Formal power series, operator calculus, and duality on Lie algebras

Philip Feinsilver\textsuperscript{a,*}, René Schott\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, USA
\textsuperscript{b} CRIN-CNRS, Université Henri Poincaré, 54506 Vandoeuvre-lès-Nancy, France

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Abstract

This paper presents an operator calculus approach to computing with non-commutative variables. First, we recall the product formulation of formal exponential series. Then we show how to formulate canonical boson calculus on formal series. This calculus is used to represent the action of a Lie algebra on its universal enveloping algebra. As applications, Hamilton’s equations for a general Hamiltonian, given as a formal series, are found using a double-dual representation, and a formulation of the exponential of the adjoint representation is given. With these techniques one can represent the Volterra product acting on the enveloping algebra. We illustrate with a three-step nilpotent Lie algebra.

1. Introduction

Lie algebras and Lie groups play an increasingly prominent rôle in many applications of mathematics, notably in areas such as computer science and control theory. Computations with Lie algebras lead to formal power series in non-commutative variables. Applications in computer science range from theoretical questions in computing [1,6,27] to algorithm analysis [10] and to practical situations such as robotic manipulation [23]. Connections with probability theory are given in [9]. It is out of the scope of this paper to mention the numerous applications in control theory. We just list some of the prominent works by Fliess [12–16], Hermes [19], Jakubczyk and Sontag [20], and Sussmann [24–26].

In this paper we present an operator calculus approach to these problems, particularly in the Lie context based on representations on the universal enveloping algebra. The paper is organized as follows. Section 2 recalls the product formulation of Volterra series. In Section 3, we consider the case of abelian (commuting) increments. Next, we formulate the boson calculus on formal series. Section 5 shows how to
represent the action of a Lie algebra on its universal enveloping algebra in terms of canonical boson operators. As applications, Hamilton’s equations for a general Hamiltonian, given as a formal series of monomials in the enveloping algebra, are found in terms of the double-dual, then it is shown how to compute the exponential of the adjoint representation. Of particular note is the fact that the calculations can be done using matrix realizations and thus can be readily implemented using a package such as MAPLE. We remark that one can use the double-dual for representing the Volterra product acting on the enveloping algebra. Section 6 illustrates these methods with some calculations for a three-step nilpotent algebra.

2. Products and iterated sums

Volterra series have been used in many areas including control theory. For example, Fliess [12], Crouch and Irving [5] used it for representing solutions of evolution equations in nonlinear control theory. Discrete Volterra series have been studied since the time of Volterra as a basic construction. Recently, a fairly complete exposition has been given by Gill and Johansen, and Gill [18,17] indicating the basic convergence theorems and showing applications to some statistical problems.

Here we indicate the basic ideas. For a given sequence \((X_1, X_2, \ldots, X_i, \ldots)\) the expression

\[
E = E(N, v) = \prod_{i=1}^{N} (1 + vX_i)
\]

with \(E(0, v) = 1\), is a discrete version of an exponential function. The simplest case \(X_i = X\) for all \(i\) gives \(E = (1 + vX)^N\) and replacing \(X\) with \(X/N\) yields \(e^{vx}\) in the limit \(N \rightarrow \infty\). The formulation as a product is interesting in that \(E\), by construction, satisfies the difference equation

\[
E(N, v) - E(N - 1, v) = vE(N - 1, v)X^n.
\]  

(1)

This shows that we think of the \(X_i\) as increments of some discrete process or as differentials \(dX\) of a deterministic or stochastic process. Compare \(E = e^{vx}\), \(dE = vE dX\).

The expansion

\[
E(N, v) = \sum_{n=0}^{N} v^n I_n(X_1, \ldots, X_N) = \sum_{n=0}^{N} v^n I_n(N)
\]

and the difference relation (1) yields

\[
I_n(N) - I_n(N - 1) = I_{n-1}(N - 1)X_N,
\]

i.e., the \(I\)'s are iterated sums of the variables \(X_1, X_2, \ldots\)

\[
I_n(N) = \sum_{j=0}^{N} I_{n-1}(j - 1)X_j
\]
or, in the language of the continuous version, \textit{iterated integrals}. The expansion of functionals of a process as a series of iterated integrals (of the process) is a Volterra series. Thus, we have here discrete Volterra series. And it is of interest to study the behavior as \( N \to \infty \) while the \( X_i \) approximate differentials of a given process. An important feature is that one can handle non-time-homogeneous processes and non-commutative variables, e.g., matrix-valued processes.

