On numbers of vertices of maximum degree in the spanning trees of a graph

Jerzy Topp\textsuperscript{a,*}, Preben D. Vestergaard\textsuperscript{b}

\textsuperscript{a} Faculty of Applied Physics and Mathematics, Gdańsk Technical University, Narutowicza 11/12, 80-952 Gdańsk, Poland

\textsuperscript{b} Department of Mathematics and Computer Science, Aalborg University, Fredrik Bajers Vej 7, 9220 Aalborg, Denmark

Received 15 November 1992

Abstract

For a connected graph $G$, let $\mathcal{T}(G)$ be the set of all spanning trees of $G$ and let $n_d(G)$ be the number of vertices of maximum degree in $G$. In this paper we show that if $G$ is a cactus or a connected graph with $p$ vertices and $p+1$ edges, then the set $\{n_d(T) : T \in \mathcal{T}(G)\}$ has at most one gap, that is, it is a set of consecutive integers or it is the union of two sets each of which consists of consecutive integers.

1. Introduction

Let $S = \{a_1, a_2, \ldots, a_m\}$ be a nonempty set of integers in increasing order. We say that $S$ has $k$ gaps, $1 \leq k < m$, if there are exactly $k$ indices $i_1, i_2, \ldots, i_k$ such that $1 \leq i_1 < i_2 < \cdots < i_k < m$ and $a_{i_j} + 1 < a_{i_{j+1}}$ for $j = 1, \ldots, k$. We also say that $S$ has no gap if $S$ consists of consecutive integers. For a graph $G$, we denote by $V(G)$ and $E(G)$, the vertex set and edge set of $G$, respectively. For a vertex $v$ of $G$, $N_G(v)$ denotes the set of vertices adjacent to $v$ in $G$. The degree of a vertex $v$ in a graph $G$ is the integer $d_G(v) = |N_G(v)|$. Let $\Delta(G)$ be the maximum degree among all vertices of $G$ and let $n_d(G)$ be the number of vertices of degree $\Delta(G)$ in $G$. For a connected graph $G$, let $\mathcal{F}(G)$ be the set of all spanning trees of $G$ and let $\mathcal{N}(G)$ be the set of numbers of vertices of maximum degree in the spanning trees of $G$, that is, $\mathcal{N}(G) = \{n_d(T) : T \in \mathcal{F}(G)\}$.

Let $f$ be an integer-valued function on the set $\mathcal{F}(G)$ of all spanning trees of a connected graph $G$. The $f$-set of the graph $G$ is the range set of the function $f$ on the set $\mathcal{F}(G)$. The function $f$ is said to interpolate over (the set of all spanning

* Corresponding author

0012-365X/96/$15.00 © 1996 Elsevier Science B.V. All rights reserved
SSDI 0012-365X(94)00389-0
trees of $G$, if the $f$-set of $G$ has no gap. For example, the number of end vertices [1,8], the independence and domination numbers [5], and some covering numbers [4,6] interpolate over any connected graph. Additional examples may be found in [3]. A set of integers is a feasible $f$-set if it is the $f$-set of some graph. Certainly, if $f$ interpolates over any connected graph, then every feasible $f$-set has no gap. However, if $f$ does not interpolate over every connected graph, the feasible $f$-sets in general are much less uniform and the question of characterization of feasible $f$-sets becomes much more difficult, see [2,7,9,10], where feasible diameter-sets have been studied. In this paper, the $n_\Delta$-set of a graph $G$ is denoted by $\mathcal{N}(G)$ and it is shown that it has at most one gap if $G$ is a cactus or if $G$ is a connected graph of order $p$ and size $p + 1$ with $p \geq 4$; a cactus is a connected graph in which each edge belongs to at most one cycle.

2. Preliminaries

Let $G$ be a connected graph. Two spanning trees $T$ and $T'$ of $G$ are said to be adjacent if the symmetric difference of their edge sets consists of two adjacent edges of $G$, that is, if there are adjacent edges $xy$ and $yz$ in $G$ such that $T' = T + yz - xy$. It is well known that for any pair of different spanning trees $T$ and $T'$ of $G$, there exists a sequence $T_0 = T, T_1, \ldots, T_k = T'$ of spanning trees of $G$ (transforming $T$ into $T'$) in which $T_i$ is adjacent to $T_{i+1}$ for $i = 0, \ldots, k - 1$.

We denote by $m(G)$ and $M(G)$ ($m$ and $M$ for short) the smallest and largest integer, respectively, of the set $\{\Delta(T) : T \in \mathcal{F}(G)\}$. It is easy to observe that $M = \Delta(G)$. For an integer $i$, $m \leq i \leq M$, we define

$$
\mathcal{F}_i(G) = \{T \in \mathcal{F}(G) : \Delta(T) = i\} \quad \text{and} \quad \mathcal{N}_i(G) = \{n_\Delta(T) : T \in \mathcal{F}_i(G)\}.
$$

Certainly, $\mathcal{F}(G) = \bigcup_{i=m}^M \mathcal{F}_i(G)$ and $\mathcal{N}(G) = \bigcup_{i=m}^M \mathcal{N}_i(G)$.

In this section we consider some properties of the sets $\mathcal{N}_i(G)$, $m \leq i \leq M$, and conditions under which the set $\mathcal{N}(G)$ has at most one gap. First, for adjacent spanning trees $T$ and $T'$ of $G$, we analyze how the maximum degree and the number of vertices of maximum degree vary as we proceed from $T$ to $T'$.

