An adaptive numerical method for solving EDQNM equations for
the analysis of long-time decay of isotropic turbulence

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Abstract
A new numerical formulation for the Eddy-Damped Quasi-Normal Markovian (EDQNM)
model is proposed in the present paper. This formulation is based on an adaptive proce-
dure that progressively modifies the spectral mesh at the large scales, forcing a resolution
requirement set by the user.

The resulting adaptive numerical method for the EDQNM model has been system-
atically tested by comparison with the classical model, covering a wide range of initial
conditions. In particular, the sensitivity of the results to the initial Reynolds number $Re_\lambda$
and to the slope of the energy spectrum at the large scales $\sigma (E(k \to 0) \propto k^{\sigma})$ has been
investigated. For all the initial conditions prescribed, the adaptive numerical method
recovers exactly the same solution of the classical version. This result has been observed
in the analysis of the main statistical quantities of interest, such as the turbulent kinetic
energy $K$ and the energy dissipation rate $\varepsilon$. The same conclusions have been drawn by
the direct comparison of the energy spectra $E$.

An extension of the adaptive numerical method to the analysis of sheared / rotating
turbulence is as well proposed and successfully assessed.

The numerical algorithm proves to be progressively more efficient when long-time
simulations are performed. In particular, a reduction up to one order of magnitude in
the computational resources required is observed. Due to its efficiency and precision, the
adaptive formulation of the EDQNM model promises to be an optimal tool to be blended
with the methods of the uncertainty quantification theory, in order to provide new insights
about isotropic turbulence decay.

Keywords: Adaptive mesh, Isotropic turbulence, EDQNM

1. Introduction

The decay of homogeneous isotropic turbulence (HIT) is one of the most classical test
cases in turbulence studies and it constitutes the simplest type of turbulent flow. This
case has been extensively studied in open literature (see the comprehensive reviews by
Batchelor\cite{1}, Hinze\cite{2}, Davidson\cite{3} and Sagaut & Cambon\cite{4}) in order to derive informa-
tion about turbulence physical nature\cite{5, 6} and for the development of subgrid scale\cite{7}
/ turbulence models\cite{8} currently used in the numerical simulation of complex industrial
flows.

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The dynamics of HIT decay are investigated in a number of theoretical works\cite{9,10,11} reported in the literature. These works rely on basic assumptions such as the Joint Gaussian Assumption (JGA) or the infinite Reynolds number hypothesis. Traditionally, these studies focused on the analysis of HIT statistical quantities such as the turbulent kinetic energy $K$, the energy dissipation rate $\varepsilon$, the Kolmogorov scale $\eta$ and the integral length scale $L$. In HIT free decay, a general statistical quantity $Q$ decays through a power-law $Q \propto t^n$. The power-law exponent $n_Q$ is not unique but it is determined by the initial conditions. Comte-Bellot & Corrsin\cite{12} (CBC) proposed analytical formulae to link the energy spectrum slope at the large scales $\sigma$ with $n_Q$. These formulae, which are reported in Table 1, have been extended to account for non-local energy transfer effects by Meldi & Sagaut\cite{13}, while Clark & Zemach\cite{14} derived the correspondent version for low Reynolds HIT decay. The values of the parameter $\sigma \in [1,4]$ studied in open literature include the two classical cases of Saffman turbulence\cite{9} ($\sigma = 2$) and Batchelor turbulence\cite{5} ($\sigma = 4$).

$$\begin{align*}
\text{High } Re_\lambda & : \frac{-2^{\sigma-p+1}}{\sigma-p+3} \quad \frac{-3(\sigma-p)+5}{\sigma-p+3} \quad \frac{2}{\sigma-p+3} \quad \frac{1}{2} \quad \frac{3(\sigma-p)+5}{4(\sigma-p+3)} \quad \frac{1-\sigma+p}{2(\sigma-p+3)} \\
\text{Low } Re_\lambda & : \frac{-\sigma+1}{2} \quad \frac{-\sigma+3}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{\sigma+3}{8} \quad \frac{1-\sigma}{4}
\end{align*}$$

Table 1: Analytical formulae for the prediction of the power-law exponents of the decay of the main HIT statistical quantities. The high $Re_\lambda$ formulae are proposed by Comte-Bellot & Corrsin\cite{12} and revisited by Meldi & Sagaut\cite{13} ($p = \max[0, 0.65(\sigma -3.2)]$), while the low $Re_\lambda$ formulae are elaborated by Clark & Zemach\cite{14}.

The comparison between the theoretical frameworks and grid turbulence experiments is a difficult issue. In fact, the presence of epistemic uncertainties and the physical domain limits preclude the possibility to observe pure HIT at very high Reynolds numbers in grid turbulence experiments. Conversely, the numerical simulation allows for an exact choice of the initial conditions of the problem, so that HIT decay can be systematically investigated. The numerical simulations of HIT free decay is particularly efficient when performed in the spectral space, because the velocity field and the pressure field are decoupled. The discretization of the spectral domain has to allow for a correct representation of the non-linear interactions between all the active scales of motion. In particular, the smallest resolved scales must be of the size of the Kolmogorov scale $\eta$. On the other hand, the mesh resolution at the large scales must be significantly larger than the integral length scale $L$, in order to exclude saturation effects. As a consequence, the largest resolved scale must satisfy the relation $L_{\text{MAX}} \gg L$. The resolution constraints are tied by the Reynolds number through the relation $L/\eta = Re_L^{3/4} = (3/20 Re_\lambda^2)^{3/4}$. $Re_L$ and $Re_\lambda$ are the Reynolds numbers based on $L$ and the Taylor microscale $\lambda$, respectively. The higher the initial Reynolds number, the larger the spectral domain to be discretized. Moreover, the integral length scale $L$ increases in time by a power-law in HIT free decay (see Table 1), while $L_{\text{MAX}}$ is fixed in time. As a consequence, the choice of the value of $L_{\text{MAX}}$ is affected as well by the total time-length of the numerical simulation.

