Abstract

Mechanism design is the study of algorithm design where the inputs to the algorithm are controlled by strategic agents, who must be incentivized to faithfully report them. Unlike typical programmatic properties, it is not sufficient for algorithms to merely satisfy the property—incentive properties are only useful if the strategic agents also believe this fact.

Verification is an attractive way to convince agents that the incentive properties actually hold, but mechanism design poses several unique challenges: interesting properties can be sophisticated relational properties of probabilistic computations involving expected values, and mechanisms may rely on other probabilistic properties, like differential privacy, to achieve their goals.

We introduce a relational refinement type system, called HOARE⁺, for verifying mechanism design and differential privacy. We show that HOARE⁺ is sound w.r.t. a denotational semantics, and correctly models (ε, δ)-differential privacy; moreover, we show that it subsumes DFuzz, an existing linear dependent type system for differential privacy. Finally, we develop an SMT-based implementation of HOARE⁺ and use it to verify challenging examples of mechanism design, including auctions and aggregative games, and new proposed examples from differential privacy.

Categories and Subject Descriptors D.3.1 [Programming Languages]: Formal Definitions and Theory—Semantics; D.2.4 [Software Engineering]: Software/Program Verification.

Keywords program logics; probabilistic programming

1. Introduction

When designing algorithms, we usually assume that the inputs are correctly reported. However, in the real world, inputs may be provided by people who may want to influence the outcome of the algorithm. Mechanism design is the field of algorithm design where the inputs to the algorithm (often called a mechanism) are controlled by strategic agents who may manipulate what their inputs. In this setting, it is not enough to design an algorithm which behaves correctly on correct input; the design of the mechanism must convince (incentivize) agents to provide their correct inputs to the algorithm.

The canonical application of mechanism design is auction design. In an auction, the algorithmic problem can be very simple: for instance, allocate some set of goods amongst a set of n agents so as to maximize their sum value for the goods. The inputs to the algorithm can be very simple: the agents’ values for the goods, but these are unknown to the algorithm designer. Instead, the mechanism must elicit bids. To incentivize agents to bid honestly, actions compute a price that each agent must pay. Auctions are generally designed so that the allocation and payment rules incentivize agents to bid their true value for the goods, no matter what their opponents do. An auction that satisfies this property is said to be dominant strategy incentive compatible, or simply truthful. This is among the most important solution concepts—predictions about behavior of strategic agents—in mechanism design.

Beyond auctions, mechanisms design can be used to handle more abstract games where agents have a variety of actions and a real-valued utility function based on the actions selected by all agents. In most settings, agents do not have any dominant strategies, and so we must be satisfied with weaker solution concepts like Nash equilibrium. Informally, a set of actions, one for each player, forms a Nash equilibrium if no player can increase her utility unilaterally deviating to a different action so long as no other player deviates.

The hope is that strategic agents will collectively decide to play at an equilibrium: no single agent can gain by deviating.

However, Nash equilibria can be an unrealistic prediction of behavior. First of all, they are generally not unique: agents must somehow coordinate to play at a single equilibrium, but different agents might prefer different equilibrium outcomes. Second of all, in games with a large number of players, agents generally do not have complete knowledge about everyone’s utility functions, and so may not even know what the Nash equilibria of the game are. To help players coordinate on an equilibrium, one approach is to design an equilibrium selection mechanism. Agents are asked to report their utility functions to some mediator, and the mediator suggests some action for them to play. Agents are strategic, so they are free to misreport their utility function, or disregard the mediator’s suggestion. A well-designed mediator will incentivize agents to report truthfully and follow the recommendation.

A promising and recent tool for mediator design is differential privacy [18]. The original goal of differential privacy was to protect individuals privacy in data mining, by ensuring that answering the same query on two databases differing in a single person’s data leads to results that are close in some sense. Seen another way, differential privacy limits any individual’s influence on the result. This can be

\[1 \text{ Contrast this with dominant strategies: in a truthful auction, no agent can gain by deviating no matter what the other players play.} \]
we discuss related work in § 7, and conclude with possible future directions in § 8.

While this is a theoretically clean idea, there can be practical issues: currently proposed mechanisms are complex enough that agents may not be able to verify the promised incentive properties of the mediator. In general, if agents do not believe the incentive properties of a mechanism, then they may behave in unpredictable ways or decline to participate. Indeed, designers of an upcoming public radio spectrum auction have stressed “obviousness” as a key feature—the incentive features should be plainly apparent to any agent [39]. While this is a desirable goal for simple mechanisms, it is hard to achieve for complex mechanisms. For these cases, we propose an alternative approach: rather than simplify the mechanism, use formal verification to automatically check incentive properties.

However, mechanism design poses serious challenges for verification. First, both differential privacy and equilibria properties are relational properties of programs, which reason about more than one run of the same program; for instance, truthfulness states that the payoff of an agent in a run when she reports truthfully is at least its payoff in a run where she reports arbitrarily. Second, equilibria properties are significantly more involved than typical program verification properties. For randomized mechanisms, properties are stated in terms of expected value rather than more standard equivalences or relations between distributions. Finally, incentive properties of mediator mechanisms rest on non-trivial interactions between game-theoretic properties and differential privacy, so their formal verification must be conducted in a framework that is expressive enough to reason about differential privacy, game-theoretic properties, and interactions of the two.

Contributions To handle these challenges, we present HOARe², a type-based framework for relational properties of higher-order probabilistic programs, like differential privacy, truthfulness, and approximate equilibrium. HOARe² is based on refinement types—an expressive type discipline that captures fine-grained properties of computations by enriching types with assertions [23]—and tightly integrates several features: relational refinements [4], refinements at higher types (where assertions constrain the behavior of functions), and a polygonal representation of approximate refinements for probabilistic computations [3]. We demonstrate the theoretical and practical relevance of HOARe² through the following contributions:

• We demonstrate that HOARe² achieves desirable meta-theoretical properties, like soundness with respect to a denotational semantics (§ 3.3) and semantic subtyping. As a contribution of independent interest, we show that the logical interpretation of non-termination adopted by existing refinement type systems is inconsistent with semantic subtyping when refinements at higher order types are allowed.

• We define a type-preserving embedding of DFuzz [24]—a linear dependent type system for differential privacy—into HOARe², and recover soundness of DFuzz from soundness of HOARe² (§ 4). The embedding illustrates how semantic subtyping and refinements at higher types combine to internalize logical relations in HOARe².

• We implement a type-checker for HOARe² and verify examples drawn from differential privacy and mechanism design. For instance, we verify truthfulness properties of randomized auctions, and an equilibrium selection algorithm for aggregative games based on differential privacy. The implementation is fully automated, discharging assertions using SMT solvers.

We discuss related work in § 7, and conclude with possible future directions in § 8.