3. Abelian processes

First we consider finite-state systems. We take the \( X_i \) to be commutative variables taking values in a fixed finite set. We are interested in the discrete model and behavior as \( N \to \infty \). The case of a single value leads to the usual exponential function as noted above.

If \( X_i \) takes two values \( \{\alpha, \beta\} \), then you can write

\[
(1 + vX_i) = (1 + \alpha v)^{(X - \beta)/(\alpha - \beta)} \left(1 + \beta v\right)^{(X - N\beta)/(\beta - \alpha)}.
\]

If \( X_i \) takes values \( \{\alpha_1, \ldots, \alpha_r\} \), use the Lagrange interpolation formula as follows. Write

\[
p(X) = \prod (X - \alpha_i).
\]

Denote by \( n_i \) the number of times the value \( \alpha_i \) is taken, \( \sum n_i = N \). Observe that

\[
\sum_i \frac{p(X_j)}{p'(^i)(X_j - \alpha_i)} = 1,
\]

while

\[
n_i = \sum_j \frac{p(X_j)}{p'(^i)(X_j - \alpha_i)}.
\]

Thus,

\[
\prod (1 + vX_j) = \prod \left(1 + \alpha_i v\right)^{\sum_j (p(X_j))/(p'(^i)(X_j - \alpha_i))}.
\]

If \( X_i \) take three values, \( \{\alpha, \beta, \gamma\} \), then, e.g.,

\[
n_\gamma = \frac{\sum X_i^2 - (\alpha + \beta) \sum X_i + \alpha \beta N}{\gamma^2 - (\alpha + \beta) \gamma + \alpha \beta},
\]

which depends on the power sums \( Y = \sum X_i^2 \) and \( X = \sum X_i \).

In general, we can write, \( a \) denoting the values taken by the \( X_i \),

\[
\prod (1 + vX_j) = \prod (1 + va)^{\sum_a \delta(X_j - a)} = \prod (1 + va)^{n_a} = e^{\sum_a n_a \log(1 + va)} = e^N \sum (n_a/N) \log(1 + va) \sim e^N \int \log(1 + va) f(v) dv,
\]

where \( f \) is the density function for the distribution of \( X_j \).
Another approach is to use power sums:

$$\prod (1 + vX_j) = e^{\sum \log(1 + vX_j)} = e^{\sum_i \sum_{j} (-1)^{j} X_{i}^{j}/j}.$$  

This form is convenient for evaluating asymptotic behavior.

**Remark 3.1.** This formulation is already interesting in the case of independent random variables, where it gives a class of basic orthogonal functionals of the process, see [8]. For general semimartingales, it gives the exponential martingale of Doléans–Dade (see [17]).

### 4. Operator calculus and formal series

Given a finite number of non-commuting indeterminates \{\xi_1, \ldots, \xi_d\}, one can consider formal series in monomials they generate. Assuming that multiplication is associative and linear with respect to an underlying set of scalars (possibly a commutative ring) one effectively has the tensor algebra. We consider the case where monomials of the form (\(n\) denoting the multi-index \((n_1, n_2, \ldots, n_d)\), \(n_i \geq 0\))

$$\psi[n] = \psi[n_1, n_2, \ldots, n_d] = \xi_1^{n_1} \cdots \xi_d^{n_d}$$

are the basis for the associative algebra generated by the \(\xi_i\), such as the case where they generate a finite-dimensional Lie algebra, according to the Poincaré–Birkhoff–Witt theorem (cf. [6]). A formal series of interest is of the form

$$\sum_n c[n] \psi[n].$$

Denote the basic multi-index having a single 1 in position \(i\) and zeros elsewhere by \(e_i\), so that \(n = \sum n_i e_i\).