Among the numerous numerical approaches developed for HIT decay analysis, DNS has traditionally been the reference tool due to the lack of physical hypothesis imposed on the behavior of the flow. Nevertheless, the computational costs required to perform a DNS presently limit its use to moderate $Re_\lambda$ and/or unsatisfactory mesh resolution. Conversely, closure methods relying on physical assumptions can be used to derive information about...
very high Reynolds numbers ($Re_\lambda \geq 10^4$) HIT. The model considered in the present paper is the Eddy-Damped Quasi-Normal Markovian (EDQNM) method[15], which is based on an efficient closure of the Lin equation. This model proved excellent characteristics of accuracy and robustness in describing HIT statistical properties decay[16, 17], requiring very low computational costs if compared to DNS.

Very recently[18], the first attempt to blend HIT studies with the theory of probability proved that the emergence and evolution of stable regimes can be analyzed by the use of stochastic methods. This probabilistic quantification is elusive for the classical numerical approaches, as the result of the simulation is determined by the initial energy spectrum imposed. This condition is chosen by the user and is not necessarily physically justified. These stochastic methods can not be used in combination with DNS at the present time, as the prohibitive computational resources required to simulate high $Re_\lambda$ HIT preclude the systematic analysis of random parameter spaces. While the EDQNM model is a suitable candidate for HIT stochastic analysis, the required resolution at the large scales, which can become even three-four times larger than the one for the inertial/dissipative region for very long decay times, can limit its application to the observation over a limited range of variation of $Re_\lambda$.

In the present paper, we propose a new formulation for the numerical resolution of the EDQNM model, which drastically reduces the computational resources required at the large scales in case of long-time HIT decay. This approach is based on an adaptive procedure applied to the spectral mesh of the EDQNM discretization, which progressively becomes more refined at the large scales. This process grants a chosen resolution at the large scale of the spectrum, so that $L_{MAX}(t) = 10^d L(t), \ d > 0$. As a consequence, saturation effects are naturally excluded. This new approach will be henceforth referred to as adaptive EDQNM model. The reader should note that the word adaptive is here referred to the numerical implementation of the EDQNM model, whose equation is not modified in the process.

The paper is organized as follows. In Section 2, the classical formulation of the EDQNM model is introduced. In Section 3, the adaptive procedure is detailed and discussed. In Section 4, the test case analyzed is presented and the initial conditions and the chosen resolution are described. In Section 5, the results obtained with the adaptive numerical method for the EDQNM model are analyzed by the comparison with the classical model and the theoretical CBC formulae. In Section 6, an extended model for the analysis of sheared turbulence is proposed and assessed. Finally, in Section 7 the conclusions are drawn.

2. The classical EDQNM model

The classical formulation of the EDQNM model is here briefly described. This model has proved to be an excellent tool for the investigation of HIT free decay statistical properties[19], and it has been recently used to investigate non-self-similar regimes[13], pressure fluctuations[20] and temperature fluctuations[21]. The reader is addressed to the comprehensive works of Orszag[15], Lesieur[22] and Sagaut & Cambon[4] for an extended discussion on the model.

As mentioned in the Introduction, the Navier–Stokes equations are usually studied in
their spectral form in HIT. We will refer to as \( NS(\vec{k}) \):

\[
\frac{\partial \hat{u}(\vec{k})}{\partial t} = \hat{u} \hat{u} - \nu k^2 \hat{u}(\vec{k})
\]  

being \( \hat{u} \) the velocity field in the spectral space, \( \vec{k} \) a general wavevector and \( \nu \) the kinematic viscosity of the flow. The term \( \hat{u} \hat{u} \) represents the non-linear convolution term. If the couple of wavevectors \( [\vec{k}, \vec{p}] \) is considered, the two-point velocity correlation can be derived by a simple manipulation of Equation 1. In fact, summing and averaging equations \( \hat{u}(\vec{p}) \cdot NS(\vec{k}) \) and \( \hat{u}(\vec{k}) \cdot NS(\vec{p}) \):

\[
\left[ \frac{\partial}{\partial t} + \nu (k^2 + p^2) \right] \langle \hat{u}(\vec{k}) \hat{u}(\vec{p}) \hat{u}(\vec{q}) \rangle = \langle \hat{u} \hat{u} \hat{u} \rangle 
\]  

Equation 2 is not closed, as the third-order moment on the right side is unknown. It is easy to observe that Equation 2 is equivalent to the Lin equation, which is the spectral counterpart of the Kármán–Howarth equation:

\[
\frac{\partial E(k,t)}{\partial t} + 2\nu k^2 E(k,t) = T(k,t)
\]  

where \( E(k,t) \) and \( T(k,t) \) are the energy spectrum and the non-linear energy transfer. The analysis of Equations 2 and 3 indicate that \( T(k,t) \) corresponds to the third-order moment, which are related to interactions between the triad of wavevectors \( [\vec{k}, \vec{p}, \vec{q}] \). These interactions are exemplified in Figure 1.

