Let $W(2n - 1, \mathbb{K})$ denote the symplectic polar space defined in the projective space $PG(2n - 1, \mathbb{K})$, where $\mathbb{K}$ denotes the field possibly non-perfect and of characteristic 2. Associated with $W(2n - 1, \mathbb{K})$ there are two ranks of interest, namely the generating rank (written as $\text{rk}_{\text{gen}}$) given as the maximum size of a maximum spanning set of points, and the embedding rank (written $\text{rk}_{\text{emb}}$), that is, the vector dimension of its universal full projective embedding. Let $d$ denote the degree of the extension $\mathbb{K}$ over $\mathbb{K}^2$. Denote by $Q_{\Lambda}(2n - 1 + d, \mathbb{K})$ the quadric of $PG(2n - 1 + d, \mathbb{K})$ associated to the quadratic form $\sum_{i=0}^{n-1} x_i x_{n+i} + \sum_{0 \leq i < \delta} \lambda_i x_{2n+i}^2$, where $\delta$ is the least ordinal number of cardinality equal to $d$ and $\Lambda = \{\lambda_i\}_{0 \leq i < \delta}$ is a basis of $\mathbb{K}$ over $\mathbb{K}^2$. This quadratic form is used to show that $W(2n - 1, \mathbb{K}) \cong Q_{\Lambda}(2n - 1 + d, \mathbb{K})$ and that $\text{rk}_{\text{emb}}(W(2n - 1, \mathbb{K})) = \text{rk}_{\text{gen}}(W(2n - 1, \mathbb{K})) = 2n + d$. As an important corollary to this result, there is an isomorphism $W(2n - 1, \mathbb{K}) \cong Q(2n, \mathbb{K})$ if, and only if, $\mathbb{K}$ is perfect (i.e. $d = 1$). If $d > 1$ then $Q(2n, \mathbb{K})$ is isomorphic to a proper subspace of $W(2n - 1, \mathbb{K})$.

The authors also show how this embedding generalizes to point-line geometries. Let $\mathcal{G} = (P, L)$ consist of a set of points $P$ and a set of lines $L$ that is a subset of $P$ such that (i) no two lines have more than one point in common and (ii) every line has at least two points. The authors define an embedding $e: \mathcal{G}_1 \rightarrow \mathcal{G}_2$ from one point-line geometry $\mathcal{G}_1 = (P_1, L_1)$ to another one $\mathcal{G}_2 = (P_2, L_2)$ as an injective mapping $e: P_1 \rightarrow P_2$ such that (i) for every line $l \in L_1$ the set $e(l)$ spans a line of $\mathcal{G}_2$, and (ii) different lines of $\mathcal{G}_1$ are mapped by $e$ into different lines of $\mathcal{G}_2$. If $e(l)$ is a line of $\mathcal{G}_2$ for every line $l \in L_1$, then we say that the embedding $e$ is full. If the set $e(P_1)$ spans a proper subspace of $\mathcal{G}_2$ then we say that the embedding is loose; otherwise $e$ is said to be strict. Among the results proved, we mention the construction of a full and isometric but loose embedding of $Q(2n, \mathbb{K})$ in $W(2n - 1, \mathbb{K})$. The dual embedding is strict and isometric, but non-full. As for the opposite embedding of $W(2n - 1, \mathbb{K})$ in $Q(2n, \mathbb{K})$, an embedding is displayed that is isometric and strict but non-full. The dual is full and isometric, but loose.

Hence, the paper treats some technical aspects linked to the theory of buildings and in particular of generalized polygons. It is grounded in the theory of classification groups. Elementary Coxeter groups originate from permutation groups of translational and reflection type [N. Bourbaki, Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Hermann, Paris, 1968; MR0240238 (39 #1590); H. S. M. Coxeter, Regular polytopes, Third edition, Dover, New York, 1973; MR0370327 (51 #6554)]. A simple observation is that tilings of the plane are generated by particular reflections that form a group. Each reflection group generates a particular tiling, and reciprocally for each tiling of a specific kind there is a group. Such an observation in dimension two (the plane) allows a generalization in arbitrary
dimensions, and the geometrical picture resembles the two-dimensional analogue. The space divides into regular polytopes separated by hyperplanes. As for permutation groups, they share a similar geometrical representation in which each element is associated projectively with a point on a sphere and the sphere decomposes according to a polytope subdivision. Each summit on the sphere gets mapped to a summit.