**Lemma 1.** For two adjacent spanning trees $T$ and $T'$ of a graph $G$ we have:

1. $|\Delta(T) - \Delta(T')| \leq 1$;
2. If $\Delta(T') = \Delta(T) + 1$, then $n_\Delta(T') = 1$;
3. If $\min\{n_\Delta(T), n_\Delta(T')\} \geq 2$, then $\Delta(T) = \Delta(T')$ and $|n_\Delta(T) - n_\Delta(T')| \leq 1$;
4. At least one of $|n_\Delta(T) - n_\Delta(T')| \leq 1$ and $\min\{n_\Delta(T), n_\Delta(T')\} = 1$ holds;
5. If $n_\Delta(T') > n_\Delta(T) + 1$, then $n_\Delta(T) = 1$ and $\Delta(T) = \Delta(T') + 1$.

**Proof.** Let $T$ and $T'$ be adjacent spanning trees of $G$ and let $A$ and $A'$ denote the sets of vertices of maximum degree in $T$ and $T'$, respectively. Assume that $T' = T + yz - xy$. Then $d_T(x) = d_T(x) - 1$, $d_T(z) = d_T(z) + 1$ and $d_T(t) = d_T(t)$ for every $t \in V(G) - \{x,z\}$. This implies that $\Delta(T') \in \{\Delta(T) - 1, \Delta(T), \Delta(T) + 1\}$ and
(1) follows. In addition, since \( z \) is the unique vertex \( t \) of \( G \) for which \( d_{T'}(t) > d_T(t) \),

it is obvious that if \( \Delta(T') = \Delta(T) + 1 \), then \( A' = \{ z \} \) and we obtain (2). To prove

(3), assume \( \min\{n_A(T), n_A(T')\} \geq 2 \). Then \( \Delta(T') = \Delta(T) \) by (1) and (2). Certainly,

since \( \Delta(T') = \Delta(T) \), we have at once that \( A' \) is one of the sets \( A, A \cup \{ z \}, A - \{ x \}, \)

(\( A \cup \{ z \} \)) and the second part of (3) follows. (4) is immediate from (3). Finally,

to prove (5), assume that \( n_A(T') > n_A(T) + 1 \). Then \( \min\{n_A(T), n_A(T')\} = 1 \) (by

(4)) and therefore \( n_A(T) = 1, n_A(T') > 2 \) and consequently \( \Delta(T') \neq \Delta(T) + 1 \) (by

(2)). Moreover, \( \Delta(T') = \Delta(T) \) cannot hold because \( \Delta(T') = \Delta(T) \) and \( n_A(T) = 1 \)

imply \( n_A(T') \leq 2 \) which is impossible. From this and from (1) it follows that \( \Delta(T') = \Delta(T) - 1 \).

This completes the proof of (5) and of Lemma 1. 

\[ \square \]

**Lemma 2.** Let \( G \) be a connected graph with \( m < M \). Then \( 1 \in \mathcal{N}_i(G) \) for every

integer \( i, m < i \leq M \).

**Proof.** For a given \( i \) with \( m < i < M \), take any \( T_0 \in \mathcal{T}_{i-1}(G) \) and \( T' \in \mathcal{T}_i(G) \).

Let \( T_0, T_1, \ldots, T_n = T' \) be a sequence of spanning trees of \( G \) in which \( T_i \) is adjacent

to \( T_{i+1} \) (\( 0 \leq i < n \)). Since \( \Delta(T_0) = i - 1 \) and \( \Delta(T_n) = i \), Lemma 1(1) implies that

there exists an index \( l, 0 \leq l < n \), such that \( \Delta(T_l) = i - 1 \) and \( \Delta(T_{i+1}) = i \). Then

Lemma 1(2) applied to \( T_l \) and \( T_{i+1} \) implies that \( n_A(T_{i+1}) = 1 \). Hence \( 1 \in \mathcal{N}_i(G) \)

as \( T_{i+1} \in \mathcal{T}_i(G) \). \( \square \)

**Lemma 3.** Let \( G \) be a connected graph. Then the set \( \mathcal{N}_M(G) \) has no gap.

**Proof.** Let \( T_0, T' \in \mathcal{T}_M(G) \) be spanning trees for which \( n_A(T_0) = \min\{n_A(T) : T \in \mathcal{T}_M(G)\} \) and \( n_A(T') = \max\{n_A(T) : T \in \mathcal{T}_M(G)\} \), respectively. The result is obvious

if \( n_A(T') \leq 2 \). Thus assume \( n_A(T') > 2 \) and let \( T_0, T_1, \ldots, T_n = T' \) be a sequence of

spanning trees in which \( T_i \) is adjacent to \( T_{i+1} \), \( 0 \leq i < n \).

If \( m = M \), then \( \Delta(T_0) = \cdots = \Delta(T_n) = M \) and it follows from Lemma 1(5) that

\( |n_A(T_{i+1}) - n_A(T_i)| \leq 1 \) \((0 \leq i < n)\). This implies that the sequence \( (n_A(T_0), n_A(T_1), \ldots, \)

\( n_A(T_n)) \) contains the sequence \( (n_A(T_0), n_A(T_0) + 1, \ldots, n_A(T_n)) \) of consecutive integers

as a subsequence, and therefore \( \mathcal{N}_M(G) \) has no gap.

