On the Pairwise Error Probability Bounds of STTC over Nakagami-m Fading Channels

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Abstract- We investigate the pairwise error probability (PEP) bounds for space-time trellis codes (STTC) in Nakagami-m fading channels with arbitrary fading severity parameter \(m\). We derived an analytical expression for PEP upper bound in the case of two transmit antennas in terms of the higher transcendental functions such as Appell hypergeometric function, or Lauricella function over uncorrelated and correlated Nakagami-m fading channels. Furthermore, we show that the probability bound for the general multiple transmit antenna scenario can also be expressed as an integral involving nested summations.

Keywords: Correlated fading, Multiple-Input Multiple-Output (MIMO), Nakagami-m fading channels, Pairwise error probability (PEP), Space-time trellis codes (STTC), Wireless communications.

I. INTRODUCTION

The main impairment in wireless communications systems is due to fading. The fading can be frequency flat or frequency selective. Both cases under adverse conditions give rise to high rate of errors in the transmission. The techniques developed to combat fading include coding, diversity, equalization etc. The space diversity using MIMO configuration together with the temporal diversity in terms of an encoder can be used to provide high spectral efficiency and better frame error rates in wireless communications systems. The pioneering work in this area is by Tarokh et. al. [1], who proposed space time trellis codes. This configuration is ideally suited for frame/packet-based communication.

Recently there has been a lot of interest in the MIMO based space-time coded systems due to the need for increased channel capacity and high bit rate applications [1]-[6]. In [1], based on the bound for pairwise error probability, code design criteria has been obtained for flat Rayleigh and Rician channels under different fading conditions. All these studies are based on the analytical results derived for the PEP, which is supposed to be the key to performance evaluation of space-time coding.

Several statistical models have been used in the literature to describe the fading envelope of the received signal. The Rayleigh and Rician distributions are used to characterize the envelope of faded signals over small geographical areas or short term fades, while the lognormal distribution is used when much wider areas are involved. But Nakagami-m distribution, proposed by Nakagami [7] is a more versatile statistical model that can model a variety of fading environments including those modeled by Rayleigh and one sided Gaussian distributions. Furthermore, [8] and [9] have demonstrated that Nakagami distribution is more flexible and more accurately fits experimental data for many physical propagation channels than other distributions. Thus, it can be summarized that analysis of PEP bounds for STTC over correlated and uncorrelated Nakagami-m channels is very useful from both theoretical and practical viewpoints. Performance of space-time codes has been discussed in [10] under Rician- Nakagami-m shadowing and in [11] over keyhole Nakagami-m channels. To the authors’ best knowledge no analytical PEP upper bounds are available for the STTC over Nakagami-m fading channels.

The main contribution of this paper is the derivation of closed form expression for the PEP bounds over the Nakagami-m fading channels. The remainder of this paper is organized as follows. The MIMO system model is presented in Section II and the PEP derivation is given in Section III. The main points are summarized in Section IV.

II. SYSTEM MODEL

We consider a communication system employing \(M_T\) transmit antennas and \(M_R\) receive antennas.

A. Channel Model

The fading coefficient \(\alpha_{i,j}\), from transmit antenna \(i\) to receive antenna \(j\) is defined as \(\alpha_{i,j} = r_{ij} e^{j\varphi_{ij}}\) where \(j = \sqrt{-1}\). Following [11] we can say that \(r_{ij}\) has a Nakagami-m distribution with fading severity parameter \(m\) and is given by

\[
f(r_{ij}) = \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^m r_{ij}^{2m-1} \exp\left( - \frac{m}{\Omega} r_{ij}^2 \right), \quad r_{ij} \geq 0 \quad (1)
\]

where \(\Gamma(.)\) is the Gamma function, \(\Omega = E(r_{ij}^2)\) and \(m \geq 1/2\). The channel phase shift \(\varphi_{ij}\) is assumed to be uniformly distributed over [0,2\(\pi\)]. If we restrict the number of transmitting antennas \((M_T)\) to 2 and assume that \(r_{ij}\) and
are correlated but identical, then the joint density function of \( r_{1j}, r_{2j} \) is given by [7]

