A linearization framework for unconstrained quadratic (0-1) problems

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Abstract

In this paper, we are interested in linearization techniques for the exact solution of the Unconstrained Quadratic (0-1) Problem. Our purpose is to propose “economical” linear formulations. We first extend current techniques in a general linearization framework containing many other schemes and propose a new linear formulation. Numerical results comparing classical, Glover’s and the new linearization are reported.

Keywords: Quadratic (0-1) problems; Linearization techniques; Polytopes

1. Introduction

The Quadratic 0-1 problem (QP) is considered to be one of the most challenging NP-hard \cite{19} combinatorial optimization problems. The problem (QP) can be stated as follows:

\[ \text{Min} \quad c^\top x + x^\top Q x \]
\[ \text{s.t.} \quad x \in \{0, 1\}^n \]

where \( c \in \mathbb{R}^n \) and \( Q = (q_{ij})_{(1 \leq i, j \leq n)} \) is an \( n \times n \) upper triangular matrix.

Boros and Hammer \cite{12} give some references to a large variety of applications such as statistical mechanics \cite{6}, clustering \cite{33}, project selection \cite{41} and graph theory \cite{18}.

Since the problem is NP-hard \cite{19}, many heuristics have been developed (see Beasley \cite{7}; Glover, Kochenberger and Alidaee \cite{21}; Alkhamis, Hassan and Ahmed \cite{3}; Lodi, Allemand and Liebling \cite{26}; Merz and Freisleben \cite{28}). Most of the exact methods are based on Branch-and-Bound schemes. The corresponding lower bounds can be roughly divided into four groups: Semidefinite Relaxations \cite{32, 25}, Lagrangean Decompositions \cite{13, 15} and...
Relaxations [1], Posiform Methods [8–10,12] and Linearization Techniques. Significant theoretical advances in cutting plane identification and generation (see Marchand, Martin, Weismantel and Wolsey [27]; Sherali and Driscoll [37] among others) have been achieved for this last group.

Linearization techniques transform the initial unconstrained quadratic problem into a constrained 0/1 linear program. The most used, and probably the most natural, linearization is the so-called “classical” (or “standard”) linearization reported below (CL). It was first proposed by Fortet [16,17], then expanded (independently) by Balas [4]; Zangwill [42] and Watters [40], and finally improved by Glover and Woolsey [22]. In the (CL) formulation below, the $z_{ij}$ variables replace the product terms $x_i x_j$, yielding the following formulation

$$
(CL) \quad \text{Min} \quad \sum_{i=1}^{n} \sum_{j=(i+1)}^{n} q_{ij} z_{ij} + \sum_{i=1}^{n} c_i x_i
$$

$$
s-t \quad (1) \quad z_{ij} \leq x_i \quad 1 \leq i < j \leq n
$$
$$
(2) \quad z_{ij} \leq x_j \quad 1 \leq i < j \leq n
$$
$$
(3) \quad z_{ij} \geq x_i + x_j - 1 \quad 1 \leq i < j \leq n
$$
$$
(4) \quad z_{ij} \geq 0 \quad 1 \leq i < j \leq n
$$
$$
(5) \quad x \in \{0, 1\}^n.
$$