If \( m < M \), then \( n_A(T_0) = 1 \in \mathcal{N}_M(G) \) by Lemma 2. Let \( l \) be the largest in-

dex \( j \) such that \( n_A(T_j) = 1 \), \( 0 \leq j < n \). Then \( n_A(T_i) > 2 \) for each \( i, l < i \leq n \),

and therefore it follows from Lemma 1(3) that \( \Delta(T_{i+1}) = \cdots = \Delta(T_n) = M \) and \( |n_A(T_{i+1}) - n_A(T_i)| \leq 1 \) \( i = l + 1, \ldots, n - 1 \). This implies that the subset \( \mathcal{N}' = \{n_A(T_i) : i = l + 1, \ldots, n\} \) of \( \mathcal{N}_M(G) \) consists of consecutive integers. Now, since \( \mathcal{N} \) has no gap and \( \mathcal{N}' \) contains the integers \( n_A(T_{i+1}) \) and \( n_A(T_n) \) (the largest in-
teger of \( \mathcal{N}_M(G) \)), in order to prove that \( \mathcal{N}_M(G) \) has no gap, it suffices to show

that \( n_A(T_{n+1}) = 2 \). On the contrary, suppose that \( n_A(T_{n+1}) > 2 \). Then \( n_A(T_{n+1}) > 2 = n_A(T_l) + 1 \) and Lemma 1(5) applied to the adjacent trees \( T_l \) and \( T_{n+1} \)
gives \( \Delta(T_l) = \Delta(T_{n+1}) + 1 = M + 1 > M \), a contradiction. This completes the

proof. \( \square \)
Corollary 1. Let \( G \) be a connected graph in which every two spanning trees have the same maximum degree, i.e. \( m = M \). Then the set \( \mathcal{N}(G) \) has no gap.

Proof. In this case \( \mathcal{N}(G) = \mathcal{N}_M(G) \) and the result follows from Lemma 3. \( \square \)

Corollary 2. Let \( G \) be a connected graph with \( m < M \). If none of the sets \( \mathcal{N}_m(G), \mathcal{N}_{m+1}(G), \ldots, \mathcal{N}_{M-1}(G) \) has a gap, then the set \( \mathcal{N}(G) \) has at most one gap.

Proof. The hypothesis together with Lemma 3 imply that none of the sets \( \mathcal{N}_m(G), \mathcal{N}_{m+1}(G), \ldots, \mathcal{N}_M(G) \) has a gap. Since the integer 1 belongs to each of the sets \( \mathcal{N}_m(G), \mathcal{N}_{m+1}(G), \ldots, \mathcal{N}_M(G) \) (by Lemma 2), the set \( \bigcup_{i=m+1}^{M} \mathcal{N}_i(G) \) has no gap. Consequently, the set \( \mathcal{N}(G) = \mathcal{N}_m(G) \cup \bigcup_{i=m+1}^{M} \mathcal{N}_i(G) \) has at most one gap. \( \square \)

Lemma 4. Let \( G \) be a connected graph and let \( i \) be an integer, \( m \leq i \leq M \). If for every pair of different spanning trees \( T_0, T \in \mathcal{F}_i(G) \) there are spanning trees \( T_0, T_1, \ldots, T_n = T \) in \( \mathcal{F}_i(G) \) with \( T_k \) adjacent to \( T_{k+1} \) for \( k = 0, \ldots, n-1 \), then the set \( \mathcal{N}_i(G) \) has no gap.

Proof. Take different spanning trees \( T_0, T \in \mathcal{F}_i(G) \). By hypothesis, there exist spanning trees \( T_0, T_1, \ldots, T_n = T \) in \( \mathcal{F}_i(G) \) with \( T_k \) adjacent to \( T_{k+1} \) for \( k = 0, \ldots, n-1 \). Since \( \Delta(T_0) = \cdots = \Delta(T_n) = i \), it follows from Lemma 1(5) that \( |n_3(T_{k+1}) - n_3(T_k)| \leq 1 \) for \( k = 0, \ldots, n-1 \). This easily implies that the set \( \mathcal{N}_i(G) \) has no gap. \( \square \)

At first glance it seems to be quite natural to suspect that for different spanning trees \( R, S \in \mathcal{F}_i(G) \) there exists a sequence \( T_0, T_1, \ldots, T_n \) transforming \( R \) into \( S \) in which all intermediate trees belong to the family \( \mathcal{F}_i(G) \) and \( T_k \) is adjacent to \( T_{k+1} \) for \( k = 0, \ldots, n-1 \). That this is not the case is illustrated in Fig. 1, which gives a graph \( G \) and two spanning trees \( R, S \in \mathcal{F}_3(G) \) for which no such sequence exists. Thus, sometimes Lemma 4 is useless in proving that the sets \( \mathcal{N}_i(G) \) have no gaps. This motivates our next definition.

Let \( G \) be a connected graph with \( m < M \). For an integer \( i \), \( m \leq i < M \), and for a pair of different spanning trees \( R, S \in \mathcal{F}_i(G) \), a sequence \( T_0, T_1, \ldots, T_n \) of spanning

![Fig. 1.](image_url)
trees of G is said to be an (R,S)-sequence in G if:

1. \( T_0 = R \);
2. \( T_n \) has at least one more edge in common with S than does \( R \);
3. \( T_k \in \mathcal{F}(G), \ 0 \leq k \leq n \), and
4. \( |n_d(T_{k+1}) - n_d(T_k)| \leq 1, \ 0 \leq k < n \).

We note that consecutive trees of an (R,S)-sequence are not necessarily adjacent, but they have the property that their respective numbers of vertices of maximum degree differ by at most 1, the maximum degree of vertices in each of them is the same as in \( R \) and \( S \), and the last tree of the (R,S)-sequence has more edges in common with \( S \) than does \( R \). For the graph \( G \) and its spanning trees \( R \) and \( S \) depicted in Fig. 1, the trees \( T_0, T_1, T_2 \), as well as \( T_0, T_1, T_2, T_3 \), form (R,S)-sequences. In these sequences the trees \( T_1 \) and \( T_2 \) are not adjacent and in the transformation of \( T_1 \) into \( T_2 \) we have applied a double exchange (made at the same time) of edges \( e_1, e_2 \) and \( e_3, e_4 \), where \( e_1, e_2, e_3, e_4 \) (in that order) form a path in \( T_1 + e_1 + e_3, e_1 \) and \( e_2 \) share one cycle of \( T_1 + e_1 + e_3 \) while \( e_3 \) and \( e_4 \) share another cycle of \( T_1 + e_1 + e_3 \). In such a double exchange of edges, the degree of one end vertex of the \( e_1, e_2, e_3, e_4 \) path is increased by one, the degree of the second end vertex of the \( e_1, e_2, e_3, e_4 \) path is decreased by one, and all other vertex degrees remain unchanged. Double exchanges of edges will be useful in our proof of Theorem 1.

