Robustness of local adaptive synchronization strategies to topological variations and delays

Pietro De Lellis*, Mario di Bernardo*

*Department of Computer and Systems Engineering
Università di Napoli Federico II
via Claudio 21 – 80125 Napoli, Italy.
Email: {pietro.delellis,mario.dibernardo}@unina.it

Abstract—This paper discusses the robustness properties of a novel decentralized strategy for synchronization of complex networks based on adaptive local gains. Namely, the strategy assumes that each vertex adaptively selects the coupling strength with all of its local neighbours. The strategy was proved to guarantee synchronization, and it is shown here to be surprisingly robust to topological variations of the underlying network. The case of time-delays on the coupling between oscillators in the network is also considered showing that, under certain conditions, the strategy still guarantees that the network will synchronize. The observations are supported by numerical simulations on the representative case of networks of Chua’s circuits.

I. INTRODUCTION

The problem of making a network of dynamical systems synchronize onto a common evolution has received much attention in the scientific community. Typically, the network consists of \( N \) identical nonlinear dynamical systems coupled through the edges of the network itself [1], [2]. Each uncoupled system is described by a set of nonlinear ordinary differential equations (ODEs) of the form

\[
\dot{x} = f(x),
\]

where \( x \in \mathbb{R}^n \) is the state vector and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a nonlinear vector field describing the system dynamics. Because of the coupling with the neighbors in the network, the dynamics of each oscillator is affected by a nonlinear input representing the interaction of all neighboring nodes with the oscillator itself. Hence, the equations of motion for the generic \( i \)-th system in the network become:

\[
\frac{dx_i}{dt} = f(x_i(t)) - \sigma \sum_{j=1}^{N} \mathcal{L}_{ij} h(x_j), \quad i = 1, 2, \ldots, N, \tag{1}
\]

where \( x_i \) represents the state vector of the \( i \)-th oscillator, \( \sigma \) the overall strength of the coupling, \( h \) the output function through which the systems in the network are coupled and \( \mathcal{L}_{ij} \) the elements of the Laplacian matrix \( \mathcal{L} \) describing the network topology. In particular, \( \mathcal{L} \) is such that its entries, \( \mathcal{L}_{ij} \), are zero if node \( i \) is not connected to node \( j \neq i \), while are negative if node \( i \) is connected to node \( j \), with \( \vert \mathcal{L}_{ij} \vert \) giving a measure of the strength of the interaction.

Traditionally, the synchronization problem has been defined in terms of finding the value of the coupling gain \( \sigma \) so that all the systems in the network synchronize on the same unknown evolution, say \( x_s(t) \). Such a coupling gain is commonly assumed to be constant and equal for all links and vertices in the network. This assumption is often unrealistic as many real-world networks are characterized instead by evolving, adapting coupling gains, which vary in time according to different environmental conditions. Examples include wireless sensor networks that gather and communicate data to a central base station [3], control networks of robots when the conditions change unexpectedly (i.e. a robot loses a sensor) [4] and the many adaptive networks found in biology, as the social insect colonies, described in [5].

To address this issue, local adaptive strategies to adjust the node coupling strength to achieve synchronization were independently presented in [6] and [7]. In those papers, an adaptive strategy named vertex-based was introduced where each vertex in the network self-selects the coupling gain to all of its neighbors on the basis of a local, fully decentralized, time-varying adaptive law. The analytical proof of asymptotic stability of such a strategy was given in [8]. In this paper, we further extend the analysis of this novel scheme by testing its robustness to topological variations and communication delays. As it will be observed in the rest of the paper we find that the adaptive nature of the scheme copes well with switches or changes in the network topology. Also, it is shown that up to certain values of the communication delay, the strategy still guarantees that the network synchronizes. Hence, we propose that adaptation of the coupling gain can be of utmost importance in all of those situations where uncertainties and delays cannot be neglected. The observations are supported by simulations on a representative network of 1000 mutually coupled Chua’s oscillators with different topologies.

II. PROBLEM STATEMENT

Let us denote with \( \mathcal{E} \) the set of edges of the network, containing pairs of indices associated to nodes connected by an existing link. For example \((i,j) \in \mathcal{E}\) will indicate there exists an edge connecting node \( i \) to node \( j \). Moreover, let us indicate with \( \mathcal{E}_i \subseteq \mathcal{E} \) the subset of all edges connected to node \( i \). We consider the following network model:

\[
\frac{dx_i}{dt} = f(x_i) - \sum_{j \in \mathcal{E}_i} \sigma_{ij} (h(x_j) - h(x_i)), \quad i = 1, 2, \ldots, N, \tag{2}
\]

where \( x_i \) represents the state vector of the \( i \)-th oscillator, \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a nonlinear vector field describing the system.
the various $\sigma_i(t)$ observe the discontinuous evolution of the time derivatives of adaptive gains and their derivatives is depicted. Here, we can of the switching are clearer in Fig. 2 where the evolution of the networks which barely affects the state evolution. The effects to a common evolution despite the switching between the two equal to 5. As we can see from Fig. 1, all circuits converge to a common evolution. The Laplacian matrix $L$ describing the network topology has always considered to be constant, i.e. the network topology has been assumed to be time-invariant. As mentioned above, in real applications the topology of the network often changes in time. Thus, an urgent question to address is whether the vertex-based strategy is robust to topological variations. To address this issue, we considered three different types of possible topological variations of the network:

1) the network switches among two fixed topologies with a given frequency $\nu$;
2) the network topology is randomly re-generated with a given frequency $\nu$;
3) periodical link failures occur in the network.