2. Relational refinements, informally

We will establish properties of programs by using refinement types, an expressive typing discipline introduced by Freeman and Pfenning [23]. As is typical in refinement type systems, we type expressions in two steps. First, we define a simply typed system in which the type of probabilistic computations are modeled using a probability monad \( \mathbb{M}[\cdot] \); for example, the expected value of a positive real-valued function w.r.t. a distribution is modeled by a function \( \mathbb{E} \) of type: \( \mathbb{M}[T] \to (T \to \mathbb{R}^+) \to \mathbb{R}^+ \), where \( \mathbb{R}^+ \) denotes the type of positive reals and \( \mathbb{R}^+ \) denotes \( \mathbb{R}^+ \) extended with \( +\infty \).

Next, we define a relational refinement type system for simply typed expressions. Relational refinements [4] specify properties of pairs of values via types of the form \( [x : T | \phi] \), where \( \phi \) is a relational assertion: a logical formula that can express facts involving the left instance \( x_0 \) and the right instance \( x_1 \) of \( x \). For instance, the type \( [x : \mathbb{N} | |x_0 - x_1| \leq k] \) models pairs of natural numbers which differ by at most \( k \).

Both traditional and relational refinement type systems (e.g., [4]) often forbid refinements at higher types, like \( [x : T | \phi] \) where \( T \) is a function type. However, such relational refinements are convenient to model properties of probabilistic operators. For instance, given a distribution \( \mu : \mathbb{M}[T] \), the following types for \( \mathbb{E}\mu \) capture monotonicity and linearity of expectation:

\[
\begin{align*}
    \{ f : T \to \mathbb{R}^+ \mid \forall z. f_{\mu z} \leq f_{\nu z} \} & \to \{ r : \mathbb{R}^+ \mid r_\mu \leq r_\nu \} \\
    \{ f : T \to \mathbb{R}^+ \mid \forall z. f_{\mu z} = k \cdot f_{\nu z} \} & \to \{ x : \mathbb{R}^+ \mid x_\mu = k \cdot x_\nu \}
\end{align*}
\]

Relational refinements can also be used to model relations between pairs of distributions, like differential privacy. A probabilistic computation \( F : T \to U \) is \((\epsilon, \delta)\)-differentially private (w.r.t an adjacency relation \( \Phi \)) if for every \( t_1, t_2 \in T \), and for every subset of outputs \( E \),

\[
t_1 \Phi t_2 \implies \Pr_{x \sim F \; t_1} [x \in E] \leq \exp(\epsilon) \Pr_{x \sim F \; t_2} [x \in E] + \delta.
\]

The parameters \( \epsilon \) and \( \delta \) are non-negative real numbers controlling the strength of the privacy guarantee. Using relational refinement types, the type of \((\epsilon, \delta)\)-differentially private computations from \( T \) to \( U \) is:

\[
\{ x : T \mid x_\phi. \Phi x_{\geq} \to [\mu : \mathbb{M}[U] \mid \Delta_\epsilon(\mu_1, \mu_2) \leq \delta] ,
\]

where

\[
\Delta_\epsilon(\mu_1, \mu_2) = \max_{E \subseteq U} \Pr_{x \sim \mu_1} [x \in E] - \exp(\epsilon) \Pr_{x \sim \mu_2} [x \in E]
\]

is the \( \epsilon \)-distance [3] between two distributions \( \mu_1 \) and \( \mu_2 \) over \( U \).

However, this modeling is not appropriate for practical program verification; indeed, the definition of \( \epsilon \)-distance uses probabilities and exponentials; it is possible to formalize basic properties for these concepts, but more advanced reasoning, which is required for some examples, is beyond the abilities of SMT solvers. Instead, we introduce a probabilistic monad [33], with a type constructor of the form \( \mathbb{M}_{\epsilon, \delta}[\cdot] \) and two operators unit and bind with respective types:

\[
\begin{align*}
    \text{unit} &: T \to \mathbb{M}_{\epsilon, \delta}[T] \\
    \text{bind} &: \mathbb{M}_{\epsilon, \delta}[T] \to (T \to \mathbb{M}_{\epsilon, \delta}[U]) \to \mathbb{M}_{\epsilon+\delta, \delta}[U].
\end{align*}
\]

The main advantage of the monad versus the explicit formalization of \( \epsilon \)-distance is that all reasoning about probabilities is confined to the definition of valid judgment, and to the proof of soundness of the monadic rules. On the other hand, the refinement types \( T \) and \( U \) remain standard relational refinements, and do not need to refer to probabilities or exponentials.

The interpretation of \( \mathbb{M}_{\epsilon, \delta}[\cdot] \) is based on a lifting operator \( \mathbb{L}_{\epsilon, \delta}(\cdot) \) that turns a relation \( \mathbb{R} \) on \( T_1 \times T_2 \) into a relation \( \mathbb{L}_{\epsilon, \delta}(\mathbb{R}) \) on \( \mathbb{M}[T_1] \times \mathbb{M}[T_2] \): we will provide the formal definition in § 3.4.
A useful property of lifting [3] is that two distributions are related by \( \mathcal{L}, \mathcal{S} \) iff their \( \epsilon \)-distance is upper bounded by \( \delta \). In particular, \((\epsilon, \delta)\)-differentially private computations can be modeled by the relational refinement:

\[
\{x : T \mid x \in \Phi \} \to \exists \mathcal{S}, \mathcal{S}\{y : U \mid y_1 = y_2\},
\]

which leads to simpler verification conditions over individual outputs than sets of outputs.

Another advantage of this polymonadic approach is that the type of the bind operator captures the sequential composition theorem of differential privacy [18], and that it leads to an elegant type system in which quantitative reasoning related to differential privacy is confined to the rules of the polymonad. This is in contrast to some prior work where quantitative reasoning is pervasive in all rules of the system [3].

The last component of HOAr2 is a monad \( \mathcal{C}[\cdot] \) to model diverging computations. While this is quite standard, this approach is key to reconcile semantic subtyping, and refinements at higher types. We elaborate on this point next.

2.1 An aside on non-termination and refinement types

Exploiting the power of refinement types requires the ability to draw useful inferences from assertions. These inferences are typically represented in typing derivations via a subtyping relation \( \preceq \), and applied with a special typing rule, called subsupposition, that changes the type of an expression to an arbitrary supertype. Ideally, the subtyping relation should be complete w.r.t. the denotational semantics of the type system—a property known as semantic subtyping. However, extending most existing refinement type systems with semantic subtyping can lead to an inconsistency, because semantic subtyping conflicts with typical logical treatments of non-termination.

