Generating functions for the generalized Gauss hypergeometric functions

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abstract

Formulas and identities involving many well-known special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) play important roles in themselves and in their diverse applications. Various families of generating functions have been established by a number of authors in many different ways. In this paper, we aim at establishing some (presumably new) generating functions for the generalized Gauss type hypergeometric function

\[ F_p(a; b; j; l) \]

which is introduced here. We also present some special cases of the main results of this paper.

1. Introduction

Many important functions in applied sciences (which are popularly known as special functions) are defined via improper integrals or infinite series (or infinite products). During last four decades or so, several interesting and useful extensions of many of the familiar special functions (such as the Gamma and Beta functions, the Gauss hypergeometric function, and so on) have been considered by several authors (see, for example, [4–7]; see also the very recent work [2]). The above-mentioned works have largely motivated our present study.

Recently, Parmar [8] introduced and studied some fundamental properties and characteristics of the generalized Beta type function \( B_p^{(a,b)}(x,y) \) defined by Parmar (see [8, p. 37, Eq. (19)]):

\[
B_p^{(a,b)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}F_1\left( a; b; -\frac{p}{t(1-t)} \right) dt,
\]

\((\Re(p) \geq 0; \min(\Re(x), \Re(y), \Re(z), \Re(b)) > 0; \Re(\mu > 0)),\)

which, in the special case when \( \mu = 1 \), reduces immediately to the generalized Beta type function defined earlier as follows (see, for details, [7, p. 4602, Eq. (4)]; see also [6, p. 32, Chapter 4]):

\[
B_p^{(a,b)}(x,y) = B_p^{(a,b,1)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}F_1\left( a; b; -\frac{p}{t(1-t)} \right) dt,
\]

\( \frac{\partial}{\partial x}(\Re(p) \geq 0; \min(\Re(x), \Re(y), \Re(z), \Re(b)) > 0). \)
(ℜ(p) ≥ 0; min{ℜ(z), ℜ(y), ℜ(μ), ℜ(β)} > 0).

For α = β, (1.2) reduces obviously to the extended Beta type function \( B_p(x, y) \) due to [4] is defined by Chaudhry et al. (see [4, p. 20, Eq. (1.7)]; see also [5, p. 591, Eq. (1.7)]):

\[
B_p(x, y) = B_p^{(x,y)}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp \left( -\frac{p}{t(1-t)} \right) dt \quad (ℜ(p) ≥ 0).
\]

Clearly, for the classical Beta function \( B(x, y) \), we have the following relationships:

\[
B(x, y) = B_0(x, y) = B_0^{(x,y)}(x, y) = B_0^{(x,1)}(x, y),
\]

where

\[
B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt \quad (ℜ(x) > 0; ℜ(y) > 0).
\]

In the year 2004, by making use of the extended Beta function \( B_p(x, y) \) defined by (1.3), Chaudhry et al. [5] extended the Gauss hypergeometric function \( _2F_1 \), as follows (see [5, p. 591, Eqs. (2.1) and (2.2)]):

\[
F_p(a, b; c; z) := \sum_{n=0}^{∞} \binom{a}{n} B_p(b + n, c - b) z^n \left/ \left( n! \right) \right.,
\]

\(|z| < 1; ℜ(c) > ℜ(b) > 0; ℜ(p) ≥ 0),

where \( \binom{a}{n} \) denotes the Pochhammer symbol defined (for \( a ∈ C \)) by (see [15, p. 2 and pp. 4–6]; see also [14, p. 2]):

\[
\binom{a}{n} := \frac{Γ(a+n)}{Γ(a)} = \begin{cases} 1 & (n = 0), \\ a(a+1)\cdots(a+n-1) & (n ∈ N := \{1, 2, 3, \ldots\}). \end{cases}
\]

provided that the Gamma quotient exists (see, for details, [16, et seq.] and [17, p. 22 et seq.]). Similarly, by appealing to the definition (1.2) of the generalized Beta function \( B_p^{(a,y)}(x, y) \), Özergin [6] and Özergin et al. [7] introduced and studied a further potentially useful extension of the generalized Gauss hypergeometric functions as follows (see, for example, [7, p. 4606, Section 3]; see also [6, p. 39, Chapter 4]):

\[
F_p^{(x,y)}(a, b; c; z) := \sum_{n=0}^{∞} \binom{a}{n} B_p^{(x,y)}(b + n, c - b) z^n \left/ \left( n! \right) \right.,
\]

\(|z| < 1; min\{ℜ(z), ℜ(μ)\} > 0; ℜ(c) > ℜ(b) > 0; ℜ(p) ≥ 0),

where \( \binom{a}{n} \) denotes the Pochhammer symbol defined by (1.6).