An expression for \( T(k,t) \) can be obtained if the procedure used to derive Equation 2 is applied to the triad \( [\vec{k}, \vec{p}, \vec{q}] \). The resulting equation is:

\[
\left[ \frac{\partial}{\partial t} + \nu (k^2 + p^2 + q^2) \right] \langle \hat{u}(\vec{k}) \hat{u}(\vec{p}) \hat{u}(\vec{q}) \rangle = \langle \hat{u} \hat{u} \hat{u} \rangle 
\]
Equation 4 can be used to solve Equation 3. The observation of Equations 2 and 4, though, clearly indicates that the hierarchy of high-order velocity moments is not closed\[23\]. Thus, a closure hypothesis has to be made in order to derive an expression for $\langle \hat{u} \hat{u} \hat{u} \rangle$. The standard quasi-normal approximation (QN) model is based on the assumption that the velocity field $u$ behaves like a Gaussian random variable. For this class of random processes $\psi$, the fourth-order moment $\langle \psi \psi \psi \psi \rangle$ can be decomposed in the sum of products of second-order velocity moments $\Sigma \langle \psi \psi \rangle \langle \psi \psi \rangle$. The difference between the actual fourth-order velocity moment and its Gaussian correspondent, which is usually referred to as cumulant, is taken into account in the EDQNM model by the addition of a linear damping term. Performing a spherical shell integration, the EDQNM equation reads as:

$$
\frac{\partial E(k, t)}{\partial t} + 2\nu k^2 E(k, t) = \int_{\Delta_k} dp dq \Theta_{kpq} \frac{k}{pq} b(k, p, q) E(q, t) [k^2 E(p, t) - p^2 E(k, t)]
$$

where $\Delta_k$ represents the integration over spherical shells (see Figure 1(a)) and $b(k, p, q)$ is a geometrical coefficient of the form:

$$
b(k, p, q) = \frac{p}{k} (xy + z^3)
$$

The terms $x$, $y$ and $z$ are the cosines of the angles of the triangle constituted by the triad $[\vec{k}, \vec{p}, \vec{q}]$ and they respectively face the sides $k$, $p$ and $q$, as shown in Figure 1(b). Their analytical expression is:

$$
x = \cos(\alpha_k) = \frac{p^2 + q^2 - k^2}{2pq}
$$

$$
y = \cos(\alpha_p) = \frac{k^2 + q^2 - p^2}{2kq}
$$

$$
z = \cos(\alpha_q) = \frac{k^2 + p^2 - q^2}{2kp}
$$

The term $\Theta_{kpq}$, which is derived by the application of a markovianization procedure, is the characteristic time of relaxation:

$$
\theta_{kpq} = \frac{1 - e^{(\mu_{kpq} + \nu (k^2 + p^2 + q^2))t}}{\mu_{kpq} + \nu (k^2 + p^2 + q^2)}
$$

where $\mu_{kpq}$

$$
\mu_{kpq} = \mu_k + \mu_p + \mu_q, \quad \mu_i = \sqrt{\int_0^t i^2 E(i, t) \, di}, \quad i = k, p, q
$$

is a characteristic eddy-damping rate. The markovianization procedure neglects the term $(\mu_{kpq} + \nu (k^2 + p^2 + q^2))^{-1}$ with the respect to the characteristic evolution time $\sum \langle uu \rangle \langle uu \rangle$.

The numerical implementation of EDQNM relies on a logarithmic discretization in wavenumber space, so that for each element $k_i$ of the mesh the relations $k_{i-1} = k_i/r$, $k_{i+1} = k_i r$ hold true. The parameter $r$ is chosen by the user. The modes $k_1$ and $k_N$, which are respectively the smallest and the largest element of the spectral mesh, are tied by the relation $k_N = r^{N-1} k_1$. $N$ is the total number of elements. The EDQNM method proves
characteristics of robustness and precision in the prediction of the three-point velocity correlations between the elements of triads $[\vec{k}, \vec{p}, \vec{q}]$. Due to the logarithmic discretization, the energy transfer between non-local very elongated triads is progressively less efficient. This can lead to a loss of accuracy in the prediction of the physical quantities analyzed. The original model is thus modified by the addition of a non-local transfer term $T_{NL}(k, t)$ to Equation 5, which exactly estimates the distant non-local triadic interactions [22, 24]. More precisely, the solved equation is modified in:

$$\frac{\partial E(k, t)}{\partial t} + 2\nu k^2 E(k, t) = \int_{\Delta_k} dpdq \frac{k}{pq} b(k, p, q) E(q, t)[k^2 E(p, t) - p^2 E(k, t)] + T_{NL}$$

(12)

where

$$T_{NL}(k, t) = -\frac{2}{15} k^2 E(k, t) \int_{k/a}^{+\infty} \Theta_{kpp} \left[ 5E(p, t) + p \frac{\partial E(p, t)}{\partial p} \right] dp + \frac{14}{15} k^4 \int_{k/a}^{+\infty} \Theta_{kpp} \frac{E^2(p, t)}{p^2} dp$$

(13)

is the non-local energy transfer and $a = r - 1$ is a parameter related to the spectral grid discretization. It is also necessary to mention that the discretization procedure implies that $E(k, t) = 0, T(k, t) = 0$ for $k < k_1, k > k_N$. The shape of the energy spectrum and of the non-linear transfer, which are reported in Figure 2 for a classical case of Saffman turbulence, show that these assumption are legitimate, if the initial resolution chosen is satisfactory.

3. Adaptive mesh procedure

In this Section, the adaptive mesh procedure for the EDQNM model is illustrated. This formulation allows to impose a fixed spectral resolution of $d$ decades at the large scales, so that $L_{MAX}(t) = 1/k_1(t) = 10^{dL}(t)$. The value of the parameter $d$ is set as an initial condition and it does not change throughout the numerical simulation.

First of all, let us assume that the energy spectrum, for $k < k_1$, behaves like $E(k \to 0, t) = A(t)k^\sigma$, where the parameter $\sigma$ is set prescribing the initial energy spectrum and
Figure 3: Ratio between the non-linear energy transfer $T$ and the energy spectrum $E$, in the case of Saffman turbulence.

the coefficient $A(t)$ complies with the relation $E(k_1, t) = E_1(t) = A(t)k_1^\sigma$. This hypothesis implies that the slope of the energy spectrum is conserved for the scales larger than $L_{\text{MAX}}$; this is consistent with a number of theoretical works reported in open literature. In addition, this assumption is more plausible from a physical point of view than the brutal approximation $E(k, t) = 0$, $k < k_1$ usually made in numerical discretizations.