Inconsistency can arise both in non-relational and relational settings; let us consider the non-relational case. Here, inconsistency manifests itself as an expression of type \( \{x : N \mid \bot\} \) in the empty context, with the expression reducing to a value. Existing refinement type systems such as F7, F+ and LIQUIDHASKELL assign the type \( \{x : N \mid x \neq 0\} \to \{y : N \mid \bot\} \) to any recursive function \( f \) that does not terminate on values \( x \neq 0 \), such as for instance \( \text{letrec} \ g \ x = \text{case} \ x \ \text{with} \ {0 \Rightarrow 0 \ s \ y := g \ x} \). On the other hand, semantic subtyping validates the equivalence FUN-SUB:

\[
\{x : T \mid \phi\} \to \{y : U \mid \psi\} \equiv \{f : T \to U \mid \forall x : T. \phi \Rightarrow \psi[f x/y]\}
\]

where we write \( T \approx U \) iff \( T \preceq U \) and \( U \preceq T \). It follows that \( f \) has the problematic type:

\[
\begin{align*}
\vdash & f : \{x : N \mid x \neq 0\} \to \{y : N \mid \bot\} \\
\vdash & f : \{g : N \to N \mid \forall x : N. x \neq 0 \Rightarrow \bot\} \\
\vdash & f : \{g : N \to N \mid \forall x : N. x \neq 0 \Rightarrow \bot\} \\
\vdash & f : \{x : N \mid \bot\} \\
\vdash & 0 : N
\end{align*}
\]

where the first three inferences are by subsumption (the first two by FUN-SUB, and the second by the standard rule of consequence—replacing an assertion by a logically weaker one).

This example shows that a naive combination of semantic subtyping with refinements at higher types is inconsistent with the use of \( \bot \) to model diverging computation. We will follow a more semantically correct approach, by modeling non-termination as a (monadic) effect using a monad \( \mathcal{C}[\cdot] \) that distinguishes non-terminating and terminating computations. In this way, the inconsistency is avoided.

We conclude this discussion by noting that this counterexample is independent of the evaluation strategy. Accordingly, it complements the recent observation by Vazou et al. [52] that a logical interpretation of non-termination is unsound for call-by-name, even in languages without higher-order refinements. Vazou et al. [52] solve the issue by internalizing size types into refinement types to enforce termination.

3. The HOAr2 System

3.1 Expressions

HOAr2 is a relational type discipline for a \( \lambda \)-calculus with inductive types, unbounded recursion and monads for probabilities and partiality. For readability, we only present our calculus with some inductive types.

Let \( \mathcal{X} = \{x, y, \ldots\} \) be a countably infinite set of variables. The set \( \text{PCF}(\mathcal{X}) \) of expressions with variables in \( \mathcal{X} \) is defined as follows:

\[
e ::= \phantom{x} x \mid n \in \mathbb{N} \mid \alpha \in \mathbb{R}^+ \mid () \mid e \mid :: = \phantom{x} \text{false} \mid \text{true} \\
\mid e \mid \lambda x. \ e \mid \text{let} \ x = e \ \text{in} \ e \mid \text{letrec}_m \ f \ = \ e \\
\mid \text{if} \ e \ \text{then} \ e \ \text{else} \ e \mid \text{case} \ e \ \text{with} \ {x \Rightarrow e \mid \ldots}\end{align*}
\]

where \( m \in \{\cdot, \cdot\} \) and \( \mathbb{R}^+ \) stands for \( \mathbb{R}^+ \) augmented with the infinity \( \infty \). We write \( \text{PCF} \) for \( \text{PCF}(\mathcal{X}) \) when \( \mathcal{X} \) is clear from the context.

Most of the syntax is standard. The expressions \( \text{unit}_m \ e \) and \( \text{bind}_m \ x = e \ \text{in} \ e \) corresponds to the unit and the multiplication of the probabilistic monad. Similarly, \( \text{e}_1 \) and \( \text{let}_1 \ x = e \ \text{in} \ e \) corresponds to the unit and the multiplication of the partiality monad. Finally, we have two expressions for building recursive definitions: one for terminating programs, the other for non-terminating ones. We distinguish between them by means of the superscript \( m \in \{\cdot, \cdot\} \).

HOAr2 distinguishes between expressions and relational expressions. The former are used in the subject of typing judgments, and correspond to the actual programs to which we can assign semantics. The latter are used in assertions.

Definition 3.1 (Expressions and Relational Expressions). Let \( \mathcal{X}_R \) and \( \mathcal{X}_P \) be two disjoint countably infinite sets of relational and plain variables. Associated with every relational variable \( x \in \mathcal{X}_R \), we have a left instance \( x_{\leftarrow} \) and a right instance \( x_{\rightarrow} \). We write \( \mathcal{X}_R^\infty \) for \( \bigcup x \in \mathcal{X}_R \{x_{\leftarrow}, x_{\rightarrow}\} \) and \( \mathcal{X}^\infty \) for \( \mathcal{X}_R^\infty \cup \mathcal{X}_P \). The set of HOAr2 expressions \( \mathcal{E} \) is the set of expressions in \( \text{PCF}(\mathcal{X}_P) \). The set of HOAr2 relational expressions \( \mathcal{E}^\mathcal{R} \) is the set of expressions in \( \text{PCF}(\mathcal{X}^\infty) \), where only non-relational variables can be bound.

3.2 HOAr2 Types

We introduce the types of HOAr2 in two steps. First, we introduce simple types; for simplicity, we restrict instances of inductive types to base types. Then, we introduce relational refinement types, which express properties about two interpretations of an expression.

Definition 3.2 (Types). The sets \( \text{Ty} \) and \( \text{CoreTy} \) of (simple) types and core (simple) types are defined as follows:

\[
\begin{align*}
\tau, \sigma, \ldots & \in \text{Ty} := \tau \mid \mathcal{M} [\tau] \mid \mathcal{C}[\tau] \mid \tau \to \sigma \\
\tilde{\tau}, \tilde{\sigma}, \ldots & \in \text{CoreTy} := \bullet \mid \mathbb{B} \mid \mathbb{N} \mid \mathbb{R}^+ \mid \mathcal{R} \text{ list.}
\end{align*}
\]

The type \( \mathcal{M} [\tau] \) corresponds to the probability monad over the type \( \tau \), while the type \( \mathcal{C}[\tau] \) corresponds to the partiality monad over the type \( \tau \). Besides the standard function types \( \tau \to \sigma \), the type language includes the unit type, booleans, integers, reals and lists. Relational types extend the grammar of simple types with relational refinements, and use a dependent function type rather than standard function types.

Definition 3.3. The sets of relational types \( T = \{T, U, \ldots\} \) and assertions \( A = \{\phi, \psi, \ldots\} \) are defined as follows:
We define the simply typed layer of support for which Definition 3.4 interpretation of types mixes a set-theoretical and cpo semantics in The denotational semantics is largely sized types and syntactic criteria.

termination guard expression must pass a output type be in the partiality monad. On the contrary, the rule for a substitution constructors pairs inequalities over relational expressions. For instance, be relational (t for \Pi for T recursive. For the latter, the constructors constructors which must handle the relational expressions. For instance, "t \Pi \tau \rightarrow e \rightarrow \tau \sigma\theta" otherwise when \(d = [\Gamma \vdash e_1 : \tau]_\theta\)

\[\begin{align*}
\Gamma \vdash e_1 : \tau & = [\Gamma \vdash e : \tau]_\theta \\
\Gamma \vdash \text{let}_1 x = e_1 \text{ in } e_2 : [\Theta e_2 : \tau]_\theta \\
& = \begin{cases}
1 \text{ if } x = [\Theta e_0] & \text{0 otherwise}
\end{cases}
\end{align*}\]

Figure 1: Interpretation of PCF Expressions

continuous function space when the codomain is equipped with a cpo structure, and the set-theoretical function space otherwise.