**Lemma 5.** Let \( G \) be a connected graph and let \( i \) be an integer, \( m \leq i \leq M \). If for every pair of different spanning trees \( R, S \in \mathcal{F}(G) \) there exists an (R,S)-sequence in \( G \), then the set \( \mathcal{N}(G) \) has no gap.

**Proof.** The result is obvious if \( |\mathcal{N}(G)| = 1 \). Thus assume that \( |\mathcal{N}(G)| \geq 2 \). Let \( T_0, T' \in \mathcal{F}(G) \) be such that \( n_d(T_0) = \min\{n_d(T) : T \in \mathcal{F}(G)\} \) and \( n_d(T') = \max\{n_d(T) : T \in \mathcal{F}(G)\} \). The repeated application of the hypothesis implies that there exists a \( (T_0, T') \)-sequence \( T_0, T_1, \ldots, T_n \) in \( G \) with \( T_n = T' \). Now, since \( |n_d(T_{k+1}) - n_d(T_k)| \leq 1 \) for \( k = 0, \ldots, n - 1 \), it is obvious that \( \mathcal{N}(G) \) has no gap.

3. **Main results**

Armed with the above properties, we can prove that \( \mathcal{N}(G) \) has at most one gap if \( G \) is a cactus or a connected graph of order \( p \) and size \( p + 1 \).

**Theorem 1.** For any cactus \( G \), the set \( \mathcal{N}(G) \) has at most one gap.

**Proof.** The theorem is true for all cacti with \( m = M \) (by Corollary 1). Now, if \( G \) is a cactus with \( m < M \), then according to Corollary 2 and Lemma 5, it suffices to show that for every \( i, m \leq i < M \), and for every pair of different spanning trees \( R \) and
S from \( \mathcal{F}(G) \), there exists an \((R,S)\)-sequence in \( G \). To prove the existence of such \((R,S)\)-sequences, we employ induction on \( n \), the number of cycles in the cactus.

First, let \( G \) be a unicyclic graph with \( m < M \). Let \( C \) be the unique cycle of \( G \) and let \( V_a \) be the set of vertices of degree \( A(G) \) in \( G \). It easily follows from the assumption \( m < M \) that \( V_a \subseteq V(C) \), \( |V_a| \leq 2 \), and \( m = M - 1 \). If \( |V_a| = 2 \), then the vertices of \( V_a \) are adjacent, \( |\mathcal{F}_m(G)| = 1 \), and we do not have different spanning trees in \( \mathcal{F}_m(G) \).

Finally, if \( |V_a| = 1 \), say \( V_a = \{v\} \), then \( \mathcal{F}_m(G) = \{G - vu, G - vw\} \), where \( u \) and \( w \) are the neighbours of \( v \) in \( C \). In this case, \( G - vu \) and \( G - vw \) form \((G - vu, G - vw)\)- and \((G - vw, G - vu)\)-sequences in \( G \).

Now consider a cactus \( G \) with \( n \) cycles (\( n \geq 2 \)) and \( m < M \). Let \( \mathcal{E} \) denote the set of all cycles of \( G \). Since any spanning tree of \( G \) is obtained by deleting exactly one edge from each cycle of \( G \), for any spanning tree \( T \in \mathcal{F}(G) \) and for any cycle \( C \in \mathcal{E} \), there exists exactly one edge of \( C \), denoted by \( e_{CT} \), that does not belong to \( T \). If \( T \in \mathcal{F}(G) \) and \( C \in \mathcal{E} \), then \( T + e_{CR} \) denotes the unicyclic graph obtained from \( T \) by adding the edge \( e_{CR} \).

Let \( R \) and \( S \) be different spanning trees of \( G \) with \( A(R) = A(S) = i \), \( m \leq i < M \). If \( e_{CR} = e_{CS} \) for some \( C \in \mathcal{E} \), then both \( R \) and \( S \) are spanning trees of the cactus \( G - e_{CR} \) and, by the induction hypothesis, there exists an \((R,S)\)-sequence in \( G - e_{CR} \) and so in \( G \). Thus, assume that \( e_{CR} \neq e_{CS} \) for each \( C \in \mathcal{E} \). We consider the following two cases.

**Case 1:** There exists a tree \( T \in \{R,S\} \), say \( T = R \), and a cycle \( C \in \mathcal{E} \) such that

\[ \forall x \in V(C) \ d_{R+e_{CR}}(x) \leq i. \]

Then we distinguish two subcases depending on degrees of vertices which do not belong to \( C \) in the unicyclic graph \( H = R + e_{CR} \).

**Case 1.1:** There exists a vertex \( y_0 \in V(G) - V(C) \) such that \( d_H(y_0) = i \). Let \( e_0, e_1, \ldots, e_l \) be a sequence of consecutive edges of \( C \) starting with \( e_0 = e_{CR} \) and ending with \( e_l = e_{CS} \). Then

\[ T_0 = H - e_0, \quad T_1 = H - e_1, \quad \ldots, \quad T_l = H - e_l \]

is a sequence of spanning trees of \( G \) with \( T_j \) adjacent to \( T_{j+1} \) (\( 0 \leq j < l \)). Moreover, since \( e_{CT} = e_l = e_{CS} \), \( T_l \) has one more edge in common with \( S \) than does \( R = T_0 \).