As a consequence, $T(k \to 0, t) \propto k^{\sigma+2}$ for $k < k_1$. This result can be derived by the manipulation of the Lin equation[17] and is as well observed in numerical EDQNM simulations for $k < k_L$, see Figure 3.

The use of the previous hypothesis allows to exactly recover the total turbulent kinetic energy $K$:

$$K(t) = \int_0^{k_1} A(t)k^\sigma \, dk + \int_{k_1}^{k_N} E(k, t) \, dk + \int_{k_N}^{\infty} E(k, t) \, dk = \frac{A(t)}{\sigma + 1} k_1^{\sigma+1} + \int_{k_1}^{k_N} E(k, t) \, dk$$

(14)

being the term $\int_{k_N}^{\infty} E(k, t) = 0$, if $k_N$ is sufficiently larger than the inverse of the Kolmogorov scale $k_\eta = 1/\eta$ (see Figure 2 (a)). The two integral terms of Equation 14 are illustrated in Figure 4. Equivalently, the integral length scale $L$ and the energy dissipation rate $\varepsilon$ are computed as:

$$L(t) = \frac{3\pi A(t) k_1^{\sigma}}{4} + \int_{k_1}^{k_N} E(k, t)/k \, dk$$

(15)

$$\varepsilon(t) = 2\nu A(t) k_1^{\sigma+3} + 2\nu \int_{k_1}^{k_N} k^2 E(k, t) \, dk$$

(16)
3.1. Adaptive procedure: criteria

The classical numerical approaches, such as DNS, are based on the use of a spectral discretization of the domain which is created at the beginning of the simulation. In the case of free HIT decay, this implies that the initial resolution at the large scales must be sufficiently large to preclude saturation effects for the time-length investigated, because the resolution $L_{MAX}/L(t)$ diminishes through the simulation. The value of this ratio can be approximated by the use of the CBC formulae:

$$\frac{L_{MAX}}{L(t)} = \frac{L_{MAX}}{L(0)} t^{-n_L}, \quad n_L > 0$$ \hspace{1cm} (17)

The adaptive mesh procedure fixes the right term of Equation 17, which is the resolution at the large scales. As a consequence, $L_{MAX}$ evolves in time following the integral length scale dynamics:

$$\frac{L_{MAX}(t)}{L(t)} = \text{const.} = 10^d$$ \hspace{1cm} (18)

where $d$ is the large scales resolution (in decades) and it is chosen by the user. This procedure implies two main benefits. First, the effects of saturation are naturally excluded, if $d$ is sufficiently large. Second, the computational requirements are drastically reduced, because the time-dependence of the resolution at the large scales is canceled.

3.2. Numerical implementation

The application of the adaptive procedure to the EDQNM model is now detailed. The user will be required to furnish the initial conditions of the flow ($Re_\lambda$, $K$, shape of the energy spectrum...) and two parameters for the resolution at the large scales. The first one, which will be referred to as $d_I$, represents the resolution at the large scales imposed as initial condition. The second, $d$, is the resolution requirement conserved by the adaptive
procedure, once it is triggered. Let us assume to fix an initial resolution at the large scales $d_I > d$, so that $\log_{10}(k_L(0)/k_1(0)) = d_I$. At the beginning of the first time step, the spectral mesh is generated. The number of elements $N$ is determined so that $k_N(0) = \beta/\eta(0) = r N^{-1} k_1(0)$, where the parameter $\beta \geq 1$ sets the resolution at the small scales. After the mesh is build, the geometry coefficient $b(k, p, q)$ are computed and stored, like in the classical EDQNM model. The adaptive model resolves the Equation 12 at each time step, so that a stable decay regime emerges after a transient. In particular, $k_L = 1/L$ will evolve following the power-law $k_L(t) \approx k_L(0) t^{-\alpha L}$. The adaptive EDQNM model will not operate on the initial mesh generated until the condition $\log_{10}(k_L(t)/k_1(0)) < d$ will occur. This condition is equivalent to Equation 18. Let us indicate with $k$ and $k'$ the mesh element before and after the application of the adaptive procedure, respectively. The same notation is used for the energy spectrum $E$. The following steps are performed:

- $A(t) = E(k_1(t), t)/k_1^\sigma(t) = E_1(t)/k_1^\sigma(t)$
- $k_i'(t) = k_i(t)/r$
- $E'(k_i', t) = E_1'(t) = A(t)(k_i'(t))^\sigma$
- $k_i'(t) = k_{i-1}(t)$, $E_i'(t) = E_{i-1}(t)$, $i = 2 : N$

The total number of elements $N$ is conserved both for $k$ and for $E$, so that the values of $k_N$ and $E_N$ before the adaptive procedure are discarded. This is a legitimate approximation if a sufficient resolution at the small scales is imposed, so that $E_N(t) = E(k_N, t) \approx 0$ for every $t$. The conservation of the total number of elements $N$ during the simulation leads to a minimum increase of computational resources required to perform the adaptive procedure. In fact, if the number of elements $N$ and the ratio $r$ are constants, there is no need to recompute the geometrical correlations between the elements of the triad $[k, p, q]$, which are represented by the term $b(k, p, q)$ in Equation 5 and 12. This matrix is calculated at the initial time and then stored. The update of the spherical shell radius for the EDQNM integration is included in the generation of the new mesh.

It is very important to stress that, thanks to Equation 14 for the calculation of $K$, $L$ and $\varepsilon$, the physical quantities values are exactly conserved through the adaptive procedure. In fact, the sum of the two terms of Equation 14 does not change, so that the variation of one term is perfectly balanced by the other term.

