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Delay-dependent stabilization for linear time-delay uncertain systems with saturating actuators

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This paper discusses the problems of the delay-dependent robust stability and stabilization for a class of linear time-delay uncertain systems with saturating actuators. Some new delay-dependent stability criteria are derived by taking the relationships between the terms in the Leibniz–Newton formula into account. The stability conditions are formulated as linear matrix inequalities that can be easily solved by various convex optimization algorithms or computing software. Moreover, the stability criteria are extended to the design of a stabilizing state feedback controller. Numerical examples demonstrate that these criteria are effective and are an improvement on previous ones.

Keywords: Leibniz–Newton formula; linear matrix inequality; time delay; delay dependence; optimization algorithm

1. Introduction

In many realistic control systems, time delay and saturating actuators occur frequently and their existence often have harmful effects on system stability (Glattfelder and Schaufelberger 1983, Manitius 1984, Kolmanovskii and Nosov 1986, Su et al. 1991). Uncertainty is often encountered in various dynamical systems due to modelling misfits, measurement errors, and linearization and approximations (Su et al. 1991). All actuation and measurement devices are subjected to time delay. Specifically, time delays arise in control actuation devices (e.g. transport lag) as well as computation delay in sensor measurement processing. On the other hand, time delay often occurs in systems such as transformation and communication systems, chemical and metallurgical processes, environmental models and power networks (Tsay and Liu 1996, Liu 2009). In control system design, the limited power supply is in the form of a saturating actuator in a practical system; hence, the actuator is nonlinear. Several authors have discussed the problem of stabilization of the system, which has completely known dynamics or parametrical uncertainties, with a saturating actuator (Glattfelder and Schaufelberger 1983, Chen and Wang 1988, Chou et al. 1989, Liu 1995, 2005, 2010, Liu and Su 1995, 1999, Niculescu et al. 1996, Han and Mehdi 1998, Su and Chu 1999, Su et al. 2001, 2002). Time delays have always been among the most difficult problems encountered in process control. In practical applications of feedback control, time delay arises frequently and can severely degrade closed-loop system performance and in some cases drive the system to instability (Liu and Su 1995, Liu et al. 2001, Cao et al. 2002). Therefore, it is very

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desirable for the control system design to investigate the problem of the stabilization of systems with time delay and saturating actuators.

Actuator saturation and time delay are often observed together in control systems. Various methods of synthesizing the controller for time-delay systems with actuator saturation have been proposed. To deal with both problems effectively, appropriate design methods are required. Analysis in the time domain uses in general the Lyapunov–Krasovskii functional or Lyapunov–Razumikhin functions (Su et al. 1991, Liu and Su 1995, Tsay and Liu 1996, Han and Mehdi 1998, Liu et al. 2001, Liu 2005, 2009, 2010). Other asymptotic stability criteria have been proposed in Su and Chu (1999) and Su et al. (2002), this result was obtained via the Lyapunov function and linear matrix inequalities (LMI) techniques approach independent of the size of the delays (Boyd et al. 1994). Using Lyapunov–Krasovskii functionals combined with LMI techniques for the stability of functional differential equations, upper bounds on the time delay are given such that the considered uncertain system is robustly stabilizable, in the case of constrained input, via memoryless state feedback control laws (Liu 1995, 2005, Cao et al. 2002, Su et al. 2002).

However, in the control of time-delay saturating actuator systems, it is usually desirable to design a controller that not only robustly stabilizes the system, but also estimates the bound of delay time $h$ to keep the stabilization of the system. Furthermore, the results are somewhat conservative, especially in situations where delays are small, there is room for investigation.

This paper deals with robust stability criteria for a class of time-delay systems with saturating actuators. Based on the Lyapunov–Krasovskii functional combining with LMI techniques and the Leibniz–Newton formula, delay-dependent stability criteria are derived using a state feedback controller. The results are illustrated by given numerical examples. These results are shown to be less conservative than those reported in the literature.

2. Main result

Consider the following time-delay systems with saturating actuators described by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h) + B \text{Sat}(u(t)), \quad (1a)$$

$$x(t) = \phi(t), \quad \forall t \in [-h, 0], \quad (1b)$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input vector; $x_t$ is the state at time $t$ denoted by $x_t(s) := x(t + s)$. $A_0, A_1$ and $B$ are known constant matrices with appropriate dimensions. Delay time $h$ is the unknown constant delay term and $\phi(t)$ is a smooth vector-valued initial function.