Based upon the generalized Beta function in (1.1), Parmar [8] introduced and studied a family of the generalized Gauss hypergeometric functions defined by Parmar (see [8, p. 44])

\[
F_p^{(x,1)}(a, b; c; z) := \sum_{n=0}^{∞} \binom{a}{n} B_p^{(x,1)}(b + n, c - b) z^n \left/ \left( n! \right) \right.,
\]

\(|z| < 1; min\{ℜ(z), ℜ(μ)\} > 0; ℜ(c) > ℜ(b) > 0; ℜ(p) ≥ 0).

Clearly, we have

\[
F_p^{(x,0)}(a, b; c; z) = F_p^{(x,y)}(a, b; c; z),
\]

\[
F_p^{(x,x)}(a, b; c; z) = F_p(a, b; c; z)
\]

and

\[
F_p^{(x,x)}(a, b; c; z) = _2F_1(a, b; c; z)
\]

in terms of the familiar Gauss hypergeometric function \( _2F_1 \).

In several areas in applied mathematics and mathematical physics, generating functions play an important rôle in the investigation of various useful properties of the sequences which they generate. They are used to find certain properties and formulas for numbers and polynomials in a wide variety of research subjects such as, for example, modern combinatorics (see [1,3,17,18]). Agarwal et al. [2] gave some interesting new classes of generating functions involving the generalized Gauss type hypergeometric function \( F_p^{(x,y)} \) defined by (1.5). In the present sequel to the aforementioned and many other recent investigations (see, for example, [3,9–13,18–24]; see also the monograph on the subject of generating functions by Srivastava and Manocha [17]), we present some (presumably new) generating functions involving the following family of generalized Gauss type hypergeometric functions:
\[ F_p^{(\alpha, \beta, \kappa, \mu)} (a; b; c; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} B_p^{(\alpha, \beta, \kappa, \mu)} (b + n, c - b) \frac{z^n}{n!}, \quad (1.12) \]

\[
( |z| < 1; \min\{\Re(x), \Re(y), \Re(z)\} > 0; \Re(c) > \Re(b) > 0; \Re(p) \geq 0),
\]

where the generalized Beta function \( B_p^{(\alpha, \beta, \kappa, \mu)} (x, y) \) is defined by

\[
B_p^{(\alpha, \beta, \kappa, \mu)} (x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} F_1 \left( \alpha; \beta; -\frac{p}{(1-t)^\kappa} \right) dt,
\quad (1.13)
\]

\[
( \Re(p) \geq 0; \min\{\Re(x), \Re(y), \Re(z)\} > 0; \min\{\Re(\kappa), \Re(\mu)\} > 0).
\]

Obviously, in their special cases when \( \kappa = \mu \), the definitions in (1.13) and (1.12) would reduce immediately to those in (1.1) and (1.8), respectively. Some interesting special cases of our main results are also considered.

### 2. Main results

In this section, we derive generating functions for the generalized Gauss type hypergeometric function defined by (1.12). Our main results are asserted by Theorems 1 and 2 below. We first recall that a generalized binomial coefficient \( \binom{\lambda}{\mu} \) may be defined (for real or complex parameters \( \lambda \) and \( \mu \)) by

\[
\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda+1-\mu)} =: \binom{\lambda}{\mu} (\lambda, \mu \in \mathbb{C}),
\quad (2.1)
\]

so that, in the special case when \( \mu = n \) (\( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \)), we get

\[
\binom{\lambda}{n} = \frac{(\lambda-1) \cdots (\lambda-n+1)}{n!} = (-1)^n \binom{-\lambda}{n} \quad (n \in \mathbb{N}_0),
\quad (2.2)
\]

where \( \binom{\lambda}{n} \) (\( \lambda \in \mathbb{C} \)) denotes the Pochhammer symbol given by (1.6).

**Theorem 1.** For \( \Re(p) \geq 0 \) and \( \lambda \in \mathbb{C} \), the following generating function holds true:

\[
\sum_{n=0}^{\infty} \frac{(\lambda + n - 1)}{n!} F_p^{(\alpha, \beta, \kappa, \mu)} (\lambda + n, b; c; z) t^n = (1 - t)^{-\lambda} F_p^{(\alpha, \beta, \kappa, \mu)} \left( \lambda, b; c; \frac{z}{1-t} \right), \quad (|t| < 1),
\quad (2.3)
\]

where the function \( F_p^{(\alpha, \beta, \kappa, \mu)} (a; b; c; z) \) is given by (1.12).