After the adaptive procedure has modified the mesh, the model behaves like the standard version and the mesh is not changed until the condition $\log_{10}(k_L/k_1) < d$ is satisfied again. When this happens, the adaptive procedure is triggered. Moreover, the numerical stability requirement (CFL) related to the discretization of the time derivative imposes a maximum value of the time step which is several orders of magnitude smaller than the characteristic time of evolution of the integral length scale. In other words, referring to the time step $i$ in which the adaptive procedure is performed, the relation $L(i) < L(i + 1) < r L(i)$ is always verified. This implies that the adaptive procedure is not triggered at every time step, but a number of times equal to $\gamma = (\log_{10} r)^{-1}$ per decade of displacement of the integral length scale $L$. A clear picture of the time evolution of the elements $k_1$ and $k_N$ is reported in Figure 5, where the case $\sigma = 2$, $Re(0) = 2000$, $d = 2$, $d_I = 2.7$ has been considered. It is possible to observe that, after a first simulation time in which the grid is not modified due to the initial value of $d_I > d$, the value of $k_1$ progressively reduces as it is recursively divided by $r$. The
same exact behavior is observed for \( k_N \). Let us now consider the energy spectra plotted for \( \tau = 10, \ 10^4 \), being \( \tau = t/t_0 \) and \( t_0 = K(0)/\varepsilon(0) \) the normalized time of the simulation and the characteristic turnover time, respectively. For the energy spectrum at \( \tau = 10 \), the resolution at large scales is \( d < \log_{10}(L_{\text{MAX}}(\tau = 0)/L(\tau = 10)) \) as the adaptive procedure has not been triggered yet. Conversely, \( E(k, \tau = 10^4) \) shows a resolution of \( d \) decades at the large scales, which is conserved throughout the EDQNM simulation.

4. Test case investigated

The adaptive mesh formulation is tested comparing a database of adaptive EDQNM simulations with a similar database build using the classical approach. For all the simulations, the initial energy spectrum prescribed is inspired by the functional forms proposed by Pope[25] and by Meyers & Meneveau[26]:

\[
E(k) = C_k \varepsilon^{2/3} k^{-5/3} f_L(kL) f_\eta(k\eta)
\]

where \( C_k \) is the Kolmogorov constant and

\[
f_L(kL) = \left( \frac{kL}{[(kL)^{1.5} + c_L]^{1/1.5}} \right)^{5/3+\sigma}, \quad f_\eta(k\eta) = \exp(\beta_\eta((k\eta)^4 + c_\eta^{1/4} - c_\eta))
\]

are two functions governing the shape of the spectrum at large scales and small scales, respectively. The coefficients are set to the values \( c_\eta = 0.4 \) and \( \beta_\eta = 5.3 \), while \( c_L \) is determined so that the initial value of the integral length scale is \( L(0) = 1 \).
The initial resolution at the small scales is $k_N(0) = 10 k_\eta(0)$, $k_\eta = 1/\eta$; due to the fact that the total number of modes in the spectral discretization is conserved, the resolution at the small scales will monotonically increase during the simulation. In fact, for the adaptive version of the EDQNM model, $k_N$ moves in time with a power law exponent of $-n_L$, while $k_\eta$ evolution is governed by the exponent $-n_\eta$. It is easy to verify using the CBC formulae in Table 1 that $-n_\eta \leq -n_L$ for every $\sigma$ value, the equality being recovered in the case of $\sigma = 1$ only. The parameter $r = k_{i+1}/k_i$ is set in all simulations to $r = 1.122$, which corresponds to a discretization of twenty wavenumbers for each decade in the spectral space. That also means that the adaptive procedure is triggered twenty times for each decade of increase of $L$. The initial resolution at the large scales, as well as the initial $Re_\lambda$ ans $\mathcal{K}$, will be specifically discussed for each case analyzed.

5. Validation & results

The performance of the adaptive numerical method for the EDQNM model is now assessed by the comparison with classical EDQNM simulations. First, a sensitivity analysis of the adaptive EDQNM model to the parameters $\sigma$ and $d$ is addressed. The initial conditions set are $Re_\lambda(0) = 2000$, $\mathcal{K}(0) = 1$ and the total simulation time considered is $\tau = 10^{10}$. More specifically, two main points are herein observed:

- the consistency of the adaptive EDQNM model, which is the possibility to recover exactly the same results of the classical model, for every initial condition imposed.
- the efficiency of the adaptive EDQNM model, which can be measured as the reduction of computational resources required, if compared to the standard model.

This analysis is performed through the comparison of two databases of twelve simulations each, generated with the adaptive model and with the classical EDQNM respectively. This database can be seen as a $[4 \times 3]$ matrix for the chosen values of the parameter $\sigma = 1, 2, 3, 4$ and the final resolution at the large scales $d_F = 2, 3, 4$. The initial value of the smallest wavenumber $k_1$ is derived through different relations for the two models. For the adaptive EDQNM model, $k_1(0)$ is set so that:

$$k_1(0) = 0.2 \cdot 10^{-d_F} k_L(0)$$

in this way, the mesh will not be modified until $k_L$ will satisfy the condition $\log_{10}(k_L/k_1) = d_F = d$, which will trigger the recursive adaptive procedure. The choice to impose an initial finer resolution is justified by possible initial oscillations of $k_L$ during the transient. Conversely, the initial value of the resolution for the classical EDQNM simulations has been derived by the use of a simplified approach based on the CBC formulae. Neglecting the effects of the transient and of finite Reynolds numbers, the displacement of the integral length scale in decades will be $n_t \cdot n_L(\sigma)$, where $n_t$ is the total simulation length (in time decades). In the present case, the resulting formula is:

$$d_I = d_F + 10 \cdot n_L(\sigma), \quad k_1 = 10^{-d_I} k_L(0)$$

For sake of clarity, the value of the first wavenumber of the spectral mesh $k_1$ is summarized in Table 2.