The operator calculus on these series is given on the basis \(\psi[n]\) by the **boson operators**, which we denote by \(\mathcal{R}_i\) and \(\mathcal{V}_i\):

\[
\mathcal{R}_i \psi[n] = \psi[n + e_i], \quad \mathcal{V}_i \psi[n] = n_i \psi[n - e_i].
\]

The vector space, say over \(C\), generated by the action of the operators \(\mathcal{R}_i\) acting on \(\psi[0]\) is called in physics terminology the **boson Fock space** (usually considered in the case where there are a countably infinite number of variables). These operators satisfy the commutation relations

$$[\mathcal{V}_i, \mathcal{R}_j] = \delta_{ij} I,$$

where \(I\) denotes the identity operator. The idea is to use these operators to represent the action of left (or right) multiplication, in the associative algebra, by the basis elements \(\xi_i\), and hence to write the algebra in terms of these operators acting on \(\psi[0]\), often denoted by \(\Omega\), and called the vacuum state.
A basic fact is that any matrix Lie algebra has a boson realization in terms of the Jordan map, namely, we have the Lie isomorphism

\[ A = (A_{ij}) \leftrightarrow \sum_{\alpha, \beta} \mathcal{R}_\alpha A_{\alpha \beta} \gamma^\beta \]

as is readily verified. Another way to interpret this is to use the natural correspondence

\[ \mathcal{R}_i \leftrightarrow X_i, \quad \gamma^i \leftrightarrow \frac{\partial}{\partial X_i} \]
acting on smooth functions \( f(x_1, \ldots, x_d) \) with \( X_i f(x_1, \ldots, x_d) = x_i f(x_1, \ldots, x_d) \). In this case, the Jordan map gives a realization of matrix algebra as an algebra of vector fields.

For finite-dimensional Lie algebras, using duality for the universal enveloping algebra, one can compute representations for the algebra. This is explained in detail in the next section. (One can find representations of quotients of the enveloping algebra and of the group as well, see [7].)

**Remark 4.1.** One can consider models using the fermion Fock space as well. This can be based on the binomial model where as in Sections 2, 3 above the \( X_i \) take just two values 0 and 1. See Meyer [22] for calculations with fermions.

5. Dual representations

The 'splitting technique' which is basic to the approach was developed from a different point of view by Wei and Norman [28,29] in the sixties. Here we start from a choice of basis for the Lie algebra considered as generators for the universal enveloping algebra. In general, a basic feature is a factorization of the Lie group into subgroups.

**Remark 5.1.** We denote partial derivatives by subscripting \( \frac{\partial}{\partial} \), e.g., \( \frac{\partial}{\partial A} = \partial/\partial A \). Repeated Greek indices are assumed summed (regardless of position).

5.1. Representations on enveloping algebras

Let \( \{\xi_1, \ldots, \xi_d\} \) be a basis for a Lie algebra. Let \( \mathcal{A} \) be the corresponding universal enveloping algebra. Denote the d-tuple \( (\xi_1, \ldots, \xi_d) \) by \( \xi \). As basis for \( \mathcal{A} \) and as basis for polynomials in commuting variables \( A = (A_1, \ldots, A_d) \) we use

\[ \psi_n(\xi) = \xi_1^{n_1} \cdots \xi_d^{n_d}, \quad c_n(A) = A_1^{n_1} \cdots A_d^{n_d}/(n_1! \cdots n_d!), \]

respectively. Note that products involving \( \xi_j \) are ordered.

The elements of the group near the identity may be expressed as products of one-parameter subgroups generated by the \( \xi_i \). I.e., let

\[ g(A, \xi) = e^{A_1 \xi_1} e^{A_2 \xi_2} \cdots e^{A_d \xi_d}. \]
This may be expanded in the form
\[ g(A, \xi) = \sum c_n(A) \psi_n(\xi) \]
and interpreted variously as
1. a generating function for the \{\psi_n\},
2. a generating function for the \{c_n\} with non-commutative variables as coefficients, and
3. as a pairing \langle \mathcal{C}, \Psi \rangle of the sequences \mathcal{C} = \{c_n\}, \Psi = \{\psi_n\}.