The saturating function is defined as follows (Chou et al. 1989):

$$\text{Sat}(u(t)) = [\text{Sat}(u_1(t)), \text{Sat}(u_2(t)), \ldots, \text{Sat}(u_m(t))]^T. \quad (2)$$

The operation of $\text{Sat}(u_i(t))$ is linear for $u_{iL} \leq u_i \leq u_{iH}$ as

$$\text{Sat}(u_i(t)) = \begin{cases} 
    u_{iL} & \text{if } u_i < u_{iL}, \\
    u_i & \text{if } u_{iL} \leq u_i \leq u_{iH}, \\
    u_{iH} & \text{if } 0 < u_{iH} < u_i, \quad i = 1, 2, \ldots, m,
  \end{cases} \quad (3)$$
where the values of \( u_{il} \) and \( u_{ih} \) are chosen corresponding to the limitations with the following properties:

\[
\text{Sat}(u_i(t)) - \frac{1}{2}(1 + \alpha_i)u_i(t) = \Delta v_i u_i(t), \quad i = 1, 2, \ldots, m.
\]  

(4)

This implies that

\[
\text{Sat}(u(t)) - W u(t) = \Delta V u(t),
\]

where

\[
W = \text{diag} \left[ \frac{1}{2}(1 + \alpha_1), \frac{1}{2}(1 + \alpha_2), \ldots, \frac{1}{2}(1 + \alpha_m) \right], \quad \Delta V = \text{diag}[\Delta v_1, \Delta v_2, \ldots, \Delta v_m],
\]

where \( \Delta v_i \) is a real number which varies between \(-(1/2)(1 + \alpha_i)\) and \((1/2)(1 - \alpha_i)\), \( \alpha_i < \alpha_i < 1 \).

When the system contains uncertainty, it can be described by the following linear differential-difference equation:

\[
\dot{x}(t) = (A_0 + \Delta A_0(t))x(t) + (A_1 + \Delta A_1(t))x(t - h) + B \text{Sat}(u(t)), \quad t > 0.
\]  

(6)

The uncertainties are assumed to be the form

\[
[\Delta A_0(t) \quad \Delta A_1(t)] = DF(t) [E_0 \quad E_1],
\]

where \( D, E_0 \) and \( E_1 \) are constant matrices with appropriate dimensions and \( F(t) \) is an unknown, real and possibly time-varying matrix with Lebesgue-measurable elements satisfying

\[
F^T(t)F(t) \leq I, \quad \forall t.
\]

(8)

The main objective is to find the range of \( h \) to guarantee stability for the time-delay system (1) or with uncertainties (6). When the time delay is unknown, how long-time delay can be tolerated to keep the system stable. To do this, one definition and three fundamental lemmas are reviewed.

**Definition 1.** The class of uncertain time-delay systems (2)–(8) are said to be robustly stable if the trivial solution \( x(t) = 0 \) of the functional differential equation associated to (6) with \( u(t) = 0 \) is globally uniformly asymptotically stable for all admissible uncertainties \( \Delta A_0(t) \) and \( \Delta A_1(t) \). The class of time-delay system (1) or uncertain time-delay system (6) is said to be robustly stabilizable if there exists a static linear state feedback control law \( u(t) = -Kx(t) \) such that the resulting closed-loop system is robustly stable.

**Lemma 1 (Liu 2009).** If there exist symmetric positive-definite matrix \( X_{33} > 0 \) and arbitrary matrices \( X_{11}, X_{12}, X_{13}, X_{22}, X_{23} \) and \( X_{23} \) such that

\[
X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23}^T & X_{33}
\end{bmatrix} \geq 0,
\]

then, we obtain

\[
-\int_{t-h}^t x^T(s)X_{33}x(s) \, ds \leq \int_{t-h}^t \begin{bmatrix}
x^T(t) & x^T(t-h) & \dot{x}^T(s)
\end{bmatrix}
\begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13}^T & X_{23}^T & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x(t-h) \\
\dot{x}(s)
\end{bmatrix}
\, ds.
\]