The accuracy of the adaptive EDQNM approach is now investigated. The numerical results for the cases $\sigma = 2, 4$ and $d_F = 2$ are compared. These cases are chosen as
an example and the same conclusions have been observed for all the simulations of the database. Figure 6 (a) and (b) show that the decay of the two physical quantities $K$ and $\varepsilon$ is correctly computed by the adaptive EDQNM model over a long time simulation. In fact, the numerical results perfectly juxtapose with the results by the classical model, and they are in very good agreement with the CBC formulae. Moreover, the analysis of the energy spectrum $E$ and of the non-linear energy transfer budget term $kT$, which are reported in Figure 7 (a) and (b) for $\tau = 10^5$, confirms that the adaptive EDQNM model achieves exactly the same level of precision of the classical model.

The efficiency of the adaptive EDQNM method is now addressed. The ratio $CT_A/CT_S$, where $CT_A$ and $CT_S$ are respectively the computational time required by the adaptive and standard method, is shown in Figure 8 and reported in Table 3. The analysis of the results shows that the maximum gain is recovered for low $\sigma$ and $d_F$ values, with a minimum value of $CT_A/CT_S = 0.46$. This result is not surprising, as for lower $\sigma$ values $k_L$ evolution in time is faster. As a consequence, a greater resolution at the large scales is needed with the classical method. At the same time, lower $d_F$ values imply a lower number of total points for the adaptive procedure. Thus, the optimal $d$ value is the threshold value for which saturation effects due to a lack of large-scales resolution are negligible. An extensive analysis of adaptive EDQNM simulations indicate that the value

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<table>
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<th>$d_F$</th>
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<th>4</th>
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<tr>
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<td>$1 \cdot 10^{-8}$</td>
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<tr>
<td>$\sigma = 3$</td>
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<td>$2 \cdot 10^{-4}$</td>
<td>$2 \cdot 10^{-5}$</td>
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<td>$4.64 \cdot 10^{-7}$</td>
<td>$4.64 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>$\sigma = 4$</td>
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<td>$2 \cdot 10^{-4}$</td>
<td>$2 \cdot 10^{-5}$</td>
<td>$8.2 \cdot 10^{-6}$</td>
<td>$8.2 \cdot 10^{-7}$</td>
<td>$8.2 \cdot 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 2: Smallest wavenumber $k_1(0)$ of the spectral mesh used in the EDQNM simulations. For all the cases, $k_L(0) = 1/L(0) = 1$ and $k_N(0) = 1.45 \cdot 10^3$, where $k_L$ and $k_N$ are the wavenumber associated to the integral length scale $L$ and the largest wavenumber resolved, respectively.

Figure 6: Comparison of HIT statistical quantities recovered by the use of the adaptive and the classical EDQNM model, in the case of Saffman and Batchelor turbulence. (a) The turbulent kinetic energy $K$ and (b) the energy dissipation rate $\varepsilon$ are respectively reported and compared with the CBC formulae (Table 1). The initial Reynolds number is $Re_\lambda = 2000$. 

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<table>
<thead>
<tr>
<th>$d_F$</th>
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</table>

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Figure 7: Comparison of the results recovered by the use of the adaptive and the classical EDQNM model, in the case of Saffman and Batchelor turbulence. (a) The energy spectrum $E$ and (b) the premultiplied non-linear energy transfer $kT$ are shown, respectively. The initial Reynolds number is $Re_\lambda = 2000$.

$d_F = d = 2$ is an excellent compromise between accuracy and efficiency, and it will be chosen as a fixed parameter from now on.

<table>
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<tr>
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<th>$d_F = 3$</th>
<th>$d_F = 4$</th>
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<td>0.49</td>
<td>0.52</td>
</tr>
<tr>
<td>$\sigma = 2$</td>
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<td>0.58</td>
<td>0.61</td>
</tr>
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<td>$\sigma = 3$</td>
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<td>0.65</td>
<td>0.67</td>
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<td>$\sigma = 4$</td>
<td>0.64</td>
<td>0.67</td>
<td>0.69</td>
</tr>
</tbody>
</table>

Table 3: Ratio $CT_A/CT_S$ of the computational resources required by the adaptive and standard EDQNM model, to perform case 1 analysis. The sensitivity of the ratio to the slope of the energy spectrum $\sigma$ and to the final resolution $d_F$ is considered.

Summarizing, the analysis of the results shows that around 50% of the computational resources needed can be saved by the use of the adaptive EDQNM model, when it is applied to average time-length simulations. The efficiency is supposed to improve when the adaptive method is applied to long-time simulations, which are needed when an analysis encompassing several decades of Reynolds number has to be performed.

We now address one of these cases, considering Saffman and Batchelor turbulence decay in the range $Re_\lambda(0) = 10^4 \rightarrow Re_\lambda(t_{fin}) = 10^{-4}$. An analysis encompassing such a wide range of $Re_\lambda$ requires the observation for a very large number of time decades. A simple approximation can be derived considering the relation $Re_\lambda \propto t^n Re_\lambda$, where the expression for $n_{Re_\lambda}$ is given in Table 1. For sake of simplicity, let us assume to use the high Reynolds expression for $Re_\lambda > 1$ and the low Reynolds expression for $Re_\lambda < 1$. This approximation indicates that a total simulation time time of $10^{56}t_0$ is necessary to observe this range of decay for Saffman turbulence, while a time of $10^{26}t_0$ is necessary for the Batchelor turbulence. In this long-time decay analysis, the initial value of the
turbulent kinetic energy is set to $K = 10^{10}$. The initial resolution for the classical EDQNM simulations is chosen using the same CBC based criterion of the first case investigated and $d_F = 2$, while for the adaptive EDQNM it is set to $d = 2$.

