Energy methods in the stability problem for the $\mathfrak{so}(4)$ free rigid body

Petre Birtea and Ioan Casu
Department of Mathematics, West University of Timișoara, Bd. V. Parvan, No. 4, 300223 Timișoara, Romania
E-mail: birtea@math.uvt.ro; casu@math.uvt.ro

Abstract
We prove that the study of stability using energy methods strongly depends on the conservation laws considered. We exemplify this by studying stability using Arnold’s method in the case of $\mathfrak{so}(4)$ free rigid body. Using the classical Mishchenko’s constants of motion we do not obtain satisfactory stability results. We find new constants of motion generated by Mishchenko’s first integrals that give better results about the stability of equilibria.

Keywords: free rigid body, equilibrium, nonlinear stability, energy method, constant of motion, Cartan algebra.

1 Introduction
The goal of this paper is to prove stability using energy methods for the Cartan type equilibria of $\mathfrak{so}(4)$ free rigid body. All the Cartan type equilibria are organized in three Weyl group orbits generated by three coordinate type Cartan subalgebras $t_1$, $t_2$, $t_3$, intersected with a regular adjoint orbit.

Explicitly, the three Weyl group orbits of Cartan type equilibria are given by \( \{ M_{1a,b}, M_{1a,-b}, M_{1b,a}, M_{1b,-a}\} \), \( \{ M_{2a,b}, M_{2a,-b}, M_{2b,a}, M_{2b,-a}\} \), and \( \{ M_{3a,b}, M_{3a,-b}, M_{3b,a}, M_{3b,-a}\} \). In the paper [2], for the $t_3$-equilibria \( \{ M_{3a,b}, M_{3a,-b}, M_{3b,a}, M_{3b,-a}\} \) we have proved using Williamson’s theorem that they are stable of center-center type. Their stability can be also obtained using Arnold’s method with Mishchenko’s constants of motion. In this paper we also prove the stability of these equilibria using the new constants of motion generated by the Mishchenko’s ones.

The equilibria $M_{1a,b}, M_{1a,-b}$ are also stable of center-center type. Arnold’s method using the Hamiltonian $H$ or the other Mishchenko’s constant of motion $I$ gives incomplete results on stability. Taking a certain combination of Mishchenko’s constant of motion we will be able to prove stability using energy methods. For equilibria $M_{2a,b}, M_{2a,-b}$ we will also prove stability using energy methods in a certain subregion of the stability region established in [2].

Although the new constants of motion are functionally dependent of the Mishchenko’s first integrals, they prove their utility in applying energy methods for studying stability of equilibria. This is also important to notice as, in higher dimensions, the energy methods are the most useful ones in studying stability (see for example [5]).

2 The free rigid body on $\mathfrak{so}(4)$: equilibria, primary stability results, constants of motion
The equations of the rigid body on $\mathfrak{so}(4)$ are given by
\[
\dot{M} = [M, \Omega],
\]
where $\Omega \in \mathfrak{so}(4)$, $M = \Omega J + J\Omega \in \mathfrak{so}(4)$ with $J = \text{diag}(\lambda_i)$, a real constant diagonal matrix satisfying $\lambda_i + \lambda_j \geq 0$, for all $i, j = 1, \ldots, 4$, $i \neq j$ (see, for example, [9]). Note that $M = [m_{ij}]$ and $\Omega = [\omega_{ij}]$ determine each other if and only if $\lambda_i + \lambda_j > 0$ since $m_{ij} = (\lambda_i + \lambda_j) \omega_{ij}$ which physically means that the rigid body is not concentrated on a lower dimensional subspace of $\mathbb{R}^4$.