The convex hull of feasible solutions of the problem (CL) is called the “Boolean Quadric Polytope” ($Q P_n$). It has been fully investigated in many contributions (see Padberg [31] or Boros and Hammer [11] for examples). Besides these works, many other valid inequalities or facets for ($Q P_n$) can be deduced from valid inequalities or facets of the so-called Cut Polytope ($C U T_n$) (see Deza and Laurent [14], De Simone [38,39]). Nevertheless, the number of variables ($O(n^2)$) and the large quantity of constraints increase the problem size. As a consequence, although the corresponding linearization technique (improved by valid inequalities of $Q P_n$ and $C U T_n$) provides excellent lower bounds, the whole method only allows one to solve medium-size problems (no more than 50 variables for full density matrices $Q$). Hence, as noted by Glover [20], the expected gain derived from dealing with a linear function is nullified as the problem size increases. Finding more “economical” models is therefore a condition for getting a tractable linearization approach. In this spirit, improved linear formulations have been proposed for non-linear (quadratic or polynomial) problems. A linearization involving a linear number of variables and constraints was first proposed by Glover [20] and improved by Oral and Kettani [30,29]. The resulting formulation, reported in Section 2, is a linearization involving only $(n - 1)$ additional variables and $2(n - 1)$ linear constraints. Adams and Glover [2] show how this linearization can be strengthened using a level 1 Reformulation Linearization Technique (RLT) (see Sherali and Adams [35]). This is done by dualizing three redundant quadratic equalities for which the optimal lagrangean multipliers may be obtained from any optimal dual solution of a level-1 RLT representation. In a previous paper [24], we proposed a method for transforming a given linearization scheme and finding another one yielding the same bound but with a smaller number of variables.

In this paper, we begin by pointing out that the classical linearization scheme and that of Glover have certain mathematical features in common. We then present a general framework for constructing linearizations, and show that it includes those two schemes as special cases. More importantly, our framework can be used to define new linearizations, inequalities and algorithms. To illustrate this point, we present and examine one new linearization in detail.

This paper is organized as follows: classical and Glover’s linearizations are briefly presented in Section 2; a general linearization framework is introduced in Section 3; one of these models is studied in Section 4; valid inequalities for this new model, corresponding to stage 2 of our linearization framework, are proposed in Section 5; numerical results are reported in Section 6 and future researches dealing with the general linearization framework are discussed in the conclusion.

2. Classical and Glover’s linearization

Let us consider the linearization (CL) above. To improve the model, many cuts have been found for the polytope ($Q P_n$) (see [31]). The main result is a tightened formulation $C L$ where inequalities (6) ((7) or (8)) and (9), called triangle and cycle inequalities, are added to the formulation. These inequalities induce facets of the Boolean Quadric.
Polynome.

\[
(CL) \quad \text{Min} \quad \sum_{i=1}^{n} \sum_{j=(i+1)}^{n} q_{ij}z_{ij} + \sum_{i=1}^{n} c_i x_i \\
\text{s.t} \quad (1), (2), (3), (4) \\
(5) \quad x \in [0, 1]^n \\
(6) \quad x_i - z_{ij} - z_{ik} + z_{jk} \geq 0, \quad 1 \leq i < j < k \leq n \\
(7) \quad x_j - z_{ij} + z_{ik} - z_{jk} \geq 0, \quad 1 \leq i < j < k \leq n \\
(8) \quad x_k + z_{ij} - z_{ik} - z_{jk} \geq 0, \quad 1 \leq i < j < k \leq n \\
(9) \quad 1 - x_i - x_j - x_k + z_{ij} + z_{ik} + z_{jk} \geq 0, \quad 1 \leq i < j < k \leq n.
\]

The lower bound corresponding to this formulation may be significantly enhanced with the very large set of valid inequalities of the cut polytope [14]. Some of these inequalities are hypermetric inequalities, defined as

\[
\sum_{j=1}^{n} \sum_{j=(i+1)}^{n} b_i b_j (x_i + x_j - 2z_{ij}) \leq 0 \quad \text{(with } \sum_{j=1}^{n} b_i = 1, b \in \mathbb{Z}^n),
\]

and clique-web inequalities (see [14]).

As noted above, Glover [20] gave a more economical formulation, improved later by Oral and Kettani [30,29]. Let us denote respectively by \( Q^+ = (q_{ij}^+) \) (resp. \( Q^- = (q_{ij}^-) \)) the positive (resp. the negative) part of the matrix \( Q \). Glover’s linearization stands as follows.