In addition, since \( i = d_R(y_0) = A(T_0) = \cdots = A(T_l) \), Lemma 1(5) implies that \( |n_A(T_{j+1}) - n_A(T_j)| \leq 1 \) (\( 0 \leq j < l \)). Hence, \( T_0, \ldots, T_l \) is an \((R,S)\)-sequence in \( G \).

**Case 1.2:** For every vertex \( y \in V(G) - V(C) \), \( d_H(y) < i \). This implies that all vertices of degree \( i \) in \( R \) belong to \( C \). In addition, since \( A(H) = A(R) = i \), no such vertex is incident with \( e_{CR} \). Let \( t_0 \) be an arbitrary vertex of degree \( i \) in \( R \). Let \( C' \) be any cycle of \( G \) different from \( C \). It is no problem to observe that if \( C' \) and \( C \) are disjoint or if their common vertex is not of degree \( i \) in \( R \), then for \( R \) and \( C' \) we have already considered Case 1.1. Therefore we henceforth assume that \( C' \) and \( C \) have a vertex in common and that this vertex is of degree \( i \) in \( R \).
Case 1.2.1: There is no edge in $C$ which is incident with all vertices of degree $i$ in $H$. Then $\Delta(H - e) = i$ for every edge $e$ of $C$. Consequently, it is easy to check that if $e_0, e_1, \ldots, e_l$ is a sequence of consecutive edges of $C$ with $e_0 = e_{CR}$ and $e_l = e_{CS}$, then

$$T_0 = H - e_0, \quad T_1 = H - e_1, \ldots, \quad T_l = H - e_l$$

is an $(R, S)$-sequence in $G$.

Case 1.2.2: There is an edge in $C$ which is incident with all vertices of degree $i$ in $H$. In this case, $H$ has at most two vertices of degree $i$. If the edge $e_{CS}$ does not cover all vertices of degree $i$ in $H$, then again it is easy to find the desired $(R, S)$-sequence. If $e_{CS}$ covers all vertices of degree $i$ in $H$, then we distinguish two cases.

Case 1.2.2.1: $t_0$ is the unique vertex of degree $i$ in $H$. In this case $t_0$ is a common vertex of all cycles of $G$. In addition, there exists a cycle $C' \in \mathcal{C} - \{C\}$ in which the edge $e_{C'R}$ is incident with $t_0$; otherwise all edges of $G$ incident with $t_0$ are in $R$ and then, since $d_R(y) < i$ for every $y \in V(G) - V(C)$, it is clear from the structure of $G$ that $i = d_R(t_0) = d_C(t_0) = M$, a contradiction. Let $f$ be the edge of $C'$ adjacent to $e_{C'R}$ which is not incident with $t_0$. Let $e_0, e_1, \ldots, e_l$ be the sequence of consecutive edges of $C$, where $e_0 = e_{CR}$, $e_l = e_{CS}$, and only $e_l$ is incident with $t_0$, see Fig. 2. Then

$$T_0 = H - e_0, \quad T_1 = H - e_1, \ldots, \quad T_{l-1} = H - e_{l-1}, \quad T_l = H - e_l + e_{C'R} - f$$

is a sequence of spanning trees of $G$ with $\Delta(T_s) = i$ ($0 \leq s \leq l$). Certainly, in this sequence $T_s$ is adjacent to $T_{s+1}$ and therefore $|n_3(T_{s+1}) - n_3(T_s)| \leq 1$ for $s = 0, \ldots, l-2$ (if $l \geq 2$). The trees $T_{l-1}$ and $T_l$ are not adjacent, but $T_l$ is obtained from $T_{l-1}$ by a double exchange of edges and certainly $|n_3(T_{l-1}) - n_3(T_l)| \leq 1$. Finally, it follows from the construction of the sequence $T_0, \ldots, T_l$ that $e_{CT_l} = e_{CS}$ and therefore $T_l$ has one more edge in common with $S$ than does $R$. Consequently, $T_0, \ldots, T_l$ is an $(R, S)$-sequence in $G$.

Case 1.2.2.2: $t_0$ and $t'_0$ are the unique vertices of degree $i$ in $H$. Then $e_{CS} = t_0t'_0$ and as in Case 1.2.2.1, there exists a cycle $C' \in \mathcal{C} - \{C\}$ in which $e_{C'R}$ is incident with $t_0$ or $t'_0$. Assume that $e_{C'R}$ is incident with $t'_0$. (The proof is analogous if $e_{C'R}$ is incident with $t_0$.) Let $f$ be the edge of $C'$ adjacent to $e_{C'R}$ which is not incident with $t'_0$. Let $e_0, e_1, \ldots, e_l$ be the sequence of consecutive edges of $C$, where $e_0 = e_{CR}$ and $e_l$
is incident with \( t_0 \) but \( e_i \neq e_{CS} \) for \( i = 1, \ldots, l \), see Fig. 3. Then

\[
T_0 = H - e_0, \; T_1 = H - e_1, \; \ldots, \; T_l = H - e_l, \; T_{l+1} = H - e_{CS} + e_{CR} - f
\]

is a sequence of spanning trees of \( G \) with \( \Delta(T_s) = i \) (\( 0 \leq s \leq l + 1 \)). In this sequence \( T_s \) is adjacent to \( T_{s+1} \) and \( |n_A(T_{s+1}) - n_A(T_s)| \leq 1 \) for \( s = 0, \ldots, l - 1 \). The tree \( T_{l+1} \) is obtained from \( T_l \) by a double exchange of edges and \( |n_A(T_{l-1}) - n_A(T_l)| \leq 1 \). Finally, \( e_{CT_{l+1}} = e_{CS} \) and therefore \( T_0, \ldots, T_{l+1} \) is an \((R,S)\)-sequence in \( G \).

