.1 from [12] by taking \(R = (VA)^* V\). Since \(\text{rank}(VA) = \text{rank}(V)\) it follows that the matrix \((VA)^*\) is a full-column rank matrix, from which we obtain that \(\text{rank}(R) = s\). Now, using \(R = FG, F = (VA)^*, G = V\) as a full-rank factorization of \(R\), according to Theorem 2.3 from [12] we conclude that

\[
X_{2^k} \to X = F(GAF)^{-1}G = (VA)^* (VA(VA)^*)^{-1} V.
\]

Now, the proof follows from Theorem 2.1, part (a) and Lemma 2.1, part (a).

2. Conditions imposed in this case are equivalent with Theorem 3.1 by taking \(R = U(AU)^*\). Since \(\text{rank}(AU) = \text{rank}(U)\) follows that the matrix \((AU)^*\) is a full-row rank matrix, from which we obtain that \(\text{rank}(R) = s\). Now, using \(R = FG, F = U, G = (AU)^*\) as a full-rank factorization of \(R\), according to Theorem 2.3 from [12] we conclude that

\[
Y_{2^k} \to Y = F(GAF)^{-1}G = U((AU)^* AU)^{-1}(AU)^*.
\]

Now, the proof follows from Theorem 2.1, part (b) and Lemma 2.1, part (b).

Corollary 3.1  a) In the case \(V \in \mathbb{C}_r^{r \times m}\), \(\text{rank}(VA) = r\), under the conditions of Theorem 3.2, part 1, we have

\[
X = A_{\{2,4\},N(V)}^{(2,4)} \in A\{1,2,4\}.
\]

b) In the case \(U \in \mathbb{C}_n^{n \times r}\), \(\text{rank}(AU) = r\), under the conditions of Theorem 3.2, part 2, we have

\[
Y = A_{\{2,3\},N(A^*)}^{(2,3)} \in A\{1,2,3\}.
\]

c) If both of the conditions \(V = A^*\) and \(U = A^*\) are satisfied, under the conditions of Theorem 3.2, the identities

\[
X = Y = A^\dagger
\]

are satisfied.

Proof.  a) Since \(X\) is \(\{2\}\)-inverse of \(A\) such that

\[
\text{rank}(X) = \text{rank}((VA)^\dagger) = \text{rank}((VA)^*) = \text{rank}(V) = \text{rank}(A),
\]

it follows that \(X\) is also \(\{1\}\)-inverse of \(A\). Moreover, from \(\text{rank}((VA)^*) = \text{rank}(A^*)\) we have that \(\mathcal{R}((VA)^*) = \mathcal{R}(A^*)\) and the proof is complete.

b) Analogously.

c) In both cases, immediately follows that \(X = Y = A_{\mathcal{R}(A^*),\mathcal{N}(A^*)}^{(2)} = A^\dagger\).
4 Numerical examples

The relaxation parameter $\beta = \beta_{\text{opt}}$ defined as in [12] is used in the next examples.

Example 4.1 In this example we get an $\{2,4\}$-inverse $X$ and $\{2,3\}$-inverse $Y$ of $A_1$ in the first iteration, as it is expected. Let us consider $6 \times 5$ matrix $A_1$ given by

$$A_1 = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
1 & -1 & 0 & 1 & 1 \\
1 & 0 & 0 & -2 & 0 \\
\end{bmatrix}.$$ 

a) Now, let us choose $V = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$

One can check that the nonzero eigenvalues of the matrix $R = (VA^*)VA$ are equal to 1, thus $mRe = 1, MRe = 1$ and also $\text{Im} (\lambda) = 0$ for each eigenvalue $\lambda$. Thus, in only 1 iteration we obtain the $\{2,4\}$-inverse of $A_1$ given by

$$X_{21} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$$ 

It is not difficult to verify $X_{21} = (VA)^\dagger V$.

b) If we take $U = \begin{bmatrix} 0 & 1 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
\end{bmatrix}$,

Since all eigenvalues are real and $mRe = MRe = 4$ again only in one iteration we obtain an exact $\{2,3\}$-inverse of $A_1$ given by

$$Y = \begin{bmatrix}
-0.25 & 0 & 0.25 & 0 & 0.25 \\
0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 \\
\end{bmatrix}.$$ 

Example 4.2 Consider the following $6 \times 5$ matrix of rank 4,

$$A_2 = \begin{bmatrix}
-1 & 0 & 1 & 2 & 2 \\
-1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 3 & 4 \\
0 & 1 & -1 & -3 & 2 \\
1 & -1 & 0 & 1 & 1 \\
1 & 0 & -1 & -2 & -2 \\
\end{bmatrix}.$$ 

a) First, let us choose $V$ of rank 2, given by

$$V = \begin{bmatrix} 3 & 1 & 3 & 1 & 2 \ -1 \\
3 & 1 & 3 & 1 & 2 \ -1 \\
0 & -1 & 0 & -2 & 1 \\
\end{bmatrix}.$$ 