In this way, regular polytopes bear the structure of underlying groups, at least for some elementary groups. Considering more elaborate groups, such as Lie groups—and especially exceptional Lie groups—they do not seem, immediately at least, to allow a simple transposition of the underlying geometrical polytope interpretation, unless one adopts an abstract formulation of the geometrical notions as in [J. Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Math., 386, Springer, Berlin, 1974; MR0470099 (57 #9866)] (e.g., chambers, walls, apartments, and buildings). Once the axiomatics are laid out (graph axiomatics in the presentation of Tits [op. cit.]; other interpretations exist as well), the picture not only becomes intuitive (in the sense of coherently complementing the works of Killing, Cartan, Witt and Weyl obtained on complex simple Lie groups, e.g., $A_n$, $B_n$, $C_n$, etc.), but also leads to the discovery of new unforeseen groups isomorphic to those classified according to this axiomatic system.

Among buildings, thick irreducible spherical buildings of rank two have attracted a lot of attention. Besides classification, a complete one has been established for the Moufang subclass [J. Tits and R. M. Weiss, *Moufang polygons*, Springer, Berlin, 2002; MR1938841 (2003m:51008)], and they can be introduced independently using point-line geometries. These buildings are called generalized polygons [H. J. Van Maldeghem, *Generalized polygons*, Birkhäuser, Basel, 1998; MR1725957 (2000k:51004)]. For example, taking $V$ as a three-dimensional right vector space over a skew field $\mathbb{K}$, a point-line geometry on the two-dimensional projective space $\operatorname{PG}(V)$ can be defined by assigning the points to 1-spaces of $V$ and lines to 2-spaces of $V$. More generally, from a quadratic form $q$ with Witt index 2 (i.e. $q^{-1}(0)$ contains 2-spaces, but no higher-dimensional subspaces) the points are the 1-spaces in $q^{-1}(0)$ and the lines are the 2-spaces in $q^{-1}(0)$; the incidence relations are satisfied through symmetrized inclusion.

One may then wonder where quadratic forms take their origin in the earlier link between elementary groups and the polytopes to which they are associated. Coxeter introduced a diagram to encode the group structure, based on how polytopes (chambers) are mapped by the group, from which a matrix can be formed. Closely linked to this matrix is a quadratic form. A related diagram, called the Dynkin diagram, occurs for a Lie group, which is constructed from permutation properties of the characteristic polynomial of the adjoint representation of the associated Lie algebra. This leads to the Weyl group, the geometry of which has chambers, walls, etc., and a canonical bilinear form as well.

Hence, buildings can also be given (or defined) through the associated Dynkin or Coxeter diagram. It is then possible to spot generalized polygons. For instance, a building of type $C_2$ can be viewed as the polar space arising from a symplectic polarity in $\operatorname{PG}(3, \mathbb{K})$. A structure of a symplectic quadrangle $W(3, \mathbb{K})$ can be associated with an anti-isomorphic orthogonal quadrangle $Q(4, \mathbb{K})$ which is a building of type $B_2$ that, in turn, can be viewed as the polar space arising from a quadric in $\operatorname{PG}(4, \mathbb{K})$.

In the characteristic 2 case, the paper under review stresses the nuance between a perfect field
\( \mathbb{K} (d = 1) \) and an imperfect one \((d > 1)\) through, on the one hand, explicitly constructing isomorphisms between \( W(2n - 1, \mathbb{K}) \) and \( Q(2n - 1 + d, \mathbb{K}) \), and, on the other hand, correspondences (embeddings in both directions) between \( W(2n - 1, \mathbb{K}) \) and \( Q(2n, \mathbb{K}) \).

Reviewed by Philippe A. Müllhaupt

References


*Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.*

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