(10)
Lemma 2 (Boyd et al. 1994). The following matrix inequality
\[
\begin{bmatrix}
Q(x) & S(x) \\
S^T(x) & R(x)
\end{bmatrix} < 0,
\]
where \(Q(x) = Q^T(x), R(x) = R^T(x)\) and \(S(x)\) depend on affine on \(x\), is equivalent to
\[
R(x) < 0
\]
(12a)
\[
Q(x) < 0
\]
(12b)
and
\[
Q(x) - S(x)R^{-1}(x)S^T(x) < 0.
\]
(12c)

Lemma 3 (Boyd et al. 1994). Given matrices \(Q = Q^T, D\) and \(E\) of appropriate dimensions,
\[
Q + DF(t)E + E^TF^T(t)D^T < 0
\]
(13a)
for all \(F(t)\) satisfying \(F^T(t)F(t) \leq H\), if and only if there exists some \(\varepsilon > 0\) such that
\[
Q + \varepsilon DD^T + E^{-1}TE^T < 0.
\]
(13b)

The following theorem provides stability analysis of the nominal unforced system (1).

Theorem 1. For a given scalar \(h > 0\), if there exist positive-definite matrices
\[
P > 0, Q > 0, R > 0\text{ and } X = \begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{12}^T & X_{22} & X_{23} \\
X_{13} & X_{23}^T & X_{33}
\end{bmatrix} \succeq 0
\]
such that
\[
\Phi = \begin{bmatrix}
A_0^TP + PA_0 + Q + X_{13} + X_{13}^T + hX_{11} & PA_1 - X_{13} + X_{23}^T + hX_{12} & hA_0^TR \\
A_1^TP - X_{13}^T + X_{23} + hX_{12} & -Q - X_{23} - X_{23}^T + hX_{22} & hA_1^TR \\
hRA_0 & hRA_1 & -hR
\end{bmatrix} < 0
\]
(14a)
and
\[
R - X_{33} \succeq 0,
\]
(14b)
then the time-delay nominal unforced system (1) is asymptotically stable for any time delay \(h\) satisfying \(0 \leq h \leq \bar{h}\).
Proof. Select the following Lyapunov–Krasovskii functional to be

\[
V(x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)\,ds + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds\,d\theta. \tag{15}
\]

Calculating the derivative of (14) with respect to \(t\) along the trajectory of nominal unforced system (1) yields

\[
\begin{align*}
\dot{V}(x_t) &= x^T(t)(A_0^TP + PA_0)x(t) + x^T(t)PA_1x(t-h) + x^T(t-h)A_1^TPx(t) + x^T(t)Qx(t) \\
&\quad - x^T(t-h)Qx(t-h) + \dot{x}^T(t)hR\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds = x^T(t)(A_0^TP + PA_0) \\
&\quad + Q)x(t) + x^T(t)PA_1x(t-h) + x^T(t-h)A_1^TPx(t) - x^T(t-h)Qx(t-h) \\
&\quad + \dot{x}^T(t)hR\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s)(R - X_{33})\dot{x}(s)\,ds - \int_{t-h}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)\,ds.
\end{align*}
\]  

(16)

Using the Leibniz–Newton formula \(x(t) - x(t - h) = \int_{t-h}^{t} \dot{x}(s)\,ds\) and Lemma 1, we obtain