\[
f(r_{1j}, r_{2j}) = \frac{4(r_{1j}r_{2j})^m m^{m+1} \exp \left( -\frac{m(r_{1j}^2 + r_{2j}^2)}{\Omega(1 - \rho)} \right)}{\Gamma(m)\Omega^{m+1}(1 - \rho)(\sqrt{\rho})^{m-1}} \times I_{m-1} \left( \frac{2\sqrt{\rho}r_{1j}r_{2j}m}{\Omega(1 - \rho)} \right)
\]

(2)

where \( \Omega = E(r_{1j}^2) = E(r_{2j}^2) \), \( \Gamma(.) \) denotes the Gamma function, \( \rho \) denotes the power correlation coefficient, \( m \geq 1/2 \), \( r_{1j}, r_{2j} \geq 0 \) and \( I_m \) denotes the \( m \)th order modified Bessel function of first kind.

### B. Space-Time Codes

The space-time code generates \( M_T \) streams of modulated constellation symbols, which will be transmitted simultaneously using \( M_T \) antennas. Complex valued modulation symbol transmitted by the antenna \( i \) during time slot \( t \) is denoted by \( c_i \). The observed signal at each receive antenna is a noisy superposition of the \( M_T \) transmitted signals corrupted by frequency flat fading. All sub channels from each transmitted antenna to receive antenna, are assumed to have identical statistical descriptions and quasi-static fading channel model. The receiver employs a maximum likelihood decoder and has perfect channel state information (CSI). Thus at the receive antenna \( j \), the sampled version of one frame of the received signal is given as

\[
y_j^f = \sum_{i=1}^{M_R} \alpha_{i,j}^f e_i^f \sqrt{E_s} + n_j^f, \quad j = 1, \ldots, M_R
\]

(3)

where \( E_s \) is the energy per symbol at each transmit antenna, \( y_j^f \) is the received signal at antenna \( j \) at the time slot \( t \), \( n_j^f \) is a complex white Gaussian random noise sample at antenna \( j \) at the time slot \( t \) with variance \( N_0/2 \) per dimension, and \( \alpha_{i,j}^f \) is the fading coefficient from transmit antenna \( i \) to receive antenna \( j \).

Then we express all the signals received by the \( M_R \) antennas by employing the following MIMO channel model [12].

\[
Y = \sqrt{E_s}HC + N
\]

(4)

where \( H \) is an \( M_R \times M_T \) channel coefficient matrix, \( Y \) and \( N \) are \( M_R \times l \) complex matrices, \( C \) is an \( M_T \times l \) complex codeword matrix. Note that here \( l \) denotes the frame length and entries of \( N \) are independent samples of a zero mean complex Gaussian random variable with variance \( N_0/2 \) per dimension.

### III. Derivation of Pairwise Error Probability

Assume that the transmitted codeword is \( C_s \) and the erroneously decoded codeword is \( E_s \). The codeword difference matrix is defined as \( B_s = C_s - E_s \) and the positive definite Hermitian matrix \( A \) is defined as \( A = B_s B_s^H \) where \( H \) represents the conjugate transpose operation.

Assuming perfect channel state information (CSI) is available at the receiver, the conditional pairwise error probability is given by [2]

\[
P(C_s, E_s \mid H) = Q \left( \frac{E_s}{2N_0} d^2(C_s, E_s) \right)
\]

(5)

and it is then upper bounded by the Chernoff bound [13]

\[
P(C_s, E_s \mid H) \leq \exp \left( -d^2(C_s, E_s) \frac{E_s}{4N_0} \right).
\]

(6)

where \( d^2(C_s, E_s) = Tr(BHB^H) \). \( Tr(.) \) denotes the trace of a matrix. Following [1] it can be written that

\[
d^2(C_s, E_s) = \sum_{j=1}^{M_R} h_j A h_j^H
\]