By duality we have a Lie homomorphism \( \xi_i \rightarrow \xi_i^* \) which is determined on the basis elements via
\[ \langle \mathcal{C}, \Psi \xi_j \rangle = \langle \xi_j^* \mathcal{C}, \Psi \rangle, \]
where \( \Psi \xi_j \) denotes the sequence with components \{\psi_n \xi_j\}. The right action by multiplication of \( \xi_j \) on a basis element \( \psi_n \) is re-expressed in terms of the \( \psi_s \). The action may be calculated using the generating functions \( g(A, \xi) \):
\[ g(A, \xi) \xi_j = \xi_j^* g(A, \xi), \]
where \( \xi_j^* \) is a differential operator acting on functions of \( A \). To see this, denote \( e^{A \xi_i} \) by \( E_i \). Then, we use the relations
\[ [E_k E_{k+1} \cdots E_i, \xi_j] = \sum_r E_k E_{k+1} \cdots [E_r, \xi_j] E_{r+1} \cdots E_i, \]
\[ E_k \xi_j = (e^{A_k \text{ad} \xi_i} \xi_j) E_k, \]
where \( (\text{ad} \xi_k) \xi_j = [\xi_k, \xi_j] \). The idea is to commute \( \xi_j \) next to the factor \( E_j \) in \( E_1 \cdots E_d \).

Similarly, we have a Lie anti-homomorphism, the left dual, \( \xi_j \rightarrow \xi_j^\dagger \), by acting on the left
\[ \langle \mathcal{C}, \xi_j \Psi \rangle = \langle \xi_j^\dagger \mathcal{C}, \Psi \rangle. \]

5.2. Coordinates of the second kind

Group elements in a neighborhood of the identity can be expressed as
\[ g(x) = e^{x_\xi \delta_0} = g(A) = e^{A_i \xi_i} e^{A_2 \xi_2} \cdots e^{A_d \xi_d}. \]

The \( x_i \) are called coordinates of the first kind and the \( A_i \), coordinates of the second kind. The dual representations denote realizations of the Lie algebra as vector fields in terms of the coordinates of the second kind acting on the left or right, respectively, i.e., define the left (respectively, right) principal matrices, \( \pi^L(A) \) (respectively, \( \pi^R(A) \)) according to
\[ \xi_j g(A) = \pi^L_{j\mu}(A) \partial_\mu g(A), \quad g(A) \xi_j = \pi^R_{j\mu}(A) \partial_\mu g(A), \]}
where here and in the following \( \partial_\mu = \partial/\partial A_\mu \). We write the dual representations
\[
\zeta_j^\dagger = \pi_{j\mu}(A) \partial_\mu, \quad \zeta_j^* = \pi_{j\mu}^*(A) \partial_\mu.
\]
If \( A \) depends on a parameter \( s \), then we have, for any function \( f(A) \), the flow
\[
\dot{f} = \dot{A}_\mu \partial_\mu f.
\]
So, let \( X = \alpha_\mu \zeta_\mu \) and consider group elements
\[
g(A(s)) = e^{sX} = e^{s\alpha_\mu \zeta_\mu}.
\]
These form a one-parameter abelian subgroup. First, we have, acting by \( X \) on the left,
\[
\dot{g} = Xg = \alpha_\mu \zeta_\mu g = \alpha_\mu \zeta_j^\dagger \partial_j g
\]
and from the right,
\[
\dot{g} = gX = g \alpha_\mu \zeta_\mu = \alpha_\mu \zeta_j^* \partial_j g = \alpha_\mu \pi_{j\mu}^* \partial_\mu g.
\]
Since, as remarked above, we have, in general, \( \dot{g} = \dot{A}_\mu \partial_\mu g \), we see the following result.