**Lemma 2.1.** Let \((GLL)\) be the following problem.

\[
(GLL) \quad \text{Min} \quad \sum_{i=1}^{n-1} t_i + \sum_{i=1}^{n} c_i x_i \\
\text{s.t} \quad (10) \quad t_i \geq \left( \sum_{j=(i+1)}^{n} q_{ij}^+ \right) x_i \quad 1 \leq i \leq n - 1 \\
(11) \quad t_i \geq \sum_{j=(i+1)}^{n} q_{ij}^- (1 - x_i) \quad 1 \leq i \leq n - 1 \\
(12) \quad x \in [0, 1]^n.
\]

We have \( V(QP) = V(GLL) \).

**Proof.** Glover [20]. \( \square \)

The main advantage of this formulation is that it only adds a linear number of variables and constraints. Nevertheless, in the above form, continuous relaxation of \((GLL)\) leads to poor lower bounds. Indeed, some numerical experiments performed in [23] on randomly generated instances give the average relative deviations (Column Gap) from the upper bound listed in Table 1.

Numerical experiments were executed with CPLEX Mip 7.0, on a Sun Ultra-Sparc machine with Ilog Concert/CPLEX technology. All data (i.e., the entries of \( Q \) and \( c \)) were randomly drawn in \([-100,100]\]. Ten problem sizes \((n = 10, 20, 30, 40, 50, 60, 80, 90, 100)\) and five densities of \( Q \) \((d = 20\%, 40\%, 60\%, 80\%, 100\%)\) were considered (since it is well known that linearization techniques give better performance for low density problems). For each problem type, 10 instances were generated. For each instance, an upper bound was computed by an evolutionary heuristic (see [23]). Glover’s formulation was improved by many other valid inequalities (see [23]) and used in a branch-and-bound scheme. The column “Gap” reports the average relative deviations, from the upper bounds, at the root of the branch-and-bound tree. We only report results for instances solved by the branch-and-bound scheme within a maximal processing time of 2 hours.

During our study, we observed that Glover’s and classical linearizations belong to the same general methodology that we are going to introduce and exploit to achieve improved results.
### Table 1

Lower bounds of the Glover’s linearization

<table>
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<tr>
<th>$n$</th>
<th>$d$</th>
<th>Gap (%)</th>
</tr>
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<td>10</td>
<td>20</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>40</td>
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<tr>
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<td>20</td>
<td>20</td>
<td>19.4</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>29.1</td>
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<tr>
<td></td>
<td>60</td>
<td>50.4</td>
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<td>20</td>
<td>51.6</td>
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</table>

### 3. A linearization framework

The main observation, justifying the linearization framework that we propose below, is the fact that classical and Glover’s linearizations may be interpreted as consisting of three steps. First, the objective function is decomposed into components. Second, an integer polytope is associated with each component and a complete linear description of that polytope is derived. Third, an additional set of linear constraints is added to link the components together.

For instance, the classical linearization introduces polytopes

$$(QP_n^{ij}) = \text{Co}\{(x, z_{ij}) \in \mathbb{R}^{n+1} | z_{ij} = x_ix_j, x \in \{0, 1\}^n\}$$

while the Glover’s linearization introduces

$$(QP_L_n^{ij}) = \text{Co}\{(x, t_i) \in \mathbb{R}^{n+1} | t_i = \sum_{j=(i+1)}^n q_{ij}x_ix_j, x \in \{0, 1\}^n\}$$

where “Co” stands for “Convex hull of”.

Each of these polytopes corresponds, in turn, to a matrix:

- $Q^{ij} = \begin{cases} Q_{kl}^{ij} = 1, & \text{if } k = i \text{ and } l = j \text{ (for } x_ix_j) \\ 0, & \text{otherwise} \end{cases}$
- $Q^i = \begin{cases} Q_{kl}^i = q_{il}, & \text{if } k = i \text{ (for } \sum_{j=(i+1)}^n q_{ij}x_ix_j) \\ 0, & \text{otherwise} \end{cases}$