It is well known and easy to verify that equations (2.1) are Hamiltonian relative to the minus Lie-Poisson bracket

$$\{F,G\}(M) := \frac{1}{2} \text{Trace}(M[\nabla F(M), \nabla G(M)])$$

and the Hamiltonian function

$$H(M) := -\frac{1}{4} \text{Trace}(M\Omega).$$

Here $F, G, H \in C^\infty(\mathfrak{so}(4))$ and the gradient is taken relative to the Ad-invariant inner product

$$\langle X, Y \rangle := -\frac{1}{2} \text{Trace}(XY), \quad X, Y \in \mathfrak{so}(4)$$

which identifies $(\mathfrak{so}(4))^*$ with $\mathfrak{so}(4)$. This means that $\dot{F} = \{F,H\}$ for all $F \in C^\infty(\mathfrak{so}(4))$, where $\{\cdot, \cdot\}$ is given by (2.2) and $H$ by (2.3), if and only if (2.1) holds. Note that the linear isomorphism $X \in \mathfrak{so}(4) \mapsto XJ + JX \in \mathfrak{so}(4)$ is self-adjoint relative to the inner product (2.4) and thus $\nabla H(M) = \Omega$.

We continue with a short presentation of the Lie algebra $\mathfrak{so}(4)$ and with details on the free rigid body dynamics on it.

The Lie algebra of the compact subgroup $\text{SO}(4) = \{ A \in \mathfrak{gl}(4, \mathbb{R}) \mid A^t A = I_4, \det(A) = 1 \}$ of the special linear Lie group $\text{SL}(4, \mathbb{R})$ is $\mathfrak{so}(4)$.

We choose as basis of $\mathfrak{so}(4)$ the matrices

$$E_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad E_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad E_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$E_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}; \quad E_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad E_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and hence we represent $\mathfrak{so}(4)$ as

$$\mathfrak{so}(4) = \left\{ M = \begin{bmatrix} 0 & -x_3 & x_2 & y_1 \\ x_3 & 0 & -x_1 & y_2 \\ -x_2 & x_1 & 0 & y_3 \\ -y_1 & -y_2 & -y_3 & 0 \end{bmatrix} \mid x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \right\}. \quad (2.5)$$

This choice of basis was made for computational convenience as we will see below. Note that $E_1 = -E_{23}$, $E_2 = E_{13}$, $E_3 = -E_{12}$, $E_4 = E_{24}$, $E_5 = E_{24}$, $E_6 = E_{34}$. The multiplication for this basis of $\mathfrak{so}(4)$ is given by the following table (the convention is to calculate [row, column]):

<table>
<thead>
<tr>
<th>$\cdot, \cdot$</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
<th>$E_5$</th>
<th>$E_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$</td>
<td>0</td>
<td>$E_3$</td>
<td>$-E_2$</td>
<td>0</td>
<td>$E_6$</td>
<td>$-E_5$</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$-E_3$</td>
<td>0</td>
<td>$E_1$</td>
<td>$-E_6$</td>
<td>0</td>
<td>$E_4$</td>
</tr>
<tr>
<td>$E_3$</td>
<td>$E_2$</td>
<td>$-E_1$</td>
<td>0</td>
<td>$E_5$</td>
<td>$-E_4$</td>
<td>0</td>
</tr>
<tr>
<td>$E_4$</td>
<td>0</td>
<td>$E_6$</td>
<td>$-E_5$</td>
<td>0</td>
<td>$E_3$</td>
<td>$-E_2$</td>
</tr>
<tr>
<td>$E_5$</td>
<td>$-E_6$</td>
<td>0</td>
<td>$E_4$</td>
<td>$-E_3$</td>
<td>0</td>
<td>$E_1$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$E_5$</td>
<td>$-E_4$</td>
<td>0</td>
<td>$E_2$</td>
<td>$-E_1$</td>
<td>0</td>
</tr>
</tbody>
</table>
In the basis \( \{E_1, \ldots, E_6\} \), the matrix of the Lie-Poisson structure (2.2) is

\[
\Gamma_- = \begin{bmatrix}
0 & -x_3 & x_2 & 0 & -y_3 & y_2 \\
x_3 & 0 & -x_1 & y_3 & 0 & -y_1 \\
-x_2 & x_1 & 0 & -y_2 & y_1 & 0 \\
0 & -y_3 & y_2 & 0 & -x_3 & x_2 \\
y_3 & 0 & -y_1 & x_3 & 0 & -x_1 \\
-y_2 & y_1 & 0 & -x_2 & x_1 & 0 \\
\end{bmatrix}.
\tag{2.6}
\]