\[
\begin{align*}
-\int_{t-h}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)\,ds &\leq \int_{t-h}^{t} \begin{bmatrix} x(t) \\ x^T(t-h) \\ \dot{x}(s) \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} \,ds \\
&\leq x^T(t)hX_{11}x(t) + x^T(t)hX_{12}x(t-h) + x^T(t)X_{13}\int_{t-h}^{t} \dot{x}(s)\,ds + x^T(t-h)hX_{12}^TX_{13}\dot{x}(t) \\
&\quad + x^T(t-h)hX_{22}x(t-h) + x^T(t-h)X_{23}\int_{t-h}^{t} \dot{x}(s)\,ds + \int_{t-h}^{t} \dot{x}^T(s)\,ds X_{23}^TX_{13}\dot{x}(t) \\
&\quad + \int_{t-h}^{t} \dot{x}^T(s)\,ds X_{23}^TX_{13}\dot{x}(t-h) = x^T(t)hX_{11}x(t) + x^T(t)hX_{12}x(t-h) \\
&\quad + x^T(t-h)X_{13}[x(t) - x(t-h)] + x^T(t-h)hX_{12}^TX_{13}\dot{x}(t) \\
&\quad + x^T(t-h)\{x(t) - x(t-h)\}^TX_{13}^TX_{13}\dot{x}(t) \\
&\quad + [x(t) - x(t-h)]^TX_{23}^TX_{13}\dot{x}(t-h) = x^T(t)hX_{11}x(t) + x^T(t)hX_{12}x(t-h) + x^T(t)X_{13}\dot{x}(t) \\
&\quad + x^T(t-h)X_{13}\dot{x}(t) + x^T(t-h)hX_{12}^TX_{13}\dot{x}(t-h) + x^T(t-h)X_{23}\dot{x}(t-h) \\
&\quad + x^T(t-h)X_{13}\dot{x}(t-h) = x^T(t)h[X_{11} + X_{13} + X_{13}]x(t) \\
&\quad + x^T(t)\{hX_{12} - X_{13} + X_{23}\}x(t-h) + x^T(t-h)[hX_{12}^T - X_{13}^T + X_{23}]x(t) \\
&\quad + x^T(t-h)[hX_{22} - X_{23} - X_{23}^T]x(t-h). \\
\end{align*}
\]  

(17)

Substituting the above Equation (17) into (16) yields the following equation:

\[
\dot{V}(x_t) < \xi^T(t)\Xi(t) - \int_{t-h}^{t} \dot{x}^T(s)(R - X_{33})\dot{x}(s)\,ds, \tag{18}
\]

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where $\xi^T(t) = [x^T(t) \ x^T(t-h)]$ and
\[
\Xi = \begin{bmatrix}
A_0^TP + PA_0 + Q + X_{13} + X_{13}^T + hX_{11} + hA_0^TRA_0 & PA_1 - X_{13} + X_{23}^T + hX_{12} + hA_0^TRA_1 \\
A_1^TP - X_{13}^T + X_{23} + hX_{12}^T + hA_1^TRA_0 & -Q - X_{23} - X_{23}^T + hX_{22} + hA_1^TRA_1
\end{bmatrix}.
\]

Finally, using the Schur complements of Lemma 2, with some effort we can show that (18) guarantees $\dot{V}(x_t) < 0$ for any $\xi(t) \neq 0$. Condition (14) of the present Theorem 1 is satisfied, if $\dot{V}(x_t) < 0$ and $R - X_{33} < 0$ if and only if (14) holds. Therefore, the nominal unforced time-delay system (1) is asymptotically stable. This completes the proof.

The time-delay systems with saturating actuator (1) are said to stable in a closed loop via a memoryless state feedback control law if there exists a control law
\[
u(t) = -Kx(t),
\]
where $K \in \mathbb{R}^{n \times n}$ such that the trivial solution $x(t) \equiv 0$ of the functional differential equation associated with the closed-loop system is uniformly asymptotically stable.

From (1)–(5) and (19), we can rewrite the time-delay systems with saturating actuator (1) as follows:
\[
x(t) = (A_0 - BKx(t))x(t) + A_1x(t-h),
\]
where $\Psi$ is known-constant matrices with appropriate dimensions as $\Psi = W + \Delta V$.

According to the above Theorem 1, we describe our method for determining the robust stabilization of time-delay system with saturating actuator (20). The main aim of this paper is to develop delay-dependent conditions for stability of the time-delay saturating actuator system (1) under the state feedback control law (19). More specifically, our objective is to determine bounds for the time delay by using the Lyapunov–Krasovskii functional and LMI methods with the Leibniz–Newton formula. Theorem 2 gives an LMI-based computational procedure to determine the state-feedback controller. Then, we have the following result.

**Theorem 2.** For given scalar $h > 0$, if there exist symmetric positive-definite matrices $W > 0, U > 0, Z > 0$, a semi-positive definite matrix
\[
T = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{12}^T & T_{22} & T_{23} \\
T_{13}^T & T_{23}^T & T_{33}
\end{bmatrix} \succeq 0
\]
and a matrix $Y$ with appropriate dimensions such that
\[
\Omega = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & \Omega_{13} \\
\Omega_{12}^T & \Omega_{22} & \Omega_{23} \\
\Omega_{13}^T & \Omega_{23}^T & \Omega_{33}
\end{bmatrix} < 0
\]