Therefore, a linearization of a quadratic objective function may be defined as an aggregation of submatrices. Each submatrix induces a new variable and therefore a new polytope whose correct description and links with other polytopes are necessary conditions for getting a good lower bound in the following framework.
Linearization framework. Let \((QP)\) be the quadratic 0-1 problem:

\[
\begin{align*}
\text{(QP)} & \quad \min_{x \in \{0, 1\}^n} c^T x + x^T Q x \\
& \text{subject to} \quad x \in \{0, 1\}^n.
\end{align*}
\]

Let \(p \in \mathbb{N}, \alpha_i \in \mathbb{R}(1 \leq i \leq p)\) and \(Q_i \in \mathbb{R}^{n \times n}(1 \leq i \leq p)\) be matrices such that

\[
Q = \sum_{i=1}^{p} \alpha_i Q_i.
\]

1. Introduce \(p\) variables \(v_i\) representing quadratic expressions \(x^T Q_i x\). We obtain \(x^T Q x = \sum_{i=1}^{p} \alpha_i v_i\).

2. **(Stage 1)** Generate the inequalities (inequalities 1) describing the polytope associated to \(Q_i\) which are the following sets:

\[
\begin{align*}
\text{Co}(x, v_i) & \in \mathbb{R}^{n+1} \mid v_i \geq x^T Q_i x, x \in \{0, 1\}^n \quad 1 \leq i \leq p \text{ if } \alpha_i > 0 \\
\text{Co}(x, v_i) & \in \mathbb{R}^{n+1} \mid v_i \leq x^T Q_i x, x \in \{0, 1\}^n \quad 1 \leq i \leq p \text{ if } \alpha_i < 0
\end{align*}
\]

3. **(Stage 2)** Generate the inequalities (inequalities 2) linking the previous polytopes together, that is, the inequalities involving several variables \(v_i\).

4. Solve the problem:

\[
\begin{align*}
\text{(P}_{p, \alpha, Q}) & \quad \min_{x \in \{0, 1\}^n} c^T x + \sum_{i=1}^{p} \alpha_i v_i \\
& \text{subject to} \quad \text{inequalities 1} \\
& \quad \text{inequalities 2}
\end{align*}
\]

The classical linearization (respectively Glover’s linearization) is an instance of this framework in which \(Q_i\) are matrices \(Q_i^{ij}\) (respectively \(Q_i^j\)) described at the beginning of this section. The framework contains several other schemes still unexplored in the current literature. In practice, both sets of inequalities can be either found by some analytical process (lifting, etc...) or computed by algorithms (lift-and-project [5], etc.).

4. Cliques–Edges linearization

The vast number of linearization models allowed by the general framework and the modest results obtained by Glover’s linearization (GLL) are incentives to explore other schemes for better formulations. Given that stage 1 of the linearization framework consists of finding convex hulls, introducing a new formulation must take into account that such polytopes may be difficult to find. Therefore, the study of specific matrices for which the associated polytope can be addressed may have potential for future research.

The matrices which are of interest for this study are those whose support graphs are cliques and for which the edges are weighted by a constant \(\overline{q}\) (i.e the matrices \(Q\) such that \(Q_{ij} = \overline{q} \forall i \neq j\)). We first need the following basic definition.

**Definition 4.1.** We denote by \(G(V, E)\) the weighted support graph of the upper triangular matrix \(Q = (q_{ij}) (1 \leq i, j \leq n)\) where

\[
V = \{1, 2, \ldots, n\}
\]

and

\[
E = \{(i, j) \mid 1 \leq i < j \leq n, \ q_{ij} \neq 0\}.
\]

Each edge of \(E\) is weighted by \(q_{ij}\).

**Theorem 4.2.** Let \(Y\) and \(QP_{n}^{\Delta}\) be the following sets:

\[
Y = \left\{(x, t^{\Delta}) \in \mathbb{R}^{n+1} \mid t^{\Delta} \geq \sum_{i=1}^{m} \sum_{j=i+1}^{m} x_i x_j, x \in \{0, 1\}^n\right\}
\]

\[
QP_{n}^{\Delta} = \left\{(x, t^{\Delta}) \in \mathbb{R}^{n+1} \mid t^{\Delta} \geq -\frac{k(k-1)}{2} + (k-1) \sum_{i=1}^{m} x_i, x \in \{0, 1\}^n 2 \leq k \leq |\Delta|, k \in \mathbb{N}\right\}.
\]
We denote by $\Delta$ the clique of size $m$ which is the support graph of the quadratic expression induced by $Y$.