(7)

where \( h_j = (\alpha_{1,j}, \alpha_{2,j}, \ldots, \alpha_{M_T,j}) \). Since \( A \) is a positive definite Hermitian matrix, the eigenvectors of \( A \), \( \{v_1, v_2, v_3, \ldots, v_{M_T}\} \), form a complete orthonormal basis of an \( M_T \) dimensional vector space, and the eigenvalues \( \lambda_i \), \( i = 1, 2, \ldots, M_T \), of \( A \) are nonnegative real numbers. Let \( \beta_{ij} = h_j^H v_i \) and \( v_i = (v_{i1}, v_{i2}, v_{i3}, \ldots, v_{iM_T}) \), where \( v_{ih} = |v_{ih}|e^{j\theta_{ih}} \).

The conditional PEP is given by [1]

\[
P(C_s, E_s \mid H) \leq \exp \left( -\sum_{j=1}^{M_R} \sum_{i=1}^{M_T} \lambda_i |\beta_{ij}|^2 \frac{E_s}{4N_0} \right).
\]

(8)

#### A. The Case of Independent Fade Coefficients with Two Transmit Antennas

In this sub section we assume that \( M_T = 2 \) and random variables \( r_{ij} \) are modeled as i.i.d Nakagami-m variates with fading severity parameters \( m \). Then we can write

\[
|\beta_{ij}|^2 = |v_{i1}|^2 r_{ij}^2 + |v_{i2}|^2 r_{2j}^2 + 2|v_{i1}| |v_{i2}| r_{ij} r_{2j} \cos(\phi_{ij} - \phi_{2j} + \theta_{12} - \theta_{11})
\]

(9)
Equation (9) will yield an inequality

$$\beta_0^2 \geq |r_{ij}|^2 r_{ij}^2 + |v_{12}|^2 r_{ij}^2 - 2|v_{11}|r_{ij}r_{ij}. \quad (10)$$

Furthermore, inequality (8) can now be written as

$$P(C_s, E_s | H) \leq \exp \left( - \sum_{j=1}^{M_T} \left[ \left( \lambda_1 |v_{11}|^2 + \lambda_2 |v_{12}|^2 \right) r_{ij}^2 + \left( \lambda_1 |v_{12}|^2 + \lambda_2 |v_{11}|^2 \right) r_{ij}^2 - 2|v_{11}|r_{ij} \right) \times r_{ij}r_{ij} \left( \frac{E_s}{4N_0} \right). \quad (11)$$

Thus to compute an upper bound on the average probability of error, we simply average the inequality (11) with respect to independent Nakagami-m distributions of $r_{ij}$ to arrive at

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (12)$$

where $a = \beta_0 |v_{11}|^2 + \lambda_2 |v_{12}|^2$, $b = \lambda_1 |v_{12}|^2 + \lambda_2 |v_{11}|^2$, $c = |v_{11}| |v_{12}| (\lambda_1 + \lambda_2)$, and $\mu = E_s / 4N_0$.

Then the following transformation is applied to the integral in the right side of inequality (12).

$$r_{ij} = \frac{r}{\sqrt{a \mu + m / \Omega}} \cos(\psi) \quad r_{ij} = \frac{r}{\sqrt{b \mu + m / \Omega}} \sin(\psi)$$

where $0 \leq r < \infty$ and $0 \leq \psi \leq \pi / 2$. After some manipulations we get

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (17)$$

where $k = \frac{c \mu}{\sqrt{(a \mu + m / \Omega)(b \mu + m / \Omega)}}$. By performing the inner integral we arrive at

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (13)$$

where $k = \frac{c \mu}{\sqrt{(a \mu + m / \Omega)(b \mu + m / \Omega)}}$. By performing the inner integral we arrive at

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By substituting $t = \sin 2\psi$ and after some manipulations we get

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (15)$$

The integral in right side of inequality (15) can be expressed in closed form as

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (16)$$

where $P_1(a, b, b'; c, x, y)$ is the Appell hypergeometric function, which is a special case of more general category of functions called Lauricella functions [14].