To illustrate how the adaptive strategy reacts to each of these cases, we will consider networks of 1000 Chua's circuits [9] coupled through a scale-free topology. The vector field describing each circuit, denoting with $(p, q, r)$ the three components of the state $x$, is the following:

$$f(p, q, r) = (\beta(-q - \Phi(p)), p - q + r, \xi q)^T$$

where $\Phi(p) = m_1 p + \frac{1}{2} (m_1 - m_0) (|p + 1| - |p - 1|)$. We choose the parameters as in [10] to ensure the chaotic behavior of the system.

A. Case 1: switching between two given topologies

In all simulations, we considered the case of a switch among two topology at each 0.01s. The two topologies where generated using the Barabási-Albert algorithm to construct scale-free networks, choosing in both cases average degree equal to 5. As we can see from Fig. 1, all circuits converge to a common evolution despite the switching between the two networks which barely affects the state evolution. The effects of the switching are clearer in Fig. 2 where the evolution of the adaptive gains and their derivatives is depicted. Here, we can observe the discontinuous evolution of the time derivatives of the various $\sigma_i$ due to the topological changes in the network but nevertheless we note that all gains settle asymptotically onto some bounded constant values as desired.

B. Case 2: periodic re-generation of the topology

Given the positive result of the previous numerical test, we then looked at the case in which a completely new scale-free topology is generated at each switching time. The switching frequency was set to be $\nu = 10s^{-1}$.

As we can see from Fig. 3, synchronization is again attained with the various $\sigma_i$ settling onto constant values (see Fig. 4). This behavior was still preserved when the switching frequency was increased (for the sake of brevity we omit the simulation results for this case). Thus, a sufficient condition for synchronization seems to be that the time-varying Laplacian always remains connected. The next step is to understand if
this condition is also necessary or not.

C. Periodic link failures

As a testbed, we consider the case in which there are periodic link failures that can eventually lead to the formation of disconnected components in the network. Thus, we start from a Barabasi-Albert scale-free network with average degree equal to 5. Then, we remove randomly \( e \) edges every 0.01s. As we can notice from Figs. 5, 6 when we choose \( e = 6 \), synchronization is still achieved: in fact, under this condition, the network remains connected for a time span which is enough to achieve synchronization. Once synchronized, clearly the dynamics of the network are not affected anymore by the coupling. So if the network becomes disconnected after synchronization is attained, all oscillators will continue evolving onto the common synchronous evolution (obviously if no noise is present).

If we now increase the number of links \( e \) removed at each time interval, we notice that when \( e \) overcomes a certain threshold, the network becomes disconnected before synchronization, and so, as we can see from Figs. 7, 8, while the \( \sigma_i \) settle to constant values, the oscillators first start to converge to each other but, when the network becomes disconnected, they start to oscillate chaotically independently from each other and synchronization is lost.

IV. Communication Time Delays

We move now to another fundamental robustness issue for the synchronization of complex networks; the case where (identical) communication time-delays \( \tau \) are present. Thus, the governing equation of the network becomes:

\[
\begin{align*}
\frac{dx_i}{dt} &= f(x_i) - \sum_{j \in E_i} \sigma_{ij}(t)(x_j(t-\tau) - x_i(t-\tau)) \\
\dot{\sigma}_{ij}(t) &= \mu || \sum_{j \in E_i} e_{ij}(t-\tau) || \quad (7)
\end{align*}
\]
To test this case, we simulated networks of $N = 30$ Chua’s circuits. We observed that the stability of the synchronization manifold is now clearly affected by the communication delays as expected. In fact, when the delay becomes larger than a given threshold, then the trajectories of the systems diverge. This threshold seems to be dependent upon the topological features of the network. In particular, increasing the number of nodes of the network, the threshold decreases. In our simulations, we considered, as before, Bárabasi-Albert scale-free like networks with average degree equal to 5. We have estimated the threshold value of the delay when $N = 30$ to be $\tau_{30} = 0.08$ (see the corresponding Figure IV), while for $N = 300$ we observed the threshold to become significantly smaller. In particular, we have $\tau_{300} = 0.00125$. This dependence of the admissible delay upon the size of the network is clearly an issue of the adaptive strategy that can only be overcome by changing the adaptation law itself. This is the subject of ongoing work.

V. CONCLUSIONS

In this paper, we discussed the robustness of the vertex-based adaptive scheme for synchronization firstly presented in [7] to topological variations and communication delays. By using a network of Chua’s oscillators as a testbed example, we observed that the adaptive strategy is well suited to cope with variations in the network topology as long as the network remains connected long enough for the synchronous regime to be attained. In this case, all circuits were shown to converge towards the same evolution with the adaptive gains settling asymptotically to some bounded constant steady-state values. The case of communication delays was also considered. We found that in this case, the adaptive strategy guarantees synchronization given that the delay is below a certain critical threshold. As expected, this threshold is dramatically dependent on the network topology and, more specifically, to the number of oscillators in the network. We wish to emphasize that these observations can be extremely important for the application of this novel strategy to realistic application examples. Ongoing work is aimed at exploiting the robustness of the strategy to topological variations while improving its overall robustness to communication delays.

REFERENCES