We have $QP_n^\Delta = \text{Co}(Y)$.

**Proof.** Rikun [34], Sherali [36]. $\square$

Based on the above results and linearization framework, we can introduce a new linearization scheme. The basic idea of this new scheme is to first decompose the matrix $Q$ into $Q^+$ and $Q^-$, where

$$q^+_{ij} = \max\{0, q_{ij}\} \ (1 \leq i < j \leq n) \quad \text{and} \quad q^-_{ij} = \min\{0, q_{ij}\} \ (1 \leq i < j \leq n).$$

Hence, $Q = Q^- + Q^+$.

To linearize $Q^-$, we can introduce variables $z_{ij}$, as in classical linearization. The new scheme is then obtained by decomposing $Q^+$ into submatrices $Q^+_{\Delta_i} (i = 1, \ldots, p)$. The support graphs of the matrices $Q^+_{\Delta_i}$ correspond to several cliques $\Delta_i (i = 1, \ldots, p)$ of $G^+ = (V^+, E^+)$ which is the support graph of $Q^+$. More precisely, these cliques are chosen to be a cover of $E^+$ (i.e., each edge of $E^+$ is in at least one of the cliques). Such a cover of $E^+$ can be determined by the following algorithm:

**Edge-covering algorithm**

$i = 1$
while ($Q^+$ is not the null matrix) {  
  - Compute by a greedy algorithm a clique $\Delta_i$, of the support graph of $Q^+$, of maximum size  
  - Let $M_{\Delta_i} = \min\{Q^+_{kl} / k \in \Delta_i, l \in \Delta_i\}$  
  - $Q^+_{kl} = Q^+_{kl} - M_{\Delta_i}, \text{if } k \in \Delta_i \text{ and } l \in \Delta_i$
}

When this algorithm stops, we have $Q = Q^- + \sum_{i=1}^p M_{\Delta_i} Q^+_{\Delta_i}$ where the matrices $Q^+_{\Delta_i}$ are defined by:

$$(Q^+_{\Delta_i})_{kl} = \begin{cases} 
1 & \text{if } k \in \Delta_i \text{ and } l \in \Delta_i \\
0 & \text{otherwise.}
\end{cases}$$

By introducing a variable $t_{\Delta_i}$ for each of the $Q^+_{\Delta_i}$, we obtain the Cliques–Edges Linearization.

**Cliques-edges formulation**

\begin{align*}
\text{(CLEF) Min} \quad & \sum_{i=1}^n c_i x_i + \sum_{i=1}^n \sum_{j=(i+1)}^n q^-_{ij} z_{ij} + \sum_{i=1}^p M_{\Delta_i} t_{\Delta_i} \\
\text{s-t} \quad & (13) \ (x, z) \in \text{Co}(x, z_{ij}) \in \mathbb{R}^{n+1} \mid x \in \{0, 1\}^n, z_{ij} = x_i x_j, 1 \leq i < j \leq n \\
& (14) \ (x, t_{\Delta_i}) \in QP_n^\Delta, \quad 1 \leq i \leq p, x \in \{0, 1\}^n
\end{align*}

Facets of the polytope in (13) correspond to constraints (1), (2), (3), (4), (5), and facets of the polytope in (14) are described by using the Theorem 4.2. Thus, all the facets describing the above polytopes are known and stage 1 of the linearization framework is achieved. We still have to determine (in stage 2 of the linearization framework) instances of linking inequalities. In the next section, we report 3 inequality families linking together instances of $t_{\Delta_i}$ and $z_{ij}$.