Case 2: For the trees \( R \) and \( S \) and for every cycle \( C \) of \( G \) there are vertices \( x \) and \( x' \) in \( C \) such that \( d_{R+e_{CR}}(x) > i \) and \( d_{S+e_{CS}}(x') > i \). This implies that in every cycle \( C \) of \( G \) at least one vertex incident with the edge \( e_{CR} \) (\( e_{CS} \), resp.) is of degree \( i \) in \( R \) (\( S \), resp.) and of degree \( i + 1 \) in \( R + e_{CR} \) (\( S + e_{CS} \), resp.). Let \( D \) be a cycle of \( G \) which contains at most one vertex belonging to another cycle of \( G \). Assume that \( v_0 \) is the unique vertex of \( D \) which can belong to another cycle of \( G \). It follows from the choice of \( D \) that every vertex \( t \in V(D) - \{v_0\} \) is incident with at most one edge of \( G \) which is not in \( R \) (\( S \), resp.). This implies that \( d_{G}(t) \leq i + 1 \) for every \( t \in V(D) - \{v_0\} \). Certainly, if \( d_{G}(t) = i + 1 \) for some vertex \( t \in V(D) - \{v_0\} \), then, since \( \Delta(R) = \Delta(S) = i \), both the edges \( e_{DR} \) and \( e_{DS} \) are incident with \( t \) and the trees \( R \) and \( R + e_{DR} - e_{DS} \) form an \((R,S)\)-sequence in \( G \). If \( d_{G}(t) \leq i \) for every \( t \in V(D) - \{v_0\} \), then \( v_0 \) is the unique vertex (of \( D \)) of degree \( i + 1 \) in \( R + e_{DR} \) and \( S + e_{DS} \) and thus both the edges \( e_{DR} \) and \( e_{DS} \) are incident with \( v_0 \). Then again the trees \( R \) and \( R + e_{DR} - e_{DS} \) form an \((R,S)\)-sequence in \( G \).

Thus all possible cases have been considered and, hence, the result follows by the principle of induction. \( \square \)

**Theorem 2.** If \( G \) is a connected graph with \( p \) vertices and \( p + 1 \) edges \((p \geq 4)\), then the set \( \mathcal{N}(G) \) has at most one gap.

**Proof.** Let \( G \) be a connected graph with \( p \) vertices and \( p + 1 \) edges. Then it is easy to observe that either \( G \) is a cactus with two cycles or \( G \) consists of a subgraph \( \Theta \) and a family of trees attached to vertices of \( \Theta \), where the subgraph \( \Theta \) consists of two vertices joined by three interior-disjoint paths, see Fig. 4. Certainly, if \( G \) is a cactus (or if \( G \) contains a subgraph \( \Theta \) and \( m = M \), resp.), then \( \mathcal{N}(G) \) has at most one gap by Theorem 1 (Corollary 1, resp.). Thus assume that \( G \) contains a subgraph \( \Theta \) and \( m < M \). Let \( u \) and \( v \) be the vertices of degree three in \( \Theta \). Let \( A \) be the set
of all vertices of degree $\Delta(G)$ in $G$. Similarly, let $B$ denote the set of vertices of degree $\Delta(G) - 1$ in $G$. It is clear from the structure of $G$ that for every spanning tree $T$ of $G$ there exist two edges $e$ and $f$ belonging to different $u - v$ paths of $\Theta$ and such that their removal from $G$ results in $T$, that is, $T = G - e - f$. In addition, since the removal of an edge from a graph decreases the degrees of its end-vertices by 1 and leaves unchanged the degree of all other vertices, it is clear that $\Delta(T) \geq \Delta(G) - 2 = M - 2$ and therefore either $m = M - 2$ or $m = M - 1$. Note that if $T = G - e - f$ is a spanning tree of $G$ with $\Delta(T) < M$, then the edges $e$ and $f$ must cover all vertices of the set $A$. This implies that $A$ is a subset of $V(\Theta)$ and that $A$ contains at most four vertices. We consider the following two cases: $m = M - 2$, $m = M - 1$.

Case 1: $m = M - 1$. By Corollary 2, it suffices to prove that the set $\mathcal{N}_m(G)$ has no gap. We consider four subcases depending on the cardinality of $A$.

Case 1.1: $\vert A \vert = 4$. If $T = G - e - f \in \mathcal{F}_m(G)$, then the edges $e$ and $f$ precisely cover all vertices of $A$. From this we conclude that $\{x \in V(G) : d_T(x) = m\} = A \cup B$, so $n_A(T) = \vert A \cup B \vert = 4 + \vert B \vert$ and the set $\mathcal{N}_m(G) = \{4 + \vert B \vert\}$ has no gap.

Case 1.2: $\vert A \vert = 3$. If $T = G - e - f \in \mathcal{F}_m(G)$, then the edges $e$ and $f$ cover all vertices of $A$ and at most one vertex of $B$. Thus, either $n_A(T) = 3 + \vert B \vert$ or $n_A(T) = 2 + \vert B \vert$. Hence, $\mathcal{N}_m(G)$ is a subset of $\{2 + \vert B \vert, 3 + \vert B \vert\}$ and $\mathcal{N}_m(G)$ has no gap.

Case 1.3: $\vert A \vert = 2$, say $A = \{a_1, a_2\}$. By Lemma 4, in order to prove that the set $\mathcal{N}_m(G)$ has no gap, it suffices to show that there exists a sequence $T_0, T_1, \ldots, T_p$ consisting of all the trees belonging to $\mathcal{F}_m(G)$ in which $T_j$ is adjacent to $T_{j+1}$ for $j = 0, \ldots, p - 1$. We distinguish the following five possibilities.