Then the nominal unforced time-delay system (1) is asymptotically stable.
\[ W - T_{33} \geq 0, \]  

(21b)

where

\[
\begin{align*}
\Omega_{11} &= WA_0^T + A_0W - B\Psi Y - Y^T\Psi^TB^T + U + T_{13} + T_{13}^T + hT_{11}, \\
\Omega_{12} &= A_1W - T_{13} + T_{23}^T + hT_{12}, \quad \Omega_{13} = h(WA_0^T - Y^T\Psi^TB^T), \\
\Omega_{22} &= -U - T_{23} - T_{23}^T + hT_{22}, \quad \Omega_{23} = hWA_1^T, \quad \Omega_{33} = -hZ.
\end{align*}
\]

Then, the time-delay system (20) with the state feedback control law and 
\[ K = YW^{-1} \] is asymptotically stable for any time delay \( h \) satisfying \( 0 \leq h \leq \bar{h} \).

**Proof.** In view of Theorem 1, to prove the asymptotic stability of the closed-loop system with control \( u(t) = -Kx(t) \), it suffices to show that there exist symmetric, positive-definite matrices \( P > 0, Q > 0 \) and \( R > 0 \) such that (14) remains valid with \( A_0 \) replaced by \( A_0 - B\Psi K \). Pre- and post-multiplying both sides of (13) by \( \text{diag}\{P^{-1}, P^{-1}, R^{-1}\} \) and letting
\[ W = P^{-1}, \quad Y = KW, \quad P^{-1}QP^{-1} = U, \quad P^{-1}X_{ij}P^{-1} = T_{ij} \] 
\( i, j = 1, 2, 3 \), \( Z = R^{-1} \) and
\[ \begin{bmatrix} R^{-1} & P^{-1} \\ -X_{33} & 0 \end{bmatrix} \]

\( P^{-1} = W - T_{33} \) leads to (21). This ends the proof. \qed

### 3. Extension to stabilization for time-delay uncertain systems with saturating actuator

Now, extending Theorem 2 to system (6) with time-varying structured uncertainties yields the following Theorem 3.

**Theorem 3.** If there exist symmetric positive-definite matrices \( W > 0, U > 0, Z > 0, \) \( \epsilon > 0, \) a semi-positive definite matrix

\[
T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12}^T & T_{22} & T_{23} \\ T_{13}^T & T_{23}^T & T_{33} \end{bmatrix} \geq 0
\]

and a matrix \( Y \) with appropriate dimensions such that

\[
\Omega = \begin{bmatrix}
\Omega_{11} + \epsilon E_0^T E_0 & \Omega_{12} + \epsilon E_0^T E_1 & \Omega_{13} & WD \\
\Omega_{12}^T + \epsilon E_1^T E_0 & \Omega_{22} + \epsilon E_1^T E_1 & \Omega_{23} & 0 \\
\Omega_{13}^T & \Omega_{23} & \Omega_{33} & hWD \\
D^T W & 0 & hD^T W & -\epsilon I
\end{bmatrix} < 0 \tag{22a}
\]

and

\[ W - T_{33} \geq 0, \tag{22b} \]

where \( \Omega_{ij}(i, j = 1, 2, 3) \) are defined in (21).
Proof. Replacing $A_0$ and $A_1$ in (20) with $A_0 + DF(t)E_0$ and $A_1 + DF(t)E_1$, respectively, we find that Lemma 3 for system (6) is equivalent to the following condition:

$$\Omega + \begin{bmatrix} WD & F(t)[E_0 & E_1 & 0] + E_0^T \end{bmatrix} F^T(t) \begin{bmatrix} D^TW & 0 \ hD^W \end{bmatrix} < 0. \quad (23)$$

By Lemma 3, taking into account the fact that for any $\varepsilon > 0$ and for any matrix $F(t)$ satisfying $F^T(t)F(t) \leq I$ and $(DF(t)E)^T F(t) + DF(t)E \leq \varepsilon^{-1} D^T D + \varepsilon E^T E$ such that

$$\Omega + \varepsilon^{-1} \begin{bmatrix} WD & 0 \ hWD \end{bmatrix} \begin{bmatrix} D^TW & 0 \ hD^W \end{bmatrix} + \varepsilon \begin{bmatrix} E_0^T \ 0 \ E_0 & E_1 & 0 \end{bmatrix} < 0. \quad (24)$$