The decay of physical quantities, such $K$ (Figure 9 (a)) and $\varepsilon$ (Figure 9 (b)) is correctly obtained by the adaptive procedure even for very long decay times. The same conclusion can be drawn observing the decay of the energy spectra, both for high and low Reynolds numbers, as reported in Figure 10 (a) and (b).

The reader has to consider that, for the adaptive simulations, 55 wavenumbers only are initially used to discretize the large scales region. Conversely, the mesh discretization for the classical EDQNM consists of 401 and 221 wavenumbers in the region $k < k_L(0)$, for the cases of Saffman and Batchelor turbulence, respectively. The number of elements used to discretize the inertial/dissipative region is the same for the two approaches, and in this particular case it consists of 126 elements. As a consequence, the adaptive approach reduces the computational resources required of the 85% for the case of Saffman turbulence and of the 70% for the case of Batchelor turbulence. Moreover, the larger the size of the mesh, the larger the number of EDQNM integrations performed on very stretched triads. For this reason, we can conclude that the precision of the adaptive EDQNM model will be at least at the level of the standard model, if not better.

5.1. Energy conservation property of the adaptive EDQNM model

We now address the fulfillment of the property of energy conservation, which represent a classical measure of the precision of a numerical method used for the analysis of HIT free decay. Let us consider the integral over $k$ of the Lin Equation:

$$\int_0^{+\infty} \frac{\partial E(k,t)}{\partial t} dk = \int_0^{+\infty} T(k,t)dk - 2\nu \int_0^{+\infty} k^2 E(k,t)dk$$

(23)

knowing that $\int_0^{+\infty} T(k,t)dk = 0$, Equation 23 reduces to

$$\frac{\partial K(t)}{\partial t} = -\varepsilon(t)$$

(24)
Figure 9: Comparison of HIT statistical quantities recovered by the use of the adaptive and the classical EDQNM model, in the case of Saffman and Batchelor turbulence. (a) The turbulent kinetic energy $K$ and (b) the energy dissipation rate $\varepsilon$ are respectively reported. The initial Reynolds number is $Re_\lambda = 10^4$ and HIT decay is observed until $Re_\lambda = 10^{-4}$.

Figure 10: Comparison of the results recovered by the use of the adaptive and the classical EDQNM model, in the case of Saffman and Batchelor turbulence. The energy spectrum $E$ is shown at the sampled times (a) $\tau = 10^2$ and (b) $\tau = 10^{20}$. 
A numerical method fulfills the energy conservation property if the relation \( \int_{k_1}^{k_N} T(k,t) \, dk = 0 \) is recovered, where \( \int_{k_1}^{k_N} \) is the discrete integral over the spectral mesh of the non-linear energy transfer. This integral, normalized over \( \varepsilon \), has been investigated in EDQNM simulations for both the adaptive and the standard formulation of the model. The results are shown in Figure 11. It is possible to appreciate that the dynamic procedure applied to the mesh in the adaptive EDQNM increases the order of magnitude of the error, if compared to the standard version. Nevertheless, this error is stable over a long time evolution and its magnitude is around \( 10^{-12} \), so that it can be considered negligible. Initially, the error has the same magnitude of the classical version, because the adaptive procedure has not been triggered yet. The error magnitude increases starting from the normalized time \( \tau \approx 10 \), which coincides with the first mesh operation. Then, the error produced by the adaptive procedure grows due to the interactions with the error associated to the first-order explicit time derivative discretization for the term \( \partial E / \partial t \), until it reaches the asymptotic limit for \( \tau \approx 10^2 \).

6. Extension of the adaptive numerical scheme to forced turbulence analysis

The adaptive method presented in Section 3 proved excellent properties of accuracy and efficiency, if compared to the classical spectral discretization. Nevertheless, the method is studied for the analysis of isotropic turbulence free decay, where the turbulent kinetic energy \( K \) is decreasing in time. The model is now extended to account a possible variation of the number of total elements \( N \), granting a mesh resolution refinement at both large and small scales. This issue can arise in the case of generally forced
turbulence, such as sheared turbulence or rotating turbulence. In such a case, \( K \) and \( \text{Re}_\lambda \) can evolve in time so that the number of active wavenumbers both in the large scales direction and in the small scales direction can increase. The extended adaptive algorithm consists of three main steps:

1. The model as presented in Section 3 is applied, and the resolution at the large scales is corrected. During this step, the total number of elements \( N \) is conserved.

2. The resolution at the small scales is checked so that \( r^{-1} 10^{d_\eta} < k_{\text{max}}/k_\eta < r 10^{d_\eta} \). \( d_\eta \) is here the fixed resolution (in decades) at the small scales.

   - In the case \( k_{\text{max}}/k_\eta > r 10^{d_\eta} \), the new number of total elements is \( N_{\text{new}} = N - 1 \) and the elements \( E(N) \) and \( T(N) \) are discarded.

   - In the case \( k_{\text{max}}/k_\eta < r^{-1} 10^{d_\eta} \), the new number of elements is \( N_{\text{new}} = N + 1 \) and \( E(N_{\text{new}}) = 0 \), \( T(N_{\text{new}}) = 0 \).