Types \(\Theta \tau\) and \(\sigma \rightarrow \tau\) where \(\tau\) is interpreted as a cpo are interpreted as cpos. However, types of the form \(\Theta \tau\) are not interpreted as cpos, because their interpretation is based on discrete distributions.\(^2\)

We can now define the denotational interpretation of expressions. Definition 3.5. A valuation \(\theta\) is any finite map from \(\mathcal{X}\) to \(\bigcup_k \Theta \tau\). A valuation \(\theta\) validates an environment \(\Gamma\), written \(\theta \models \Gamma\), if \(\forall x \in \text{dom}(\Gamma), x\theta \in [\tau]\). We denote by \(\Gamma \vdash e : \tau\theta\) the interpretation of \(\Gamma \vdash e : \tau\) with respect to \(\theta \models \Gamma\).

The definition of the interpretation is mostly standard; Figure 1 gives the interpretation of the monadic constructions and of the two letrec operators. As expected, the static semantics is sound w.r.t. the denotational one.

Lemma 3.1. If \(\Gamma \vdash e : \tau\) and \(\theta \models \Gamma\), then \([\Theta e]_\theta \in [\tau]\).

3.4 Refinement Typing

The key point of relational typing is its ability to relate a pair of expressions—which we call the left and right expressions—via relational assertions that appear as refinements in types. For instance, the type

\[\Pi(x :: \mathcal{N}). \{y :: \mathcal{N} | y_{\Theta} = x_2 + 1 \land y_{\Theta} = x_0\}\]

represents a pair of integer to integer functions where the left function adds 1 to argument, and the right one returns its argument untouched.

In this section, we define the refinement type system of HOARe\(^2\) in three steps. First, we give an interpretation for assertions and refinement types. Second, we define a subtyping relation that is complete w.r.t. this interpretation. Finally, we define the refinement type system, and prove its soundness w.r.t. a denotational semantics. We start by defining relational contexts.

Definition 3.6. A relational environment \(G\) is any finite sequence of relational bindings \((x :: T)\) s.t. a variable is never bound twice and only variables of \(\mathcal{X}_R\) are bound. We use \(\emptyset\) to denote the empty environment. A relational environment defines a finite mapping from\(^2\) It would have been possible to interpret them as sub-distributions and to define another letrec operator for probabilistic computations, at the cost of replacing \(\delta + \delta'\) by \(\exp(\epsilon)\delta + \exp(\epsilon)\delta'\) in the typing rule for bind. However, our examples do not require this additional generality.
where $\overline{C}$ stands for the $C$-boolean operator.

Figure 2: Relational interpretation of assertions

variables to relational types; we write $x \vartriangleleft_G$ for the application of the finite map $\vartriangleleft_G$ to $x$.

We define a type erasure function $\lceil \cdot \rceil$ from relational to simple types, which maps dependent functions to function spaces, and erases refinements and the indexes of the probabilistic monad. The definition of $\lceil \cdot \rceil$ extends recursively to relational environments: $x \vartriangleleft_G = \lceil x \vartriangleleft_G \rceil$ for any $x \in \text{dom}(G)$. We also define the relational type erasure of $\vartriangleleft_G$, written $|G|$, by $x \vartriangleleft |G| = x \vartriangleleft_G$ iff $x \in \text{dom}(G)$, where $s \subseteq \{<,>,\}$. Note that a given a relational binding $(x : T)$, the relational type erasure $\{x : T\}$ gives the environment $(x_\omega : [T], x_\omega : [T])$.

Next, we interpret assertions and refinement types.

**Definition 3.7 (Relational interpretation of refinement types).** We say that a valuation $\theta$ validates a relational environment $G$, written $G \vdash \theta$, if $\theta \models |G|$ and $\forall x \in \text{dom}(G), (x_\omega, x_\phi, \theta) \in \{x \vartriangleleft_G \theta\}$.

Figures 2 and 3 define the relational interpretation $\{x \vartriangleleft_G \theta\}$ of a relational type $T$ w.r.t. a valuation $\theta : \Gamma \vdash (\forall \theta. \theta \models |G|)$. The existence of the normalization function immediately entails the probabilistic polymonad, and the underlying refinements. Other rules are mostly standard. The definition of subtyping validates the relational counterpart of the equivalence FUN-SUB discussed in § 2. More generally, it is possible to define a normalization function that converts any refinement type $T$ into an equivalent type $\{x : U \mid \phi\}$, where $U$ is a simple type, i.e., does not contain any refinement. The existence of the normalization function immediately entails semantic subtyping.

Finally, we present the HOARE$^2$ subtyping rules.

**Definition 3.10 (Relational Typing).** The refinement typing relation $G \vdash e : T$ is defined in Figure 5. We use $\Gamma \vdash e : T$ as a shorthand for $\Gamma \vdash e : \Gamma \vdash T$.

We briefly comment on some of the typing rules. As in relational Hoare logic [6], we distinguish between synchronous and asynchronous rules; the latter operate on both expressions of the
We have implemented a type-checker for HOARE². The type-checker generates proof obligations during type-checking; proof obligations are sent to SMT solvers via Why3. The type-checker uses a ML-like syntax and includes a few practical extensions like inductive datatypes, let expressions, as well as the ability to define logical predicates and core theories for the datatypes.

All the programs presented in § 5 and § 6, as well as some additional examples from the DP literature (private histograms, sums, two level counters and IDC) were automatically type checked by the implementation, with the only help of top-level type annotations. See Table 1 for a summary.

Both the type-checker and the Coq formalization are available at https://github.com/ejgallego/HOARE2/.

4. Embedding DFuzz

DFuzz [24] is a linear dependently typed language that has been used to verify many examples of differential private algorithms. In this section, we define a type-preserving embedding from DFuzz
We refer to Gaboardi et al. [24] for definitions and further explanations. The environment

is well typed in context

for every

if fresh in \( \Phi, \Gamma \)

\( \phi, \psi \) are constants

\( \phi, \psi \) are constants

\( R \) is a set of constraints used in pattern matching, and \( \Gamma \) is an environment containing assignments of the form \( x :! \mathbb{R} \). Figure 7 gives selected typing rules, where environments are combined by algebraic operations. The environment \( R \cdot \Gamma \) is obtained by taking \( x :! R \cdot \Gamma \) for every \( x :! R \cdot \Gamma \), in \( \Gamma \), while environment addition is defined as:

\[
(x :! R_1 \cdot \Gamma) + (x :! R_2 \cdot \Delta) = x :! (R_1 + R_2) \cdot \Gamma + \Delta \\
(x :! R \cdot \Gamma) + \Delta = x :! R \cdot (\Gamma + \Delta) \\
\Gamma + (x :! R_1 \cdot \Gamma) = x :! R_1 \cdot (\Gamma + \Gamma) \\
\Gamma + (x :! R_1 \cdot \Gamma) = x :! R_1 \cdot (\Gamma + \Gamma) \\
\]

We refer to Gaboardi et al. [24] for definitions and further explanation of the typing rules.