**Lemma 5.2 (Splitting Lemma).** Denote by \( \pi(A) \) either the left or the right principal matrices. Then we have
\[
\dot{A}_k = \pi_{jk}(A)
\]
with initial values \( A_k(0) = 0 \).

(These equations are a constant-coefficient version of the basic equations studied by Fliess [12,11].) In particular, evaluating at \( s = 1 \) gives the coordinate transformation \( A = A(1) \) corresponding to (3). With non-zero initial conditions, this yields the group law, equivalently, the matrix elements for the group, which we have shown how to calculate recursively in [7].

**Remark 5.3.** We mention the difference between the factorization given in (3) and formulae such as the Baker–Campbell–Hausdorff and the Zassenhaus formula (see [21, pp. 368–372]). The former (BCH) expresses the product of two exponentials as a single exponential — of an infinite sum of terms (various commutators), the latter (Z) expresses the exponential of the sum of two elements as a product of the exponentials of the individual elements times a factor — an infinite product. The advantage, in the present context, of the form (3) is that only exponents of finite sums and finite products are involved.

### 5.3. Properties of dual representations

Differentiating (3) directly, we have
\[
Xg = \sum_i e^{A_i \xi_i} \cdots e^{A_{i-1} \xi_{i-1}} \dot{A}_i \xi_i e^{A_i \xi_i} \cdots e^{A_0 \xi_0}.
\]
Evaluating at $s = 0$ in (4) yields, with $g(0)$ the identity,

$$X = \alpha_\mu \xi_\mu = \dot{A}_\mu(0) \xi_\mu.$$

Thus, $\dot{A}_k(0) = \alpha_k$, i.e.,

$$\dot{A}(0) = \alpha.$$

And letting $s = 0$ in Lemma 5.2, we thus have

$$0 = \text{identity}.$$

The right-dual mapping $\xi \rightarrow \xi^*$ gives a Lie homomorphism, i.e. $[\xi_i^*, \xi_j^*] = [\xi_i^*, \xi_j^*]$, while the action on the left reverses the order of operations, giving a Lie antihomomorphism $[\xi_i, \xi_j]^* = [\xi_i^*, \xi_j^*]$. In terms of the adjoint representation, we thus have, for the Lie bracket of the corresponding vector fields:

**Right dual:** Taking commutators and then evaluating at $A = 0$:

$$\pi_{i\mu}^* \partial_\mu \pi_{jk}^* - \pi_{j\mu}^* \partial_\mu \pi_{ik}^* = c_{i\mu}^{jk} \pi_{ik}^*,$$

$$\partial_i \pi_{jk}^*(0) - \partial_j \pi_{ik}^*(0) = c_{ij}^k. \quad (5)$$

**Left dual:** Similarly, we have

$$\pi_{i\mu}^{\dagger} \partial_\mu \pi_{jk}^{\dagger} - \pi_{j\mu}^{\dagger} \partial_\mu \pi_{ik}^{\dagger} = c_{i\mu}^{jk} \pi_{ik}^{\dagger},$$

$$\partial_i \pi_{jk}^{\dagger}(0) - \partial_j \pi_{ik}^{\dagger}(0) = c_{ij}^k.$$

As well, we see that by construction, the left and right actions commute, so that we have

**Combined:** Commuting the left and right yields

$$\pi_{i\mu}^* \partial_\mu \pi_{jk}^* = \pi_{i\mu}^{\dagger} \partial_\mu \pi_{jk}^{\dagger},$$

$$\partial_i \pi_{jk}^*(0) = \partial_j \pi_{ik}^{\dagger}(0).$$

Combining this last relation with (5), we have

$$\partial_i \pi_{jk}^*(0) - \partial_i \pi_{jk}^{\dagger}(0) = c_{ij}^k,$$

the transpose of the adjoint representation. This can be summarized in the phrase: the transposed adjoint representation is the linearization of the difference between the right and left duals. We thus define the extended adjoint representation as the difference of the right and left duals

$$\tilde{\xi}_j = \xi_j^* - \xi_j^\dagger,$$
which gives a representation of the Lie algebra, since \( \xi_j \rightarrow -\xi_j \) gives a Lie homomorphism, the minus sign reversing the commutators. And we set the corresponding \( \hat{\pi} \)

\[
\hat{\pi}(A) = \pi^*(A) - \pi^!(A).
\]