Since rank \( \mathfrak{so}(4) = 2 \), there are two functionally independent Casimir functions which are given respectively by

\[
C_1(M) := -\frac{1}{4} \text{Trace}(M^2) = \frac{1}{2} \left( \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 \right)
\]

and

\[
C_2(M) := -\text{Pf}(M) = \sum_{i=1}^{3} x_i y_i.
\]

Thus the generic adjoint orbits are the level sets

\[
\text{Orb}_{c_1,c_2}(M) = (C_1 \times C_2)^{-1}(c_1, c_2), \quad (c_1, c_2) \in \mathbb{R}^2.
\]

Note that if \( M \neq 0 \), then \( dC_j(M) \neq 0 \) for \( j = 1, 2 \).

The Lie algebra \( \mathfrak{so}(4) = \mathfrak{so}(3) \times \mathfrak{so}(3) \) is of type \( A_1 \times A_1 \) and, consequently, the positive Weyl chamber, which is the moduli space of (co)adjoint orbits, is isomorphic to the positive quadrant in \( \mathbb{R}^2 \). In the basis of \( \mathfrak{so}(4) \) that we have chosen above, the positive Weyl chamber is given by the set \( \{(c_1,c_2) \in \mathbb{R}^2 \mid c_1 \geq |c_2|\} \).

In all that follows we will denote by \( \text{Orb}_{c_1,c_2} \) the regular adjoint orbit \( \text{Orb}_{c_1,c_2} \), where \( c_1 > 0 \) and \( c_1 \geq |c_2| \).

Using the Lie bracket table in the chosen basis given above, it is immediately seen that the coordinate type Cartan subalgebras of \( \mathfrak{so}(4) \) are \( t_1, t_2, t_3 \), where

\[
t_1 := \text{span}(E_1, E_4) = \left\{ M_{a,b}^1 := \begin{bmatrix} 0 & 0 & 0 & b \\ 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -b & 0 & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\},
\]

\[
t_2 := \text{span}(E_2, E_5) = \left\{ M_{a,b}^2 := \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ -a & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\},
\]

\[
t_3 := \text{span}(E_3, E_6) = \left\{ M_{a,b}^3 := \begin{bmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & -b & 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.
\]

The intersection of a regular adjoint orbit and a coordinate type Cartan subalgebra has four elements which represents a Weyl group orbit. Thus we expect at least twelve equilibria for the rigid body equations (2.1) in the case of \( \mathfrak{so}(4) \). Specifically, we have the following result.

**Theorem 2.1.** The following equalities hold:

(i) \( t_1 \cap \text{Orb}_{c_1,c_2} = \left\{ M_{a,b}^1, M_{-a,-b}^1, M_{b,a}^1, M_{-b,-a}^1 \right\} \),

(ii) \( t_2 \cap \text{Orb}_{c_1,c_2} = \left\{ M_{a,b}^2, M_{-a,-b}^2, M_{b,a}^2, M_{-b,-a}^2 \right\} \).
(iii) \( t_3 \cap \text{Orb}_{c_1; c_2} = \left\{ M^a_{a,b}, M^a_{-a,-b}, M^b_{b,a}, M^b_{-b,-a} \right\} \),

where

\[
\begin{align*}
a &= \frac{1}{\sqrt{2}} \left( \sqrt{c_1 + c_2} + \sqrt{c_1 - c_2} \right), \\
b &= \frac{1}{\sqrt{2}} \left( \sqrt{c_1 + c_2} - \sqrt{c_1 - c_2} \right).
\end{align*}
\]

The intersections \( t_1 \cap \text{Orb}_{c_1; c_2}, t_2 \cap \text{Orb}_{c_1; c_2}, t_3 \cap \text{Orb}_{c_1; c_2} \) are Weyl group orbits.