In \( \text{DFuzz} \), types, \( \sigma, \tau \) are interpreted as metric spaces, with associated metrics \( d_\sigma, d_\tau \). Then, the \( \text{DFuzz} \) type system enforces metric preservation [24, 48]: if \( \mathbb{R} \) is well typed in context \( \Gamma \), for arbitrary closing substitutions \( \theta_1, \theta_2 \) for \( \Gamma \), the distance between the interpretations of \( \theta_1(e) \) and \( \theta_2(e) \) is upper bounded by the distance between \( \theta_1 \) and \( \theta_2 \). As a particular instance, the \( \text{DFuzz} \) expressions of type \( ! \mathbb{R} \sigma \rightarrow \tau \) correspond to \( R \)-sensitive functions, i.e., functions \( f \) such that for every pair of inputs \( v_1 \) and \( v_2 \), \( d(f(v_1), f(v_2)) \leq R \cdot d_R(v_1, v_2) \). We will present an embedding which captures metric preservation as a relational refinement type.

To this end, we first define the multiplication operation on sensitivities in more detail. We distinguish two sorts: sensitivities \( d_\mathbb{R} = \mathbb{R}^{\geq 0} \cup \{0_\mathbb{R}, 1\} \) and distances \( d_\mathbb{R} = \mathbb{R}^{\geq 0} \cup \{0_\mathbb{R}, 1\} \). We interpret sensitivities \( R \) in \( \text{DFuzz} \) as sensitivities, while metrics (Figure 8) are interpreted as distances. We write \( s \) and \( d \) to range over the respective sorts. To interpret multiplication, we define a associative and commutative operator \( \odot \) that maps \( \mathbb{R}_d \times \mathbb{R}_d \rightarrow \mathbb{R}_d \) and \( \mathbb{R} \times \mathbb{R}_s \rightarrow \mathbb{R}_d \). The non-standard cases are those involving \( 0_\mathbb{R}, 0_d \) and \( \infty, \perp \):

\[
s \odot \infty = \infty \\
d \odot \infty = \infty \\
s \odot \infty = \infty \\
\perp \odot \perp = \perp \\
r \odot r' = r \cdot r' \\
\]
Theorem 4.1

If \( \phi : \Phi, \Gamma \vdash \tau \vdash D e : \mathfrak{M}[R] \) then

\[ \{ x :: \tau^* \mid \mathcal{D}(x_a, x_{c0}) \leq 1 \} \vdash \Phi, \Gamma \vdash e^* : \mathfrak{M}_0, o \{ y :: \mathbb{R} \mid y_a = y_{c0} \} \].

Proof. By Theorem 4.1.

\[ x :: \tau^* \vdash e^* : \{ y :: \mathfrak{M}_0, \Phi, \Gamma \} \mid \mathcal{D}(x_a, x_{c0}) \leq \mathcal{D}(x^*, x_{c0}) \]  

with \( \Gamma \equiv x :: ! \tau \). Let \( U = \{ x :: \tau^* \mid \mathcal{D}(x_a, x_{c0}) \leq 1 \} \). By definition of \( \mathcal{D} \) and elementary reasoning about probabilities,

\[ x :: U \vdash e^* : \{ y :: \mathfrak{M}_0, \Phi, \Gamma \} \mid \mathcal{D}(y_a, y_{c0}) \leq 0 \].

Finally, by semantic subtyping:

\[ x :: U \vdash e^* : \mathfrak{M}_0, o \{ y :: \mathbb{R} \mid y_a = y_{c0} \} \].

Moreover, Theorem 4.1 and Theorem 3.1 give a direct proof of metric preservation for \( \mathcal{D} \).

Theorem 4.2 (\( \mathcal{D} \) Metric Preservation [24]). If \( \phi : \Phi, \Gamma \vdash \tau \) and \( \theta \vdash \Phi, \gamma, \theta \vdash \Gamma \) then

\[ \mathcal{D}[\Gamma \gamma] P(e^* | \phi ; o \{ e^* \}, \theta) \leq \mathcal{D}[\Gamma \gamma] P(\theta \psi) \].

Notice that the above theorem uses three valuations. The valuation \( \theta \) is used for index variables which are equal in the two executions. The other two valuations \( \theta_1, \theta_2 \) are used to substitute related values in the two executions.

5. Differential Privacy

Theorem 4.1 establishes that every differentially private algorithm that can be modeled in \( \mathcal{D} \) is also captured by \( \text{HOAR}^\mathbb{R} \). In addition, we present a few previously unverified algorithms demonstrating the features of our system.

In what follows, we will use some notational shorthands. We write \texttt{let}/\texttt{munit}/\texttt{clet}/\texttt{cunit} for the bind/return operations of the probabilistic and partiality monad. We write \( R \) for the type \( \{ x :: \mathbb{R} \mid x_a \geq x_{c0} \} \). When a relational variable \( x \) is assumed to be equal in both runs (\( x_a = x_{c0} \)), we omit the projection and write \( x \) for both \( x_a \) and \( x_{c0} \).

5.1 Private Primitives

We review two differentially private mechanisms that are used in the next algorithms. The first mechanism is the \texttt{Laplace mechanism} [7], which releases a private version of a numeric value (which can differ in the two runs) by adding noise drawn from the Laplace distribution.

Formally, the \( \epsilon \)-private Laplace mechanism takes a real number \( x \) as input and returns \( x + \nu \), where \( \nu \) is random noise drawn from the Laplace distribution, which has density function

\[ F(\nu) = \frac{e}{2} \exp \left( -\frac{\epsilon |\nu|}{} \right) \).

If \( x \) can differ by at most \( s \) in adjacent runs, then the Laplace mechanism is \( (\epsilon, s, 0) \) differentially private. We model this as an operator \texttt{tap} with the following type:

\[ \Pi(x :: \mathbb{R}) \mid \mathfrak{M}_R(x_a, x_{c0}) \vdash \mathfrak{M}_R(x_a, x_{c0}) \]

and vice versa.

The other cases are similar. Interestingly, the case of pattern matching does not require asynchronous reasoning. Indeed, the refinement type of the translation of the term under match ensures that the two runs will take the same branch.