3. If \( N_{\text{new}} \neq N \), the geometry table for the coefficient \( b(x, y, z) \) is recomputed.

The extended model grants a fixed resolution both at the large scales and at the small scales. The model is now assessed by the analysis of initially isotropic turbulence, which is subjected for a limited time to a forcing effect \( F \). The Lin equation is thus modified:

\[
\frac{\partial E}{\partial t} + 2 \nu k^2 E = T + F
\]  

(25)

This forcing term can be thought as an external effect influencing the turbulent state. As an example, we consider a power-law forcing of the form:

\[
F(k) = \begin{cases} 
0, & k < 10^{-2}, k > 2 \times 10^4 \\
10 k^{-1}, & 10^{-2} < k < 2 \times 10^4 
\end{cases}
\]  

(26)

a power-law dependence on the wavenumber is expected in fractal grid turbulence experiments and the power law-exponent of the forcing is usually included in the range \([-2, 1]\)[27]. Here, the coefficient is set to \(-1\) in order to drive the integral length scale as close as possible to the largest forced mode for \( k = 10^{-2} \). The forcing term in Equation 25 is applied for \( 10^3 t_0 \) characteristic times. After this threshold, \( F(k) = 0 \ \forall k \) and a free decay is analyzed. The choice for the premultiplying factor of 10 in Equation 26 has been made in order to significantly affect the magnitude of the total turbulent kinetic energy of the flow.

The initially isotropic turbulence state is defined by the functional form in Equations 19-20. The parameters chosen are \( \sigma = 2 \), \( \text{Re}_\lambda = 10^3 \), \( K = 1 \), \( k_L = 1 \) and, as a consequence, \( k_\eta = 6.81 \cdot 10^3 \). The initial mesh resolution imposed is \( k_1 = 2 \cdot 10^{-3} \), \( k_N = 7.08 \cdot 10^4 \), \( r = 1.122 \) and \( N = 152 \). The adaptive procedure will conserve the values of \( d = 2 \) and \( d_\eta = 1 \) decades of resolution at the large scales and at the small scales, respectively.

The evolution of the energy spectrum is reported in Figure 12 (a). The forcing term significantly modifies the energy spectrum and an asymptotic shape is observed for \( \tau = 10^3 t_0 \). In this state, the total dissipation is in equilibrium with the energy input rate associated with the forcing model. The observation of the vertical lines, which are associated to the wavenumbers \( k_1 \) and \( k_N \), indicate that the mesh resolution has been refined both at large and small scales in order to grant the fixed requirements. For \( \tau > 10^3 t_0 \), a free decay is released. The small scales progressively loose energy and the peak of the energy spectrum
Figures 12 and 13: Forced turbulence decay, analyzed by the extended adaptive EDQNM model. (a) The energy spectra sampled at $\tau = 0, 10^3, 10^7$ are shown. Vertical lines refer to the wavenumbers $k_1, k_N$. (b) The total number of elements of the spectral mesh $N$ as a function of $\tau$ is presented.

The conclusions drawn by the observation of the energy spectrum are confirmed when the main statistical quantities are analyzed. The data reported in figure 13 (a) and (b) show the evolution in time of the turbulent kinetic energy $K$ and the integral length scale $L$. $K$ progressively reaches an asymptotic limit at $\tau = 10^3 t_0$. A monotonic decrease is observed for $\tau > 10^3 t_0$.

The time evolution of $N$ is reported in figure 12 (b). As expected, $N$ increases and reaches an asymptotic limit at $\tau = 10^3 t_0$. A monotonic decrease is observed for $\tau > 10^3 t_0$. Moves towards the large scales. New mesh elements are added at the large scales, while the largest wavenumbers are removed. As the total energy of the system decreases, the number of active modes diminishes and the adaptive procedure reduces the total number of elements $N$. The time evolution of $N$ is reported in figure 12 (b). As expected, $N$ increases and reaches an asymptotic limit at $\tau = 10^3 t_0$. A monotonic decrease is observed for $\tau > 10^3 t_0$. When the forcing term fades a transient state emerges, after which the classical power law decay is observed. A similar behavior is obtained for $L$. 

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7. Conclusions

A new formulation for the EDQNM model is proposed in the present paper. This formulation is based on an adaptive procedure that progressively modifies the resolution of the spectral mesh at the large scales, forcing a resolution requirement set by the user.

The resulting adaptive EDQNM model has been tested by comparison with the classical model, selecting different values of the initial $Re_{\lambda}$ and of the slope of the energy spectrum $\sigma$. In the first case considered, HIT decay with initial $Re_{\lambda} = 2000$ has been analyzed, for a total simulation time of $10^{10} t_0$. The analysis has been performed by the comparison of two databases, in which the value of the parameter $\sigma$ and of the final resolution $d_F$ have been systematically checked. For all the initial condition chosen, the adaptive EDQNM model recovers exactly the same results of the classical model. The computational resources required by the adaptive method are always significantly lower if compared to the standard model, but the magnitude of the improvement is dependent on the parameters analyzed. The analysis of very long-time decay has then been addressed, considering Saffman and Batchelor turbulence decay in the range $10^{-4} \leq Re_{\lambda} \leq 10^4$. In this case, the large scales resolution parameter has been fixed to $d = 2$, which proved to be sufficiently high to exclude saturation effects. The numerical results confirmed the accuracy of the adaptive EDQNM method, even when moderate to very low Reynolds numbers are considered. This result has been observed in the analysis of both the main statistical quantities of interest and in the direct comparison of the energy spectra.

The most important point is that the model proved to be progressively more efficient when long-time simulations have to be performed. In particular, a reduction of the order of 85% in the computational resources needed has been observed, when the case of very long-time screening of Saffman turbulence decay is addressed. This reduction in computational time, along with the excellent precision observed, shows that the adaptive EDQNM model is a promising tool for stochastic analysis of HIT free decay.

The adaptive EDQNM model has been successfully extented to the analysis of forced turbulence. The procedure has been complexified, in order to take into account a variation of the active wavenumbers both in the large scales and in the small scales. The extended adaptive procedure has been assessed by the analysis of an initial isotropic state, perturbed by a fractal-like forcing model. The results show that the fixed requirements for the mesh resolution have been respected both at the large scales and at the small scales, and the total number of elements is updated accordingly to the variation of turbulent kinetic energy of the flow.

References