5. **Linking inequalities of Cliques–Edges linearization**

In this section, we present three lemmata defining linking inequalities for the Cliques–Edges linearization scheme. We first extend the Boolean Quadric Polytope triangle inequalities to valid inequalities of $QP_n^\Delta$ (Lemma 5.1). We denote here by $\Delta$ a clique associated to a variable $t_{\Delta}$ of the (CLEF) linearization.

The underlying ideas of the next 2 lemmata (Lemmas 5.3 and 5.5) follow. In the (CLEF) linearization, a set of cliques covers the support graph and each associated polytope is completely described. The support graph contains many other cliques and simple structures (like cycles) not described by the linearization variables. Hence, if these structures are covered by a variable combination, some valid inequalities describing the associated polytopes may be found.
5.1. Triangle cuts

**Lemma 5.1.** Let $\Delta$ be a clique of the Cliques–Edges linearization and $j \in \bar{\Delta}$ (complementary of $\Delta$ in $V$). We denote $I = \{i \in \Delta \mid q_{ij} < 0\}$.

The following inequalities are valid

$$t_{\Delta} \geq (k - 1) \sum_{i \in I} z_{ij} - \frac{k(k - 1)}{2} x_j$$

$$t_{\Delta} \geq (k - 1) \sum_{i \in \Delta} x_i - \frac{k(k - 1)}{2} - (k - 1) \sum_{i \in I} z_{ij} + \frac{k(k - 1)}{2} x_j \quad 2 \leq k \leq |\Delta|.$$ 

**Proof.** Let $\Delta$ be a clique of the cliques-edges linearization. From Theorem 4.2 we have

$$t_{\Delta} \geq (k - 1) \sum_{i \in \Delta} x_i - \frac{k(k - 1)}{2} (2 \leq k \leq |\Delta|).$$

Let $j \in \bar{\Delta}$. Multiplying the above inequality by $x_j$ yields:

$$x_j t_{\Delta} \geq (k - 1) \sum_{i \in \Delta} x_i x_j - \frac{k(k - 1)}{2} x_j.$$ 

Given that variables $z_{ij}$ of the Cliques–Edges Formulation are the products $x_i x_j$ (when $q_{ij} < 0$), we have $\sum_{i \in \Delta} x_i x_j \geq \sum_{i \in I} z_{ij}$.

Now, $x_j \leq 1$ implies that the following inequality is valid:

$$t_{\Delta} \geq (k - 1) \sum_{i \in I} z_{ij} - \frac{k(k - 1)}{2} x_j.$$ 

By multiplying the previous clique facet by $1 - x_j$, we can also conclude that the inequality

$$t_{\Delta} \geq (k - 1) \sum_{i \in \Delta} x_i - \frac{k(k - 1)}{2} - (k - 1) \sum_{i \in I} z_{ij} + \frac{k(k - 1)}{2} x_j$$

is valid. □

5.2. Clique cuts

In order to present valid inequalities (clique cuts) in Lemma 5.3, we first need a definition.

**Definition 5.2.** We denote:

$$m_h = \begin{cases} z_h & \text{if } h \in \{(i, j) \mid q_{ij} < 0, \ 1 \leq i < j \leq n\} \\ t_h & \text{if } h \in \Delta_1, \Delta_2, \ldots, \Delta_p \end{cases}$$

where $\Delta_1, \Delta_2, \ldots, \Delta_p$ are the cliques obtained by the edge-covering algorithm.

Note that this definition will be also used in the Lemma 5.5 of the next subsection.

**Lemma 5.3.** Let $\Delta$ be a clique of the $Q$ support graph (G(V, E)) and $R(\Delta)$ an edge-covering of $\Delta$ by some cliques of the (CLEF) linearization (i.e. cliques $\Delta_i$ (1 \leq i \leq p) and edges (i, j) of $Q^-$ support graph).