Hence typed \( \mathcal{D} \) expressions are differentially private.
then the Exponential mechanism satisfies $(\epsilon s, 0)$-differential privacy. We model this as an operator $\expMech$ of type
\[
\{(a :: A \mid a_0 \Phi a_0) \to \mathcal{G} \to \mathbb{R}_{\leq 0}\} \to \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}
\]
with the type $\mathcal{G}$ of score functions defined as
\[
\{b :: B \mid b_0 = b_0\} \to \{a :: A \mid a_0 \Phi a_0\} \rightarrow r : \mathbb{R}_{\geq 0} | [r_0 - r_0 | \leq s].
\]

5.2 Dual Query Release

We first focus on the problem of privately answering a large set of queries. The Laplace mechanism is a simple solution, but it's known that this will add noise to each query proportional to $\sqrt{k}$ for $k$ queries under $(\epsilon, \delta)$-privacy. When $k$ is large, the large noise will make the released answers completely useless. Fortunately, there is a line of algorithms where noise is added in a carefully correlated manner, guaranteeing privacy while adding noise proportional only to $\log k$. We have verified the privacy of one such algorithm, called DualQuery [25]. The algorithm is parameterized by a natural number $s$ and a set $q$ of queries to answer accurately. The input is the number of rounds $t$ and database $db$, and the output is a private synthetic database that is accurate for the given queries. The code of the algorithm is given below:

\[
\text{let rec dualquery t db = match t with}
| 0 & \rightarrow \text{munit [ ]}
| 1 + t' & \rightarrow \text{mlet curdb = dualquery t' db in}
\text{let quality = build_quality t' curdb in}
\text{mlet e = expmech db quality in}
\text{mlet new_qry = sampleN s e in}
\text{let newrecord = opt new_qry in}
\text{munit (newrecord :: curdb)}
\]

We encode the database as a list of natural numbers; adjacent databases are lists of the same length whose distance w.r.t. $D_{\text{list}}$ is smaller than 1. Here we consider $D_{\text{list}}$ to be defined similarly to the distance $D_{\text{list}[n]}$ for list of size $n$ defined in Figure 8 but where the $n$ is provided implicitly by the length of the lists. We represent the output of the mechanism as a list of selected records, each encoded as a natural number.

The algorithm performs $t$ steps, producing one record of the synthetic database in every round. For each round, we first build a quality score $quality$—a function from queries to real numbers—based on the previously produced records, using the auxiliary function $build_quality$. If we think of the current records as forming an approximate database, the quality score measures how poorly the approximation performs on each query. We then sample $s$ queries using the exponential mechanism with this quality score; queries with higher error are more likely to be selected. These queries are fed into an optimization function $opt$, which chooses the next record to add to the approximate database.

The only private operation is the exponential mechanism. The quality score we generate at each round $i$ has sensitivity $i$, and so a draw from the exponential mechanism is $i$-private. Since $i$ is upper bounded by $t$ and there are $s$ samples per round, the privacy cost per round is bounded by $s \cdot t \cdot e$. With $t$ rounds in total, the whole algorithm is $s \cdot t^2 \cdot e$-private. This guarantee is reflected in the type of dualquery:

\[
\{t :: N : t_0 = t_0\}
\rightarrow \{db :: N \text{list} \mid D_{\text{list}}(db_0, db_0) \leq 1\}
\rightarrow \mathbb{R}_{\leq 0} \rightarrow \{\{t :: N \text{list} \mid t_0 = t_0\}\}
\]

The state types that for two runs with adjacent databases, dualquery will return synthetic databases that are $s \cdot t^2 \cdot e$ apart, where $t$ is the number of iterations and $s$ is the number of samples used.

5.3 Private Counters and the Partiality Monad

Our second example is a private counter. The program takes in a list of real numbers, and releases a list of running counts. This algorithm is also known as the binary mechanism due to Chan et al. [11] and has not been verified before; previous verification work focused on the two-level counter from the same paper.

Suppose the input stream has length $T = 2^n$. The binary mechanism will return a list of noisy sums, reusing noise to reduce the improve the accuracy of the sums. The algorithm proceeds via branching recursion. In the base case, we add Laplace noise to the single element of the input stream and return. In the recursive case, we split the input stream into a first and a second half and perform the recursive call on each half; we then return the noise sum of the whole stream together with the result of the recursive calls. Each output list contains the sums for one “level” of the tree; the first list contains a single sum of length $2^n$, the next contains the two sums of length $2^n - 1$, and so on.

The algorithm terminates, but the simple guard condition implemented in our tool does not capture termination. Hence its formalization is based on the partiality monad and its associated $\text{cunit}$ and $\text{clet}$ operations.

\[
\text{let rec binary n ls = match ls with}
| [] & \rightarrow \text{cunit [ ]}
| x :: xs & \rightarrow \text{clet (cunit (mlet sum = lap x in \text{munit ([sum] :: [ ]}))}
| y :: ys & \rightarrow \text{let (left, right) = split l in}
\text{clet clefTN = binary (n - 1) left in}
\text{clet crightN = binary (n - 1) right in}
\text{cunit (mlet leftN = clefTN in mlet rightN = crightN in mlet sum = lap (sum l) in \text{munit ([sum] :: ([leftN ++ rightN]))}}
\]

The algorithm binary takes as input a natural number $n$ and a list $ls$ of reals with length $2^n$ and returns a list of lists of reals. Formally, binary has type
\[
\{n :: N : n_0 = n_0\}
\rightarrow \{t :: \text{list} \mid \text{sz}(l_0) = \text{sz}(l_0) = 2^n \land D_{\text{list}}(l_0, l_0) \leq k\}
\rightarrow C(\mathbb{R}_{\leq 0} (n + 1)) \{\{t :: \text{list} \text{list} \mid l_0 = l_0\}\}
\]

where we write $n$ for readability (since it is assumed equal in both runs) and where we use $D_{\text{list}}$ to the distance of lists at the type $\text{list}$ (defined analogously to $D_{\text{list}[n]}$).

6. Auctions and Algorithmic Game Theory

We now study the verification of mechanisms with incentive properties. We start by describing the truthfulness property for deterministic mechanisms, then we proceed to the randomized case. The closing examples illustrates the problem of computing an approximate Nash equilibrium using differential privacy.

6.1 Truthful auctions

In the digital goods setting, there is an infinite supply of identical goods to be sold in auction. For instance, when selling music downloads, goods can be reproduced for free. We assume every agent (or bidder) $i$ has a secret value $v_i$, which is the price she values the item, and submits a single bid $b_i$ to the mechanism. Once all bids have been submitted, the mechanism selects a set of winning bidders and prices $p_i$ for each winner. Bidders aim to maximize their utility, which is 0 if they do not win and $v_i - p_i$ if they win and at price $p_i$.

\footnote{It is of course possible to prove termination using known techniques, but we want to demonstrate the partiality monad.}
We want our mechanism to be truthful: given fixed bids of the other agents $b_{-i}$, the utility of agent $i$ is maximized when she bids her true valuation $b_i = v_i$. This feature makes bidding easy for bidders, and provides the algorithm designer with some assurance that she will see the correct inputs (the true values of the players).