Applying the Schur complement of Lemma 2 shows that (24) is equivalent to (22). This completes the proof. \hfill \square

Remark 1. As in the stabilization problem, the upper bound $\bar{h}$ which ensures that time delays (6) are stabilizable for any $h$ can be determined by solving the following quasi-convex optimization problem when the other bound of time delay $h$ is known.

$$\begin{align*}
\text{Maximize} & \quad h \\
\text{Subject to} & \quad W > 0, \ U > 0, \ Z > 0, \ \varepsilon > 0 \text{ and } T \succeq 0.
\end{align*} \quad (25)$$

Inequality (25) is a quasi-convex optimization problem and can be obtained efficiently using MATLAB LMI Toolbox. Then, the controller $K = YW^{-1}$ stabilizes system (6).

To show the usefulness of our result, let us consider the following numerical examples.

4. Examples

Example 1. Let us consider the uncertain time-delay system with saturating actuator

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t - \tau) + B \text{ Sat}(u(t)), \quad (26)$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -0.8 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_0 = E_1 = \text{diag}\{0.2, 0.2\}, \quad D = I.$$

Assume that the saturated range $\alpha_i$ is inside the sector $[0.5, 1]$. The problem is to design a state feedback controller (6) to estimate the delay time $h$ such that the system (26) is asymptotically stable.

Solution: We choose $\alpha_i = 0.5$, solving the following quasi-convex optimization problem (25), the maximum upper bound, $h$, for which the system is stabilized by the corresponding state feedback $K = YW^{-1} = [4.1398 \ 46.0295]$ and $h < 7.0817$. The simulation of the
above-closed system for \( h = 7.0 \) is shown in Figure 1. The obtained system (22) would be stable if the delay time \( h \) is less than 7.0 for the saturated range \( \alpha_i \) that is inside the sector \([0.5, 1]\). For a comparison with the results of other researchers, see Table 1.

From Table 1, the proposed criteria are less conservative than these (Niculescu et al. 1996, Su et al. 2001, 2002, Liu 2005, 2010). Hence, our result gives a less conservative bound than those obtained using delay-dependent stabilization criteria (Niculescu et al. 1996, Su et al. 2001, 2002, Liu 2005, 2010).

Example 2. Consider a time-delay system with an actuator saturated at level \( \pm 1 \) and dynamics described as follows:

\[
\dot{x}(t) = A_0 x(t) + A_1 x(t - h) + B \text{Sat}(u(t)),
\]

where

\[
A_0 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 \\ 0.7 \end{bmatrix}.
\]

It is intended to stabilize the controlled system and find the range of delay time \( h \) using a state feedback controller \( K \) to guarantee that the above system is asymptotically stable.

Table 1. Some comparisons for allowable time-delay upper bound of example 1.

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<tbody>
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<td>( h )</td>
<td>0.2841</td>
<td>0.3819</td>
<td>0.5075</td>
<td>0.5522</td>
<td>4.0813</td>
<td>7.0817</td>
</tr>
</tbody>
</table>

Figure 1. The simulation of the example 1 for \( h = 7.0 \) s.
Solution: The nonlinear saturating characteristics belong to the sector $[0.8 \ 1]$. Applying the condition (21) of Theorem 2 using the LMI Toolbox in MATLAB (with accuracy 0.01), the solutions of the LMI given in (27) are found to be $h \leq 3.6194$ for which the system is stabilized by the corresponding state feedback $K = W^{-1} = [449.0925 \ 23.8450]$. On the other hand, the delay bound for guaranteeing the asymptotic stability of the system (27) is $h < 0.2064$ (Liu and Su 1995). Hence, for this example, the stabilization criterion of this paper is less conservative than the existing result (Liu and Su 1995).

Example 3. Consider a fourth-order linearized model of the match number in a wind tunnel, borrowed from Manitius (1984) and described by

$$
\dot{x}(t) = \begin{bmatrix}
-a & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -\omega^2 & -2\xi\omega & \omega^2 \\
0 & 0 & 0 & 0
\end{bmatrix} x(t) + \begin{bmatrix}
0 & k_a & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} x(t-h) + \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \text{Sat}(u(t)),
$$

where $x(t) = [\delta M, \delta \theta, \delta \dot{\theta}, \delta \theta_a]^T$ consists of the variation in the Mach number $\delta M$, the guide vane angle $\delta \theta$, the variation rate guide vane angle $\delta \dot{\theta}$, and the guide vane angle actuator $\delta \theta_a$, with $(1/a) = 1.964 \text{s}^{-1}$, $\omega = 6 \text{rad/s}$, $\xi = 0.8$ and $k_a = -0.0117 \text{deg}^{-1}$.