1. $a_1$ is adjacent to $a_2$ and $\{a_1, a_2\} \cap \{u, v\} = \emptyset$. If $e_1, \ldots, e_n$ are the consecutive edges of the unique cycle of $H = G - a_1a_2$, then the trees $T_i = H - e_i$, $1 \leq i \leq n$, form the set $\mathcal{F}_m(G)$ and $T_j$ is adjacent to $T_{j+1}$ for every $j$, $1 \leq j < n$.

2. $a_1$ is adjacent to $a_2$ and $\{a_1, a_2\} = \{u, v\}$. Let $e_1, \ldots, e_n$ be the consecutive edges of the unique cycle of $H = G - uv$, where $e_1$ and $e_n$ are incident with $u$ while $e_l$ and $e_{l+1}$ are incident with $v$ (for some $l$ with $1 < l < n$). Then the trees $T_1 = G - e_1 - e_{l+1}$, $T_2 = H - e_1$, \ldots, $T_{n+1} = H - e_n$, $T_{n+2} = G - e_l - e_n$ form the set $\mathcal{F}_m(G)$ and $T_j$ is adjacent to $T_{j+1}$ for every $j$, $1 \leq j \leq n + 1$. 

Fig. 4.
3. \(a_1\) is not adjacent to \(a_2\) but \(\{a_1, a_2\} = \{u, v\}\). Assume that \(N_\Theta(u) = \{u_1, u_2, u_3\}\), \(N_\Theta(v) = \{v_1, v_2, v_3\}\), and suppose that \(u_i\) and \(v_i\) \((i = 1, 2, 3)\) belong to the same \(u-v\) path in \(\Theta\). Then the trees

\[
T_1 = G - uu_1 - vv_3, \quad T_2 = G - uu_1 - vv_2, \quad T_3 = G - uu_3 - vv_2,
\]

\[
T_4 = G - uu_3 - vv_1, \quad T_5 = G - uu_2 - vv_1, \quad T_6 = G - uu_2 - vv_3
\]

form \(\mathcal{T}_m(G)\) and \(T_j\) is adjacent to \(T_{j+1}\) for \(j = 1, \ldots, 5\).

4. \(a_1\) and \(a_2\) belong to different \(u-v\) paths of \(\Theta\) and \(\{a_1, a_2\} \cap \{u, v\} = \emptyset\). For \(i = 1, 2\), let \(b_i\) and \(c_i\) be the neighbours of \(a_i\) in \(\Theta\), where \(b_i\) belongs to the \(a_i-u\) path in \(\Theta - v\), while \(c_i\) belongs to the \(a_i-v\) path in \(\Theta - u\). Now it is easy to observe that the trees

\[
T_1 = G - a_1b_1 - a_2b_2, \quad T_2 = G - a_1b_1 - a_2c_2, \quad T_3 = G - a_1c_1 - a_2c_2,
\]

\[
T_4 = G - a_1c_1 - a_2b_2
\]

form \(\mathcal{T}_m(G)\) and \(T_j\) is adjacent to \(T_{j+1}\) for \(j = 1, 2, 3\).

5. \(|\{a_1, a_2\} \cap \{u, v\}| = 1\), say \(a_1 \notin \{u, v\}\) and \(a_2 = u\). Assume that \(N_\Theta(a_1) = \{b_1, c_1\}\), where \(b_1\) belongs to the \(a_1-u\) path in \(\Theta - v\), and let \(b_2\) and \(c_2\) be the neighbours of \(u\) which belong to the unique cycle of \(\Theta - a_1\). If \(a_1\) is not adjacent to \(a_2\), then the trees

\[
T_1 = G - a_1b_1 - a_2b_2, \quad T_2 = G - a_1b_1 - a_2c_2, \quad T_3 = G - a_1c_1 - a_2c_2,
\]

\[
T_4 = G - a_1c_1 - a_2b_2
\]

form \(\mathcal{T}_m(G)\) and \(T_j\) is adjacent to \(T_{j+1}\) for \(j = 1, 2, 3\). If \(a_1\) is adjacent to \(a_2\) and \(e_1, e_2, \ldots, e_r\) are the consecutive edges of the unique cycle of \(H = G - a_1a_2\) with \(e_1 = a_2b_2\) and \(e_r = a_2c_2\), then the trees

\[
T_1 = G - a_1c_1 - e_r, \quad T_2 = G - a_1c_1 - e_1, \quad T_3 = H - e_1,
\]

\[
T_4 = H - e_2, \ldots, \quad T_{r+2} = H - e_r
\]

form \(\mathcal{T}_m(G)\) and \(T_j\) is adjacent to \(T_{j+1}\) for \(j = 1, \ldots, r + 1\).

Case 1.4: \(|A| = 1\), say \(A = \{a\}\). Assume first that \(a \notin \{u, v\}\). Let \(f_1\) and \(f_2\) be the edges of \(\Theta\) incident with \(a\) and let \(H = G - f_1, F = G - f_2\). Now if \(e_1, \ldots, e_n\) are the consecutive edges of the unique cycle of \(\Theta - a\), then the trees

\[
T_1 = H - e_1, \ldots, \quad T_n = H - e_n, \quad T_{n+1} = F - e_n,
\]

\[
T_{n+2} = F - e_1, \ldots, \quad T_{2n} = F - e_{n-1}
\]

form \(\mathcal{T}_m(G)\) and \(T_j\) is adjacent to \(T_{j+1}\) for \(j = 1, \ldots, 2n - 1\).