5.4. Double dual

Expanding out Eq. (3) in series, we have

\[
g(A) = \sum_{n_1, \ldots, n_d} \frac{A_1^{n_1} \cdots A_d^{n_d}}{n_1! \cdots n_d!} = \sum_n \frac{1}{n!} \psi_n^n,
\]

using multi-index notation, \( n = (n_1, \ldots, n_d) \), i.e., the group element is the generating function for the monomials \( \psi_n \). Now, we observe the action of the operators of multiplication by \( A_j \) and \( \partial_j \), differentiation with respect to \( A_j \) on \( g(A) \) dualized to acting on the basis \( \psi_n \): \( A_j g(A) \) is the same as mapping \( \psi_n \rightarrow n_j \psi_{n-e_j} \), and \( \partial_j g(A) \) is the same as mapping \( \psi_n \rightarrow \psi_{n+e_j} \). As in Section 4, we define operators \( \mathcal{R}_j, \mathcal{V}_j \), on \( \psi_n \), thus,

\[
\mathcal{R}_j \psi_n = \psi_{n+e_j}, \quad \mathcal{V}_j \psi_n = n_j \psi_{n-e_j}.
\]

Now, we can take the right and left duals acting on functions of \( A \) and convert them to operators acting on the enveloping algebra, and ‘functions of \( \xi \),’ e.g., exponential functions, and by extension to Fourier integrals. Thus, the left and right double duals

\[
\hat{\xi}_j = \mathcal{R}_{\mu} \pi^{\mu}_{j}(\mathcal{V}), \quad \hat{\xi}^*_j = \mathcal{R}_{\mu} \pi^{\mu*}_{j}(\mathcal{V}),
\]

where we drop the dagger for the left double dual and just call it the ‘double dual’. For the left action, the double dual thus gives a Lie homomorphism. These give the left and right multiplication by the basis elements \( \xi_j \) on the enveloping algebra in terms of the basis \( \psi_n \), i.e.,

\[
\hat{\xi}_j \psi_n = \hat{\xi}_j \pi_n, \quad \psi_n \hat{\xi}_j = \hat{\xi}^*_j \psi_n.
\]

We set

\[
\hat{\pi} = (\pi^\dagger)^!, \quad \hat{\pi}^* = (\pi^*!)^!
\]

so that

\[
\hat{\xi}_j = \mathcal{R}_{\mu} \hat{\pi}_{\mu j}(\mathcal{V}), \quad \hat{\xi}^{\mu*}_j = \mathcal{R}_{\mu} \hat{\pi}^{\mu*}_{j}(\mathcal{V}).
\]

We can formulate this in terms of vector fields. We apply the algebraic version of Fourier transform, interchanging variables \( A \) with their derivatives \( \partial_A \), cf., the duality \( A \leftrightarrow A^\dagger \), \( \partial_A \leftrightarrow \mathcal{R}_A \), to the left dual. We use \( (y_1, \ldots, y_d) \) as variables with the corresponding meaning \( \partial_j = \partial / \partial y_j \). For polynomials \( f \), we have, as well as \( [\partial_j, f(y_j)] = f'(y_j) \),

\[
[f(\partial_j), y_j] = f'(\partial_j).
\]
and similarly for $f(\partial_1, \ldots, \partial_d)$. These extend directly to smooth functions. Now, we form dual to a vector field $X = a_\mu(y) \partial_\mu$

$$\hat{X} = y_\mu a_\mu(\partial).$$

We have, with $Y = y_\mu b_\mu(\partial)$, subscripts preceded by a comma denoting partial derivatives,

$$[\hat{Y}, \hat{X}] = [y_\mu b_\mu(\partial), y_\mu a_\mu(\partial)]$$

$$= y_\mu (a_\ell(\partial)b_{\mu, \ell}(\partial) - b_\ell(\partial)a_{\mu, \ell}(\partial))$$