The following inequalities are valid:

$$\sum_{h \in R(\Delta)} m_h \geq (k - 1) \sum_{i \in \Delta} x_i - \frac{k(k - 1)}{2}, \quad 2 \leq k \leq |\Delta|.$$
Proof. According to Theorem 4.2, introducing a variable $v_\Delta$ for the clique $\Delta$ would lead to the following facets:

$$v_\Delta \geq (k - 1) \sum_{i \in \Delta} x_i - \frac{k(k - 1)}{2} \quad 2 \leq k \leq |\Delta|.$$  

Given that $R(\Delta)$ is a cover of $\Delta$, we have: $\sum_{h \in R(\Delta)} m_h \geq v_\Delta$. Hence, the following inequalities are valid:

$$\sum_{h \in R(\Delta)} m_h \geq (k - 1) \sum_{i \in \Delta} x_i - \frac{k(k - 1)}{2} \quad (2 \geq k \geq |\Delta|). \quad \Box$$

Note that there exists an exponential number of such inequalities since the support graph has an exponential number of cliques. Therefore, the associated separation problem is difficult (NP-hard) and, in the cut generation process, a heuristic has to be used (see [23]).

5.3. Cycle cuts

In order to define cycle cuts (Lemma 5.5), we first need to find some valid inequalities for the quadratic expressions associated with cycles. In the theorem below, we consider quadratic expressions whose support graphs are cycles with edge weights all equal to 1. More formally, we are interested in the set $Z$ of integer points for which we try to describe the convex hull.

$$Z = \{(x, t) \in \mathbb{R}^{n+1} | t \geq x_1 x_2 + x_2 x_3 + \cdots + x_n x_1, x \in \{0, 1\}^n\}.$$  

We propose some valid inequalities in the following theorem of Co($Z$).

**Theorem 5.4.** Let $Z$ be the following set

$$Z = \{(x, t) \in \mathbb{R}^{n+1} | t \geq x_1 x_2 + x_2 x_3 + \cdots + x_n x_1, x \in \{0, 1\}^n\}.$$  

The following inequalities are valid for Co($Z$)

(i) $t \geq \sum_{i=1}^{n-1} \lambda_i(x_i + x_{i+1} - 1) + \lambda_n(x_n + x_1 - 1) \quad \lambda \in \{0, 1\}^n$

(ii) $t \geq \sum_{i=1}^{n-1} x_i - \frac{n-1}{2}$ when $n$ is odd.

**Proof.** Let us consider separately inequalities (i) and (ii).

(i) Let $\lambda \in \{0, 1\}^n$. To show that the inequality $t \geq \sum_{i=1}^{n-1} \lambda_i(x_i + x_{i+1} - 1) + \lambda_n(x_n + x_1 - 1)$ is valid, it is sufficient to prove that it is satisfied by any point of $Z$. Thus, let $(x, t) \in Z$. As $\lambda \in \{0, 1\}^n$, we have

$$t \geq x_1 x_2 + x_2 x_3 + \cdots + x_n x_1 = \sum_{i=1}^{n-1} x_i x_{i+1} + x_n x_1 \geq \sum_{i=1}^{n-1} \lambda_i x_i x_{i+1} + \lambda_n x_n x_1.$$  

Moreover, given that $x$ is a binary vector, we also have

$$x_i x_{i+1} \geq x_i + x_{i+1} - 1 \quad (1 \leq i \leq n - 1) \quad \text{and} \quad x_n x_1 \geq x_n + x_1 - 1.$$  

Hence, it follows that $t \geq \sum_{i=1}^{n-1} \lambda_i(x_i + x_{i+1} - 1) + \lambda_n(x_n + x_1 - 1)$

(ii) Let $(x, t) \in Z$. There are two cases.

• $x_n = 0$

  By first considering that $t \geq x_1 x_2 + x_2 x_3 + \cdots + x_n x_1 \geq \sum_{i=1}^{n-1} x_i x_{i+1}$ and that $x_i x_{i+1} \geq x_i + x_{i+1} - 1$ ($1 \leq i \leq n - 1, i \text{ odd}$), we can write

  $$t \geq (x_1 + x_2 - 1) + (x_3 + x_4 - 1) + \cdots + (x_{n-2} + x_{n-1} - 1).$$
Table 2
Numerical results of the linearization (CLEF)

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<th>Gap$_2$ (%)</th>
<th>ncuts</th>
<th>$T_{lb}$ (s)</th>
<th>Nodes</th>
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where, $n$ being odd, there are $\frac{n(n-1)}{2}$ odd indices between 1 and $n - 1$.