B. The Case of Correlated Fade Coefficients with Two Transmit Antennas

In this sub section we assume that $M_T = 2$ and $r_{ij}, r_{ij}$ are correlated Nakagami-m random variables having the bivariate density function given by (2). In addition to that we assume that they are independent for different $j$ values.

Thus to calculate the PEP upper bound, inequality (11) is averaged with respect to (2) to arrive at

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (17)$$

where $C_\rho = \frac{4\rho^{m+1}}{\Gamma(m)\Omega^{m+1}(1-\rho)\sqrt{\rho}}$. By replacing the Bessel function term in its series expansion and following the same line of arguments as in the previous case we get

$$P(C_s, E_s) \leq \prod_{j=1}^{M_T} \left( \frac{2}{\Gamma(m)} \left( \frac{m}{\Omega} \right)^{2m} \right) \int_0^{\infty} \left( a \mu + \frac{m}{\Omega} \right) r_{ij}^2 - \left( b \mu + \frac{m}{\Omega} \right) r_{ij}^2 + 2c \mu r_{ij}r_{ij} \left( r_{ij}r_{ij} \right)^{2m-1} dr_{ij}dr_{ij} \quad (18)$$

where $D(k) = \frac{\sqrt{\rho \mu}}{\Omega(1-\rho)} \frac{2k^{2m+1}}{\Gamma(2k+2m)} \frac{1}{\Gamma(2k+2m)\Gamma(k+1)\Gamma(k+m)} \times \frac{1}{(a \mu + m / \Omega)(b \mu + m / \Omega)} \frac{c \mu}{\sqrt{(a \mu + m / \Omega)(b \mu + m / \Omega)}}$.
The integral in the right side of inequality (18) can be expressed in closed form as
\[
P(C_s, E_s) \leq \prod_{j=1}^{M_r} C_\rho \sum_{k=0}^{\infty} D(k) \frac{\Gamma(2m+2k)\sqrt{\pi}}{\Gamma(2m+2k+1/2)} \times F_1\left(2m+2k, \frac{1}{2}; 2k+2m; 2m+2k+\frac{1}{2}; -1, k^t\right)
\]
where \( F_1(a,b,b'; c; x,y) \) is the Appell hypergeometric function.