From a verification point of view, truthfulness is a relational property of programs: if the mechanism maps bids to outcomes and all but one of the bidders bids the same in both runs, then the remaining bidder should have higher utility when bidding truthfully than when bidding non-truthfully.

We start with the fixed-price auction, a very simple mechanism for this setting. First, we pick a price $p$ (the reserve price). Bidders then submit their bids, and we select all bidders who bid above $p$ to be winners. Each winning bidder is charged price $p$.

Informally, this process is truthful: a bidder’s price does not depend on her own bid, so lowering her bid will never lower the price—it can only cause her to lose the item at a price that she would have wanted to pay. Similarly, increasing her bid above her value is never beneficial: if her truthful bid is winning, raising her bid does not change the outcome (she still wins, and pays the same price). If her truthful bid is losing, raising her bid can only cause her to win the item at a price that is higher than her value.

To model this with code, we will model a single bidder’s utility when she deviates. Note that each bidder is treated independently—her utility depends solely on her value, her bid, and the reserve price, and not on what any of the other bidders do. So, we can model this auction with the following function, which calculates a single bidder’s utility:

```haskell
let fixedprice b p = if b > p then v * p else 0
```

For clarity, we treat $v$ as a parameter declared in context with refinement type $\{v :: \mathbb{R} | v_0 = v\}$. Truthfulness of this auction follows from the type of `fixedprice`:

```haskell
\{b :: \mathbb{R} | b_0 = v\} \rightarrow \{p :: \mathbb{R} | p_0 = p\} \rightarrow \{u :: \mathbb{R} | u_0 \geq u\}.
```

The relational variable $b$ is required to be equal to $v$ in the first run, and arbitrary on the second run. Then, the final utility $u$ cannot be higher on the second run, demonstrating truthfulness.

This example also demonstrates a boolean version of the asynchronous typing rule $\text{ACase}$ from Figure 5. Since the bid $b$ is arbitrary in the second run, the two runs may take different branches. Indeed, these are the most interesting cases of the reasoning: If the same branch is taken in both runs, then the utility is the same in both runs (since the price is the same in both runs). When different branches are taken, we verify that truthfulness holds even when deviating from truthful bidding changes the outcome of the auction.

### 6.2 Universal Truthfulness and Randomized Mechanisms

While the fixed-price auction is very simple, it has poor revenue properties since the price is set independently of the bids. Setting it too high will lead to very few goods sold (and hence low revenue), and setting it too low may sell many goods, but at a price that is too high will lead to very few goods sold (and hence low revenue), however, picking the price as a function of the bids and setting it too low may sell many goods, but at a price that is too high will lead to very few goods sold (and hence low revenue).

To model the random sampling auction, we will treat as parameters the value $v : \mathbb{R}$ of the single deviating bidder $*$ and the bids of the other bidders $b : \mathbb{R}$ list; these are again assumed to the same on both runs. We define a (deterministic) utility function that takes the bid $b$ for the deviating bidder $*$, a coin $(\text{mygrp})$ indicating the group of $*$, and a list of coins $(\text{othergrp})$ indicating the groups of the other bidders. Then, utility for $*$ is computed using the fixed-price auction with reserve price from the bids in the other group; the optimal reserve price function is denoted by `optfixed`:

```haskell
let utility b (mygrp, othergrp) =
  let (g1, g2) = split (mygrp :: othergrp) (b :: bs) in
  if mygroup then fixedprice b (optfixed g1)
  else fixedprice b (optfixed g2)
```

Universal truthfulness can be seen from the type of utility:

```haskell
\{b :: \mathbb{R} | b_0 = v_0\} \rightarrow \{c :: \mathbb{B} \times \mathbb{R} \times \text{list} | c_0 = c_0\}
\rightarrow \{u :: \mathbb{R} | u_0 \geq u\}
```

The type shows that for any realization of the randomness, the utility is maximized by truthful reporting.

The main auction takes in the real-valued bid $b$ of $*$, draws the boolean indicating the groups, and uses the expectation operation to compute the expected utility of $*$ on this distribution:

```haskell
let auction b =
  mlet me = flip in
  mlet others = repeat N flip in
  let coins = munit (me, others) in
  E coins (utility b)
```

Above, `flip` returns a uniformly random boolean, and has type

```
\mathcal{M}_{0,0}([c :: \mathbb{B} | c_0 = c_0])
```

The repeat function is used to generate a list of $N$ random booleans (where $N + 1$ is the total number of bidders) that are then used to split the other bidders into two groups.

Truthfulness for the random sampling auction is reflected by the type for `auction`, which computes the expected utility of $*$:

```
\{b :: \mathbb{R} | b_0 = v_0\} \rightarrow \{u :: \mathbb{R} | u_0 \geq u\}
```

To verify the truthfulness of this auction we rely on monotonicity of expectation, as captured by the refinement type from § 2.

### 6.3 Nash Equilibrium via Differential Privacy

In this section, we move beyond auctions and consider the more general setting of *games*. A game is played by a collection of $N$ agents indexed by $i$, each with a set of possible actions $A_i$ (the *action space*). Given a vector of actions (one for each player) $a = (a_1, \ldots, a_N)$, each agent receives a (possibly randomized) payoff $P_i(a_1, \ldots, a_N)$; agents seek to maximize their (expected) payoff. For an example, auctions can be considered as games where each agent’s action space is the space of possible bids, and the payoff of each agent is their utility for the chosen outcome.

So far, we have considered mechanisms where one action (truthfully reporting) is a dominant strategy: a maximum payoff strategy no matter how the opponents play. In general games, like rock-paper scissors, dominant strategies usually do not exist. In this section, we consider a weaker solution concept: approximate Nash equilibrium.
where we present and verify the approximate Nash equilibrium property we want to prevent. Second, payoff functions may consist of sensitive information, and agents may be unwilling to reveal their payoffs to agents may prefer different equilibria among the Nash equilibria; agents may gain more than \(\alpha\) for the signal function and \(\dev\) that map recommended actions to actual actions. Agents also have a true payoff, \(\pay^*\), and \(\dev\), also considered as parameters since they are the same in both runs. The mechanism will not use these functions in the code; they are only referred to by the refinements.

The function \(\expay\) performs the following steps:

1. Use \(\mkSums\) to compute a noisy list \(\sums\) of signals, using the Laplace mechanism;
2. compute a signal \(s^*\) such that if both agents choose their (self-reported) best action \(b^*\) and \(b^*\) for signal \(s^*\), then the true signal based on strategy profiles \(a\) and \(a^*\) (defined next) is close to \(s^*\);
3. apply the deviation functions \(\dev\) and \(\dev^*\) to the recommended action \(b^*\) and \(b^*\) of each player to produce the strategy profile \(a\) and \(a^*\);
4. calculate the true payoff \(p^*\) for the deviating agent on the strategy profile;
5. compute the expectation of the payoff \(p^*\) for the deviating agent.