$$= [X, Y].$$

For $\xi_j = \pi_{j\mu}^\dagger \partial_\mu$, we have, with $y$ as a row vector multiplying from the left,

$$\hat{\xi} = y \tilde{\pi}(\partial).$$

### 5.4.1. Orbits for general Hamiltonians

We can use the double duals to find 'Hamilton's equations' for the Lie algebra. Let $H(\xi)$ be given as a formal series of monomials $\psi_n$. We want to solve

$$\frac{\partial u}{\partial t} = [H, u]$$

for functions of $\xi$. Consider $u(0) = \xi_j$. We have

$$\dot{\xi}_j = [H, \xi_j] = H\xi_j - \xi_j H = (\hat{\xi}_j^* - \hat{\xi}_j)H$$

$$= \mathcal{R}_{\mu} \tilde{\pi}_{j\mu}(\psi)H,$$

cf., Eq. (6). Note that this involves exponentiation of the difference between the right and left duals, which commute, so that one can exponentiate them separately, then multiply the results together.

### 5.4.2. Coadjoint orbits

For calculating the coadjoint orbits, or effectively what is the same, to calculate the exponential of the adjoint representation, the principal matrices suffice. Denote the matrices of the adjoint representation in the basis $\xi_i$ by $\xi_j$. Define $\tilde{\pi}$ to be the matrix of the group element $g$ given by exponentiating the adjoint representation. Then we have

**Theorem 5.4.** The exponential of the adjoint representation, $g(A, \xi)$, is given by the relation

$$\tilde{\pi}^* = \tilde{\pi} g(A, \xi),$$

t.e.,

$$\tilde{\pi} = \tilde{\pi}^{-1} \tilde{\pi}^*.$$
Proof. Start with

\[ g \tilde{\xi}_j = \tilde{\xi}_j^* g = \gamma_{\mu j} \xi_\mu g \]

for some matrix \( \gamma \). We know that this exists, since

\[ \gamma_{\mu j} \xi_\mu = g \tilde{\xi}_j g^{-1} = g(A, \text{ad} \tilde{\xi}_j). \]

On the other hand,

\[ \gamma_{\mu j} \xi_\mu \theta = \gamma_{\mu j} \tilde{\xi}_j^* \theta, \]

so that \( \gamma_{\mu j} \xi_\mu = \xi_\mu^* \), or,

\[ \gamma_{\mu j} \pi_{j e} e^c = \pi_{j e}^* e^c, \]

i.e., \( \pi^* = \gamma^t \pi^t \) and the result follows upon taking transposes.

5.5. Volterra products on enveloping algebras

Using the double-dual representation, we have the form of the product \( \prod (1 + vX_j) \) acting on the enveloping algebra as

\[ \prod (1 + vX_j) \psi_n \]

and similarly for acting on a formal series as in Section 4. One can consider limit theorems for increments by suitably scaling the \( X_j \). For example, if there are dilation automorphisms of the Lie algebra, then these can be implemented and then a limit of the corresponding product taken.

6. Example of representations on enveloping algebras

Consider the three-step nilpotent Lie algebra (cf. [11]) generated by the operators \( d/dx \) and \( X^2/2 \), acting on smooth functions \( f(x) \). Identifying these two operators as \( \xi_4 \) and \( \xi_3 \), respectively, we can formulate the corresponding abstract Lie algebra with commutation relations

\[ [\xi_4, \xi_3] = \xi_2, \quad [\xi_4, \xi_2] = \xi_1 \]

with other commutators (among the basis elements) zero. A matrix realization is given by

\[ X = a \xi_1 + b \xi_2 + \gamma \xi_3 + \delta \xi_4 = \begin{pmatrix} 0 & \delta & 0 & \alpha \\ 0 & 0 & \delta & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
with corresponding Jordan map, cf. (2),

\[ \xi_1 = \mathcal{R}_1 \mathcal{V}_4, \quad \xi_2 = \mathcal{R}_2 \mathcal{V}_4, \quad \xi_3 = \mathcal{R}_3 \mathcal{V}_4, \quad \xi_4 = \mathcal{R}_1 \mathcal{V}_2 + \mathcal{R}_2 \mathcal{V}_3. \]