The control $\text{Sat}(u(t))$ represents the actuator rate $\delta \theta_a$.

It is found that the nominal system (28) is unstable and it is intended to stabilize the controlled system using an output feedback. Now, find the range of $h$ with the state feedback controller $K$ to guarantee that the above system is asymptotically stable.

Solution: By taking the nonlinear saturating characteristics belonging to the sector $[0.5 \ 1]$, we get Theorem 2 which remains feasible for any decay rate $h \leq 0.9685$. In the case of $h = 0.9685$, solving Theorem 2 yields the following set of feasible solutions:

$$
W = \begin{bmatrix}
37.4255 & -0.0037 & -0.0011 & -0.0004 \\
-0.0037 & 14.5004 & 0.4737 & 0.4685 \\
-0.0011 & 0.4737 & 0.4302 & 0.3739 \\
-0.0004 & 0.4685 & 0.3739 & 35.3154
\end{bmatrix}, \quad U = \begin{bmatrix}
16.7852 & -0.0082 & -0.0030 & 0.0102 \\
-0.0082 & 13.6900 & 2.4900 & -4.4827 \\
-0.0030 & 2.4900 & 2.6208 & -3.5836 \\
0.0102 & -4.4827 & -3.5836 & 21.2992
\end{bmatrix},
$$

$$
Z = \begin{bmatrix}
34.7329 & 0.0078 & -0.0073 & 0.0089 \\
0.0078 & 66.0260 & 7.4414 & -30.3855 \\
-0.0073 & 7.4414 & 33.9947 & -2.6681 \\
0.0089 & -30.3855 & -2.6681 & 61.8132
\end{bmatrix}, \quad T_{11} = \begin{bmatrix}
12.8540 & 0.0049 & -0.0011 & 0.0031 \\
0.0049 & 10.6746 & 1.2943 & -1.9397 \\
-0.0011 & 1.2943 & 1.5826 & -1.4652 \\
0.0031 & -1.9397 & -1.4652 & 14.8603
\end{bmatrix},
$$

$$
T_{12} = \begin{bmatrix}
-6.0572 & 0.0170 & 0.0007 & 0.0020 \\
0.0004 & -4.9503 & 0.1339 & -1.4230 \\
0.0000 & 0.1500 & -0.4760 & -0.9404 \\
0.0024 & -1.3147 & -0.7835 & -4.4330
\end{bmatrix}, \quad T_{13} = \begin{bmatrix}
-6.0713 & -0.0115 & -0.0001 & -0.0004 \\
-0.0015 & -4.2485 & -0.0955 & -1.8850 \\
-0.0001 & 0.3272 & -0.2592 & -1.4236 \\
0.0007 & -0.8734 & -0.1331 & -4.1237
\end{bmatrix},
$$

$$
T_{11} = \begin{bmatrix}
-6.0713 & -0.0115 & -0.0001 & -0.0004 \\
-0.0015 & -4.2485 & -0.0955 & -1.8850 \\
-0.0001 & 0.3272 & -0.2592 & -1.4236 \\
0.0007 & -0.8734 & -0.1331 & -4.1237
\end{bmatrix}.
$$
which gives $K = YW^{-1} = [0.0003 - 3.1732 33.2562 2.1701]$. 

Applying the criteria in Liu (2005) and Manitius (1984), the upper bounds on time delay for robust stability are $h \leq 0.2715$ and 0.33, respectively. This example shows that the criteria in this article give a much less conservative result than those in Manitius (1984) and Liu (2005).

5. Conclusion

This paper deals with the problem of robust stabilization criteria for a class of linear time-delay saturating actuator systems via the Lyapunov–Krasovskii functional combined with LMI techniques; simple and improved delay-dependent stability is proposed. Compared with the several existing stability criteria, the allowable bound of delay time is significantly improved. The significance of the results obtained is demonstrated by two illustrated examples.

Notes on contributor

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References