**Proof.** For convenience, let the left-hand side of the assertion (2.3) of Theorem 1 be denoted by \( S \). Then, by substituting the series expression from (1.12) into \( S \), we find that

\[
S = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k=0}^{\infty} \binom{\lambda}{n} \frac{B_p^{(\alpha, \beta, \kappa, \mu)} (b + k, c - b)}{B(b, c - b)} \frac{z^k}{k!} \right) t^n,
\quad (2.4)
\]

which, upon changing the order of summation and after a little simplification, yields

\[
S = \sum_{k=0}^{\infty} \binom{\lambda}{k} \frac{B_p^{(\alpha, \beta, \kappa, \mu)} (b + k, c - b)}{B(b, c - b)} \left[ \sum_{n=0}^{\infty} \frac{(\lambda + n + k - 1)}{n!} \frac{z^k}{t^n} \right],
\quad (2.5)
\]

Finally, if we apply the following generalized binomial expansion:

\[
\sum_{n=0}^{\infty} \frac{(\lambda + n - 1)}{n!} t^n = (1 - t)^{-\lambda} \quad (|t| < 1; \lambda \in \mathbb{C}),
\quad (2.6)
\]

for evaluating the inner sum in (2.5), we get the desired assertion (2.3) of Theorem 1. \( \square \)

**Remark 1.** For \( \kappa = \mu = 1 \) (and for \( \kappa = \mu = 1 \) and \( p = 0 \)), the generating function (2.3) asserted by Theorem 1 was derived earlier by Agarwal et al. [2].

**Definition.** In terms of the generalized Gauss type hypergeometric function given by (1.12), we define a sequence \( \{ \Omega_{a,b,c}^{(\alpha, \beta, \kappa, \mu)} \}_{n=0}^{\infty} \) as follows:
Theorem 2. For each \( N \in \mathbb{N} \), the following generating function holds true:

\[
\sum_{n=0}^{\infty} \binom{\lambda+n}{N} \binom{\lambda+n}{m} (1-t)^{-m-n} \frac{z^{n}}{(1-t)^{n}}.
\]

(2.8)

Proof. Using the definitions (2.7) and (1.12) and changing the order of summation, the left-hand side \( \zeta \) of the result (2.8) is given by

\[
\zeta = \sum_{k=0}^{\infty} \binom{\lambda+m}{N} \binom{\lambda+m+1}{N} \cdots \binom{\lambda+m+N-1}{N} \frac{B_{\varphi}(a+b+k,c-b)}{B(a,b-c)} \left[ \sum_{n=0}^{\infty} \binom{\lambda+m+Nk+n-1}{n} t^{n} \right]^{\varphi} \frac{z^{k}}{k!}.
\]

(2.9)

Now, by appealing once again to the generalized binomial expansion, we easily arrive at the desired result (2.8) asserted by Theorem 2.

Remark 3. If we set \( N = 1 \) and replace \( \lambda \) by \( \lambda - m \) (\( m \in \mathbb{N}_{0} \)) in (2.8), we readily obtain the assertion (2.3) of Theorem 1.

3. Concluding remarks and observations

It is interesting to mention here that, whenever a generalized Gauss type hypergeometric function \( F_{\varphi}(a,b;c;z) \) reduces to the Gauss hypergeometric function and other related hypergeometric functions, the results become relatively more important from the application viewpoint. Most of the special functions of mathematical physics and engineering, such as the Jacobi and Laguerre polynomials, can be expressed in terms of the Gauss hypergeometric function and other related hypergeometric functions. Therefore, the numerous generating functions involving extensions and generalizations of the Gauss hypergeometric function are capable of playing important rôles in the theory of special functions of applied mathematics and mathematical physics.

We conclude our present investigation by remarking further that the results obtained here are of general character and (upon specialization) can yield the corresponding generating functions for each of the families of the generalized Gauss type hypergeometric functions \( F_{\varphi}(a,b;c;z) \) defined by (1.8) including such relatively more familiar hypergeometric functions as (for example) the generalized Gauss type hypergeometric function \( F_{\varphi}(a,b;c;z) \) defined by (1.7), the extended Gauss hypergeometric function \( F_{\varphi}(a,b;c;z) \) defined by (1.5), other special functions which are expressible in terms of the Gauss hypergeometric function, and so on.

References