Thus, we have $t \geq \sum_{i=1}^{n} x_i - \frac{n-1}{2}$.

• $x_n = 1$

As in the previous case, we have

$t \geq x_1 x_2 + x_2 x_3 + \cdots + x_n x_1 \geq \sum_{i=1, i \text{ even}}^{n-1} x_i x_{i+1} + x_1$.

Moreover, given that $x_i x_{i+1} \geq x_i + x_{i+1} - 1$ (2 $\leq i \leq n - 1$, $i$ even), we can write:

$t \geq (x_2 + x_3 - 1) + (x_4 + x_5 - 1) + \cdots + (x_{n-1} + x_n - 1) + x_1$.

Now, taking into account that $n$ is odd, we know that there exists $\frac{n(n-1)}{2}$ even indices between 1 and $n - 1$. Thus, we have $t \geq \sum_{i=1}^{n} x_i - \frac{n-1}{2}$. 

}\end{verbatim}
Table 3
Numerical results of the linearization (CL)

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<th>Gap₂ (%)</th>
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It follows that \( t \geq \sum_{i=1}^{n} x_i - \frac{n-1}{2} \) is a valid inequality \( \Box \)

Based on this theorem, we propose, in the following lemma, valid inequalities for (CLEF) linearization.

**Lemma 5.5.** Let \( C \) be an odd length cycle of the support graph \( G(V, E) \) and \( R(C) \) an edge-covering of \( C \) by some cliques of the (CLEF) linearization (i.e. cliques \( \Delta_i \) \( 1 \leq i \leq p \) and edges \( (i, j) \) of \( Q \) support graph).

The following inequality is valid

\[
\sum_{h \in R(C)} m_h \geq \sum_{i \in C} x_i - \frac{|C| - 1}{2}.
\]

**Proof.** By using Theorem 5.4, the proof is similar to that of Lemma 5.3. \( \Box \)

### 6. Numerical tests

In order to evaluate the quality of Cliques–Edges linearization (CLEF) and compare it to classical linearization (CL) and Glover’s, numerical experiments were performed with the same instances used for Glover’s linearization in Section 2. The same machine was also used. Our aim was to evaluate the lower bound performances in a commercial Branch-and-Bound scheme (CPLEX Mip 7.0).

Tables 2 and 3 report, respectively, the performance of the (CLEF) linearization and the performance of classical linearization. For each table, the first 2 columns give the number of variables and the density of the problem. The third
one gives the number of additional variables induced by linearization. The fourth one, the average relative deviations from the upper bounds, obtained before generating the linking inequalities (valid inequalities of Section 5) and the next column indicates the average relative deviations taking into account linking inequalities. The upper bounds were obtained by the evolutionary heuristic used for Table 1. The last 4 columns report the number of generated cuts, the time spent to compute the lower bound with the linking inequalities, the number of nodes in the branch-and-bound and the whole CPU time for solving the problem.

From these experiments, it appears that the Cliques–Edges linearization is competitive, at least with respect to classical and Glover’s linearizations.

7. Conclusion and further developments

The general framework introduced in this paper yields a new interpretation of existing linearization techniques and paves the way for new ones. For instance, linearization schemes involving cycles, forests and trees (or other well-studied structures) could be investigated in order to improve (CLEF) results. Compared with other techniques for solving unconstrained 0-1 Quadratic programs, especially with Semidefinite Programming, Linearization Techniques appear to be less efficient, since Semidefinite Programming allows one to solve larger instances. Nevertheless, advances in commercial linear programming packages, as well as the introduction of new linearization schemes, might make the linearization techniques more efficient for solving unconstrained 0-1 quadratic problems.

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References