At this point one can compute directly with matrices. Calculate the group element as the product of exponentials of the corresponding matrices \( \xi_i \) as follows:

\[
g = e^{A \xi_1} e^{B \xi_2} e^{C \xi_3} e^{D \xi_4} = \begin{pmatrix} 1 & D & D^2/2 & A \\ 0 & 1 & D & B \\ 0 & 0 & 1 & C \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7}
\]

Now, consider the relations \( \dot{g} = Xg = gX \)

\[
\begin{pmatrix} 0 & \dot{D} & D\dot{D} & \dot{A} \\ 0 & 0 & \dot{D} & \dot{B} \\ 0 & 0 & 0 & \dot{C} \\ 0 & 0 & 0 & 0 \end{pmatrix} = Xg = \begin{pmatrix} 0 & \delta & D\delta & B\delta + \alpha \\ 0 & 0 & \delta & C\delta + \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
= gX = \begin{pmatrix} 0 & \delta & \delta D & \alpha + \beta D + \gamma D^2/2 \\ 0 & 0 & \delta & \beta + \gamma D \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

We thus read off

\[
\dot{A} = B\delta + \alpha = \alpha + \beta D + \gamma D^2/2, \quad \dot{B} = C\delta + \beta = \beta + \gamma D
\]

along with \( \dot{C} = \gamma, \dot{D} = \delta \). Solving, with initial conditions \( A(0) = A, B(0) = B, C(0) = C, D(0) = D \), we find \( C(s) = \gamma s, D(s) = \delta s, \) and

\[
A(s) = \alpha s + \beta \delta s^2/2 + \gamma \delta^2 s^3/6,
\]

\[
B(s) = \beta s + \gamma \delta^2 s^2/2.
\]

The coordinates of the second kind follow by setting \( s = 1 \). A direct exponentiation of \( X \) (e.g., using MAPLE) gives

\[
\begin{pmatrix} 1 & \delta & \delta^2/2 & \delta^2 \gamma/6 + \delta \beta/2 + \alpha \\ 0 & 1 & \delta & \gamma \delta/2 + \beta \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

which yields, upon comparison with the group element, Eq. (7), the coordinates of the second kind directly:

\[
A = \alpha + \frac{1}{2} \beta \delta + \frac{1}{6} \gamma \delta^2, \quad B = \beta + \frac{1}{2} \gamma \delta, \quad C = \gamma, \quad D = \delta.
\]
The splitting lemma, Lemma 5.2, applies as follows: first for the action on the left, second for the right action

\[(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (x, \beta, \gamma, \delta) (\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ B & C & 0 & 1 \end{array}) = (x, \beta, \gamma, \delta) \pi^1,\]

\[(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = (x, \beta, \gamma, \delta) (\begin{array}{cccc} 1 & 0 & 0 & 0 \\ D & 1 & 0 & 0 \\ D^2/2 & D & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}) = (x, \beta, \gamma, \delta) \pi^* .\]

The resulting dual representations are given in Table 1.

With the $\pi$-matrices given by

\[\pi^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ B & C & 0 & 1 \end{pmatrix}, \quad \pi^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ D & 1 & 0 & 0 \\ D^2/2 & D & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\]

\[\pi = \begin{pmatrix} 1 & D & D^2/2 & -B \\ 0 & 1 & D & -C \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\]

the exponential of the adjoint representation calculated via Theorem 5.4. One can work directly with the adjoint representation from

\[\hat{X} = x^2 + \beta x + \gamma y + \delta y^2 = \begin{pmatrix} 0 & \delta & 0 & -\beta \\ 0 & 0 & \delta & -\gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .\]

7. Concluding remarks

This approach gives a theoretical basis for explicit computation of representations of Lie algebras and Lie groups. At the same time, the operator calculus presented in this
paper is well suited for symbolic computations. Besides applications in control theory, we are employing these methods in probability theory and stochastic analysis.

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References