C. The Case of Independent Fade Coefficients with \( M_T \) Number of Transmit Antennas

In this sub-section we assume that the number of transmit antennas is \( M_T \) and random variables \( r_{ij} \) are i.i.d Nakagami-m variates with fading severity parameters \( m \). Following inequality (9), we can generalize the result to yield
\[
\left| \beta_{ij} \right|^2 \geq \mathbf{r} \mathbf{A}_i \mathbf{r}^T > 0 \tag{20}
\]
where \( \mathbf{A}_i \) is \( M_T \times M_T \) positive definite symmetric matrix, \( \mathbf{r} = \left( r_{1j} \ r_{2j} \ \ldots \ r_{M_Tj} \right) \) and \( (\cdot)^T \) denotes the transpose of a matrix. Hence inequality (8) can be re-written as
\[
P(C_s, E_s | H) \leq \exp\left( -\sum_{j=1}^{M_s} \mathbf{r} \mathbf{D} \mathbf{r}^T \right) \tag{21}
\]
where \( \mathbf{D} = \sum_{i=1}^{n} \lambda_i \mathbf{A}_i \) and \( n \) is the rank of the matrix \( \mathbf{A} \). It is clear that the matrix \( \mathbf{D} \) is a positive definite symmetric matrix. Hence by using eigen decomposition, we can write \( \mathbf{D} = \mathbf{P} \mathbf{Q} \mathbf{P}^T \), where \( \mathbf{P} \) is an orthogonal matrix and \( \mathbf{Q} \) is a diagonal matrix having the eigenvalues of \( \mathbf{D} \) along the main diagonal. To find the PEP upper bound inequality (21) is averaged with respect to independent Nakagami-m distributions of \( r_{ij} \) as
\[
P(C_s, E_s) \leq \prod_{j=1}^{M_s} \left( \frac{2}{\Gamma(m)} \right)^{M_s} \frac{m^{mM_s}}{\Omega} \int_{0}^{\infty} \ldots \int_{0}^{\infty} \exp(-\mathbf{r} \mathbf{P} \mathbf{Q} \mathbf{P}^T \mathbf{r}^T \left( r_{1j} r_{2j} \ldots r_{M_Tj} \right)^2 \mathbf{P} \mathbf{r}^T \mathbf{r}) \ d\mathbf{r}_{1j} d\mathbf{r}_{2j} \ldots d\mathbf{r}_{M_Tj} \tag{22}
\]
Let \( y = \mathbf{r} \mathbf{P} \) and \( \mathbf{S} = \mathbf{Q} + \frac{m}{\Omega} \mathbf{I} \), then inequality (22) can be written as
\[
P(C_s, E_s) \leq \prod_{j=1}^{M_s} \left( \frac{2}{\Gamma(m)} \right)^{M_s} \frac{m^{mM_s}}{\Omega} \int_{0}^{\infty} \ldots \int_{0}^{\infty} \exp(-\mathbf{y} \mathbf{S} \mathbf{y}^T) \times \left[ \prod_{j=1}^{M_s} \sum_{k=1}^{M_T} \mathbf{y}_{kj} \mathbf{p}_{ik} \right]^{2m-1} d\mathbf{y}_{1j} d\mathbf{y}_{2j} \ldots d\mathbf{y}_{M_Tj}. \tag{23}
\]
The integral in the right side of inequality (23) can be evaluated using the multinomial theorem and Gamma function. Although it is straightforward it involves tedious calculations. It is important to note that here we assumed that the matrix \( \mathbf{D} \) has full rank and the range of \( M_T \) fold integrals are given in general but may change depending on the sign of the parameters.

D. The case of Orthogonal Space-time Block Codes

For the special case when all the eigenvalues of \( \mathbf{A} \) in (7) are equal, i.e., when the collection of the signal matrices is drawn from an orthogonal design [3], \( \mathbf{A}_i \) matrix in inequality (20) has only one nonzero element at its \( (i,i) \) position which is unity and the inequality becomes an equality. Then \( \mathbf{D} = \lambda_i \mathbf{I} \) and \( n \) becomes \( M_T \). Furthermore the inequality (23) can be written as
\[
P(C_s, E_s) \leq \prod_{j=1}^{M_s} \prod_{i=1}^{M_T} \left( \frac{2}{\Gamma(m)} \right)^{M_T} \frac{m^{mM_T}}{\Omega} \exp(-\lambda_i) \times r_{ij}^{2m-1} d\mathbf{r}_{ij} \tag{24}
\]
and by evaluating the integral, the pairwise error probability bound reduces to
\[
P(C_s, E_s) \leq \left( \frac{1}{1 + \mu \lambda} \right)^{mM_T M_s}. \tag{25}
\]
This inequality is same as that given in [6] when \( m=1 \) (Rayleigh fading) and \( \Omega = 1 \), (i.e. when each component Gaussian variable has a variance of \( \frac{1}{2} \) [7]).

IV. CONCLUSIONS

In this paper, we have derived analytical PEP upper bounds for space-time trellis codes over slow fading quasi static Nakagami-m fading channels. We couldn’t use the traditional approach given in [1] since no closed form formula exists for the pdf of linear sum of Nakagami-m random variables [7], [15]. So we changed the approach and performed a detailed analysis in the case of two transmits antennas as a special case and also extended it to the general case. Moreover, these results can be used to extract parameters for designing good space-time codes.

REFERENCES