Finally assume that \(a \in \{u, v\}\), say \(a = u\) and \(N_\Theta(u) = \{u_1, u_2, u_3\}\). Let \(f_1, \ldots, f_t (h_1, \ldots, h_m)\) and \(k_1, \ldots, k_n\), resp.) be the consecutive edges of the \(u-v\) path in \(\Theta - uu_2 - uu_3\) \((\Theta - uu_1 - uu_3\) and \(\Theta - uu_1 - uu_2,\) resp.) with \(f_1 = uu_1 (h_1 = uu_2\)
and \( k_1 = uu_3 \), resp.), see Fig. 4. Now, if \( F = G - f_1, H = G - h_1 \) and \( K = G - k_1 \), then in the sequence
\[
F - h_1, F - h_2, \ldots, F - h_m, F - k_n, F - k_{n-1}, \ldots, F - k_1,
\]
\[
H - f_1, H - f_2, \ldots, H - f_1, H - k_n, H - k_{n-1}, \ldots, H - k_1,
\]
\[
K - f_1, K - f_2, \ldots, K - f_1, K - h_m, K - h_{m-1}, \ldots, K - h_1
\]
every two neighbouring trees are adjacent and they form the set \( \mathcal{S}_m(G) \).

Case 2: \( m = M - 2 \). By Corollary 2, it suffices to show that neither \( \mathcal{N}_m(G) \) nor \( \mathcal{N}_{m+1}(G) \) has a gap. Take any spanning tree \( T = G - e - f \) of \( G \) with \( \Delta(T) = m \). The assumption \( \Delta(T) = m = \Delta(G) - 2 \) implies that every vertex of \( A \) must be covered by the edges \( e \) and \( f \), so \( e \) and \( f \) are adjacent and consequently either \( A = \{u\} \) or \( A = \{v\} \), say \( A = \{u\} \) and assume that \( N_0(u) = \{u_1, u_2, u_3\} \). Since \( \Delta(T) < M - 1 \), every vertex of \( B \) must be covered by \( e \) or \( f \). Thus \( |B| \leq 2 \) and \( B \subseteq N_0(u) \). We distinguish three subcases depending on the cardinality of \( B \).

Case 2.1: \( B = \emptyset \). Then \( \mathcal{S}_m(G) \) consists of the three mutually adjacent trees \( G - uu_1 - uu_2, G - uu_1 - uu_3, G - uu_2 - uu_3 \) and so \( \mathcal{N}_m(G) \) has no gap (by Lemma 4). Since \( u \) is the unique vertex of degree at least \( \Delta(G) - 1 = m + 1 \) in \( G \), \( u \) is the unique vertex of degree \( m + 1 \) in every \( T \in \mathcal{F}_{m+1}(G) \) and therefore \( \mathcal{N}_{m+1}(G) = \{1\} \).

Case 2.2: \( |B| = 1 \), say \( B = \{u_1\} \). In this case the edge \( uu_1 \) cannot occur in any spanning tree \( T \in \mathcal{F}_m(G) \), so \( \mathcal{F}_m(G) = \{G - uu_1 - uu_2, G - uu_1 - uu_3\} \) and by Lemma 4, \( \mathcal{N}_m(G) \) has no gap. Since \( u \) and \( u_1 \) are the only vertices of degree at least \( \Delta(G) - 1 = m + 1 \) in \( G \), only \( u \) or \( u_1 \) can be of degree \( m + 1 \) in any \( T \in \mathcal{F}_{m+1}(G) \) and it is no problem to observe that \( \mathcal{N}_{m+1}(G) = \{1, 2\} \).

Case 2.3: \( |B| = 2 \), say \( B = \{u_1, u_2\} \). Then \( \mathcal{F}_m(G) = \{G - uu_1 - uu_2\} \) and this trivially implies that \( \mathcal{N}_m(G) \) has no gap. Finally, since \( u, u_1, u_2 \) are the only vertices of degree at least \( \Delta(G) - 1 = m + 1 \) in \( G \), only \( u, u_1 \) or \( u_2 \) can be of degree \( m + 1 \) in any \( T \in \mathcal{F}_{m+1}(G) \). This implies that \( \mathcal{N}_{m+1}(G) \) is a subset of \( \{1, 2, 3\} \). Finally, it is easy to observe that if \( 3 \in \mathcal{N}_{m+1}(G) \), then \( 2 \in \mathcal{N}_{m+1}(G) \) and therefore \( \mathcal{N}_{m+1}(G) \) has no gap.

This completes Case 2 and finishes the proof of Theorem 2.

4. Conclusion

Let \( A = \{a_1, \ldots, a_n\} \) be a set of positive integers with \( a_1 < a_2 < \cdots < a_n \). It follows from Lemmas 2 and 3 that if \( A \) is a feasible \( n_A \)-set, then \( A \) is a set of consecutive integers or \( a_1 = 1 \). On the other hand, the graph \( G_{a,b,c} \) in Fig. 5 with \( a = 0, b = a_1 \) and \( c = n - 1 \) shows that every set of consecutive integers, \( A = \{a_1, a_1 + 1, \ldots, a_1 + n - 1\} \), is an \( n_A \)-set. The same graph with \( a = i, b = g \) and \( c = n - i - 1 \) shows that a set \( A \) containing 1 and having exactly one gap, \( A = \{1, \ldots, i + g, i + 1 + g, \ldots, n - 1 + g\} \), is an \( n_A \)-set. For a long time we believed that every \( n_A \)-set has at most one gap. That this is not the case shows the graph in
Fig. 6 (delivered by a referee) for which the $n_d$-set $\{1, 2, 3, 4, 6, 8\}$ has two gaps. In fact it is easy to see that joining together many such graphs one can get a graph for which the $n_d$-set has an arbitrary number of gaps. In connection with this it would be desirable to completely characterize feasible $n_d$-sets. One can ask if it is possible to find a nontrivial upper bound on the number of gaps in $n_d$-sets. And in fact in connection with Theorems 1 and 2 it would be interesting to know if the number of gaps in the $n_d$-set of a graph can be bounded by the cyclomatic number of the graph.

References