Players have two opportunities to deviate; they could misreport their best response function, or they could choose a deviation function to play differently than their recommendation. We want to show that reporting the true best response function and following the recommendation (i.e., using the identity function for deviation) is an approximate Nash equilibrium. As before, we perform the verification by assigning \(\expay\) a relational refinement type where \(\ast\) behaves truthfully in the left execution, while in the right execution \(\ast\) behaves arbitrarily. Assuming that \(\br^\ast\) is the true best response function corresponding to \(\pay^*\), and that \(\dev^\ast\) is the identity function, and that \(\br\) and \(\dev\) coincide on both runs (we thus omit subscripts), we want to prove (according to Definition 6.1):

\[
\expay \br^\ast, \dev^\ast \br \dev \geq \expay \br^\ast, \dev \br \dev - \alpha
\]

for some value \(\alpha\). We do this by checking that \(\expay\) has type:

\[
\begin{align*}
&\br^* :: \mathbb{R} \rightarrow A & \forall s, a, \pay^* (\br^* s) \geq \pay^* a \\
&\dev^* :: A \rightarrow A & \forall x, \dev^* x = x \\
&\br :: \mathbb{R} \rightarrow A & \br = \br^\ast \\
&\dev :: A \rightarrow A & \forall a, \dev a = \dev^\ast a = a \\
&\u :: \mathcal{R}^\ast & \u \geq \u^\ast - \alpha
\end{align*}
\]

We briefly comment on the two most interesting steps in the verification. First, when calculating \(s^*\), the algorithm computes \(\sums\) by adding Laplace noise to each induced best response: we want to ensure that agents have a limited influence on the chosen signal (and hence have limited incentive to misreport their best response strategy). For the accuracy guarantee we need to show that this noise is not too large (which is the case if the signal and payoff functions satisfy certain low-sensitivity conditions captured with refinement types). This is modeled by assigning to the Laplace mechanism \(\lap\) a refinement type capturing accuracy:

\[
\Pi(x :: A). \mathbb{M}_{|x_{\ast} - x_{\ast}|, \beta} [\{ u :: \mathbb{R} | u_{\ast} = u_{\ast} \land |x_{\ast} - u_{\ast}| < T \}],
\]

where \(T\) is defined as \(|x_{\ast} - x_{\ast}| + \frac{1}{2} \log \frac{2}{\beta} \). Informally, this states that the added noise is less than \(T\) with probability \(1 - \beta\).

The second interesting point is taking the expected value. The expression payoff \(p^*\) is randomized, and has type

\[
\mathbb{M}_{\alpha', \beta} [\{ u :: \mathbb{R}^+ | u_{\ast} \geq u_{\ast} - \alpha' \}]
\]
for some concrete values of $\epsilon'$, $\delta'$ and $\alpha'$. The payoff above is a probability distribution on real numbers, related by the lifted inequality relation. We wish to take the expected value of these distributions, in order to relate the expected payoff on the two runs. However, a priori, it is not clear how the expected values of these distributions are related. Fortunately, the expected values are related in a rather simple way, as seen in the following refinement for $E$:

$$
\begin{align*}
\forall \epsilon' > 0, \delta' > 0, \alpha' > 0 : \\
\{ x : \mathbb{R}^+ \mid x_0 \geq x_0 - \alpha' \} \\
\rightarrow \{ f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \mid f_{\epsilon} = f_{\epsilon'} = \text{id} \} \\
\rightarrow \{ u : \mathbb{R}^+ \mid u_0 \geq u_0 - \alpha' \},
\end{align*}
$$

where $\alpha'$ is an expression computed from $\epsilon'$, $\delta'$, and $\alpha'$. That is, taking expectation of two distributions related by the lifted inequality relation yields two real numbers that are approximately related by the unlifted, standard inequality relation on real numbers. Though not obvious, the soundness of this refinement can be derived from the definition of expectation and lifting.

From this refinement on the expected payoff for $*$ computed by expay, we conclude that truthful reporting and following the recommended action is an approximate Nash equilibrium.

7. Related Work

Our work lies at the intersection of differential privacy, mechanism design, probabilistic programming languages, and verification. We briefly comment on the first three areas (which are too enormous to be covered here) and elaborate on the most relevant work in program verification.

Differential privacy Differential privacy, first proposed by Blum et al. [7] and formally defined by Dwork et al. [20], has been an area of intensive research in the last decade. We have touched on a handful of private algorithms, including an algorithm for computing running sums [11] (part of a broader literature on streaming privacy), answering large classes of queries [25] (part of a broader literature on learning-theoretic approaches to data privacy). We refer readers interested in a more comprehensive treatment to the excellent surveys by Dwork [18, 19].

Mechanism Design Mechanism design was introduced to the theoretical computer science community (with a new focus on efficient implementations) by the seminal work of Nisan and Ronen [41]; see Nisan et al. [42] for a textbook introduction. It is understood that truthfulness guarantees can be difficult to prove and verify, so there is a literature giving generic reductions from mechanism design to algorithm design in limited settings but it is known that this is not possible in full generality [2, 12, 16, 32]. Differential privacy was first proposed as a tool in mechanism design by McSherry [36], and has since found many applications; see Pi and Roth [44] for a survey of this area.

Probabilistic programs There is a long line of work that develops models of probabilistic programs. The monadic representation of distributions originates from Giry [26] and was further developed in a programming language setting by later work [8, 34, 45, 47]. The connections with machine learning have recently triggered a surge of interest in probabilistic programming languages. We refer the reader to recent introductory articles [28, 29] for further information.

Verification of higher-order programs The refinement type discipline was introduced by Freeman and Pfennig [23], and further developed by others [14, 17, 55]. Advances in SMT solvers have allowed practical systems that support refinement types through SMT back-ends, for instance F7 [5], F* [51], and LIQUIDHASKELL [49]. Our work is mostly related to a recent variant of F* called RF* [4]. Like HOARE*, RF* supports relational reasoning of probabilistic computations. However, RF* lacks support for approximate relational refinement types and higher-order refinements, which are both critical for verifying differential privacy and game-theoretic properties.

8. Future Directions

HOAREx is an expressive system of relational refinement types that captures differential privacy, game-theoretic properties and other relational properties of probabilistic computations. An exciting direction for further work is to formally verify more complex mechanisms whose truthfulness guarantees are less standard. For example, it would be interesting to verify mechanisms that are merely truthful in expectation, for which universally truthful analogues do not exist (e.g., Dobzinski and Dughmi [15]). Such mechanisms use randomization in a crucial way for their incentive properties, and in addition to being interesting challenges for verification, are mechanism for which truthfulness is non-obvious. We also intend to develop a non-relational version of HOAREx for reasoning about the accuracy of probabilistic computations.

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