Iterative scheme (3.1) with $P_X$ and $Q = Q_X$ defined as in (3.13) gives the following $\{2, 4\}$-inverse of $A_2$:

$$X_{25} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0.220992 & -0.431257 & 0.220992 & 0.0740306 & -0.862515 & 0.431257 \\
-0.0105758 & 0.0681551 & -0.0105758 & -0.00552526 & 0.13631 & -0.0681551 \\
-0.243243 & 0.567568 & -0.243243 & -0.0810811 & 1.13514 & -0.567568 \\
0.320799 & -0.400705 & 0.320799 & 0.106933 & -0.80141 & 0.400705
\end{bmatrix}.$$ 

Additionally, for the same matrix $A_2$ if we take, for example, the matrix $V = \begin{bmatrix}
3 & 1 & 0 & 1 & 0 & -1 \\
1 & 0 & 3 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & -2 & 4
\end{bmatrix}$, which has rank 4, we obtain the following $\{1, 2, 4\}$-inverse of $A$ after 25 iterations

$$X_{25} = \begin{bmatrix}
-0.238095 & -0.114286 & 1 & -0.2 & -1.31429 & 0.961905 \\
-0.119048 & 0.0761905 & 1.16667 & -0.2 & -2.12381 & 0.914286 \\
-0.285714 & 0.0761905 & 0.666667 & -0.2 & -1.12381 & 0.247619 \\
-0.452381 & 0.0761905 & 0.166667 & -0.2 & -0.12381 & -0.419048 \\
0.52381 & -0.0285714 & 0 & 0 & 0.171429 & 0.32381
\end{bmatrix}.$$ 

b) For the same matrix $A_2$ we apply the iterative scheme (3.2) for the choice of matrices $P_Y$ and $Q = Q_Y$ which are defined by (3.14), where

$$U = \begin{bmatrix}
3 & 5 \\
1 & 7 \\
-3 & 2 \\
1 & -2 \\
2 & -2
\end{bmatrix}.$$ 

After 10 iterations, we get the $\{2, 3\}$-inverse

$$Y_{210} = \begin{bmatrix}
-0.213456 & -0.0607649 & 0.0827195 & 0.313031 & 0.0607649 & 0.213456 \\
-0.216572 & -0.00325779 & -0.0328612 & 0.259207 & 0.00325779 & 0.216572 \\
0.0225921 & 0.0830737 & -0.16204 & -0.109773 & -0.0830737 & -0.0225921 \\
0.0288244 & -0.0319405 & 0.0691219 & -0.00212465 & 0.0319405 & -0.0288244 \\
0.00311615 & -0.0575071 & 0.115581 & 0.0538244 & 0.0575071 & -0.00311615
\end{bmatrix}.$$ 

After the usage

$$U = \begin{bmatrix}
1 & 3 & 0 & 0 \\
2 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 \\
2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix},$$

the generated $\{1, 2, 3\}$-inverse of $A$ is equal to

$$Y_{210} = \begin{bmatrix}
-0.6 & 0.6 & 1 & -0.2 & -0.6 & 0.6 \\
-0.836364 & 1.56364 & 1.63636 & -0.490909 & -1.56364 & 0.836364 \\
0.372727 & -0.327273 & -0.272727 & 0.381818 & 0.327273 & -0.372727 \\
-0.353636 & 0.563636 & 0.636364 & -0.490909 & -0.563636 & 0.353636 \\
0.1 & -0.1 & 0 & 0.2 & 0.1 & -0.1
\end{bmatrix}.$$
5 Conclusion

We interpreted full rank representation of \( \{2, 4\} \) and \( \{2, 3\} \)-inverses through the full rank representation of outer inverses. In this way, the methods developed for computing outer inverses with prescribed range and null space can be applied in computation of \( \{2, 4\} \) and \( \{2, 3\} \)-inverses with prescribed range and null space.

Modifying and applying the SMS algorithm from [2, 12] accordingly to derived representations, we presented two algorithms for computing generalized inverses. The first is intended for computing \( \{2, 4\} \)-inverses and the second one for computing \( \{2, 3\} \)-inverses. These algorithms actually show the utility of the successive matrix squaring algorithm (SMS algorithm). The first application is straight whereas for the second one we did some modifications and showed the respective convergence results.

Additionally, we indicate some necessary conditions for obtaining \( \{1, 2, 3\} \), \( \{1, 2, 4\} \)-inverses and finally the Moore-Penrose inverse. In this way we filled the gap of finding \( S \)-inverses, where \( S \subset \{1, 2, 3, 4\} \) using SMS method for computing \( \{2\} \)-inverses with prescribed range and null space from [12].

References


Full-rank representations of \( \{2,4\} \), \( \{2,3\} \)-inverses and successive matrix squaring algorithm