We will work from now on with a generic \( \mathfrak{so}(4) \)-rigid body, that is, \( \lambda_i + \lambda_j > 0 \) for \( i \neq j \) and all \( \lambda_i \) are distinct. The relationship between \( \Omega = [\omega_{ij}] \in \mathfrak{so}(4) \) and the matrix \( M \in \mathfrak{so}(4) \) in the representation \( \rho \) is hence given by

\[
\begin{align*}
(\lambda_3 + \lambda_2)\omega_{32} &= x_1, & (\lambda_1 + \lambda_3)\omega_{13} &= x_2, & (\lambda_2 + \lambda_1)\omega_{21} &= x_3 \\
(\lambda_1 + \lambda_4)\omega_{14} &= y_1, & (\lambda_2 + \lambda_4)\omega_{24} &= y_2, & (\lambda_3 + \lambda_4)\omega_{34} &= y_3
\end{align*}
\]

and thus the equations of motion (2.1) are equivalent for \( n = 4 \) to the system

\[
\begin{align*}
\dot{x}_1 &= \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_1 + \lambda_3} \right) x_2 x_3 + \left( \frac{1}{\lambda_3 + \lambda_4} - \frac{1}{\lambda_2 + \lambda_4} \right) y_2 y_3 \\
\dot{x}_2 &= \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_2 + \lambda_4} \right) x_1 x_3 + \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_4} \right) y_1 y_3 \\
\dot{x}_3 &= \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_4} \right) x_1 x_2 + \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_4} \right) y_1 y_2 \\
\dot{y}_1 &= \left( \frac{1}{\lambda_3 + \lambda_4} - \frac{1}{\lambda_2 + \lambda_4} \right) x_2 y_3 + \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_4} \right) x_3 y_2 \\
\dot{y}_2 &= \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_3 + \lambda_4} \right) x_1 y_3 + \left( \frac{1}{\lambda_1 + \lambda_3} - \frac{1}{\lambda_2 + \lambda_4} \right) x_3 y_1 \\
\dot{y}_3 &= \left( \frac{1}{\lambda_2 + \lambda_4} - \frac{1}{\lambda_3 + \lambda_4} \right) x_1 y_2 + \left( \frac{1}{\lambda_3 + \lambda_4} - \frac{1}{\lambda_2 + \lambda_4} \right) x_2 y_1.
\end{align*}
\]

The Hamiltonian (2.3) has in this case the expression

\[
H(M) = -\frac{1}{4} \text{Trace}(M\Omega)
\]

\[
= \frac{1}{2} \left( \frac{1}{\lambda_2 + \lambda_3} x_1^2 + \frac{1}{\lambda_1 + \lambda_3} x_2^2 + \frac{1}{\lambda_1 + \lambda_2} x_3^2 + \frac{1}{\lambda_1 + \lambda_3} y_1^2 + \frac{1}{\lambda_2 + \lambda_4} y_2^2 + \frac{1}{\lambda_3 + \lambda_4} y_3^2 \right).
\]

The Hamiltonian nature of system (2.8) can be checked in this case directly, writing \((\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{y}_1, \dot{y}_2, \dot{y}_3)^T = \Gamma_-(\nabla H)^T\), where the Poisson structure \( \Gamma_- \) is given by (2.6) and

\[
(\nabla H)^T = \left( \frac{x_1}{\lambda_2 + \lambda_3}, \frac{x_2}{\lambda_1 + \lambda_3}, \frac{x_3}{\lambda_1 + \lambda_2}, \frac{y_1}{\lambda_1 + \lambda_4}, \frac{y_2}{\lambda_2 + \lambda_4}, \frac{y_3}{\lambda_3 + \lambda_4} \right)^T.
\]

Theorem 2.2. [3] If \( \mathcal{E} \) denotes the set of the equilibrium points of (2.8), then \( \mathcal{E} = t_1 \cup t_2 \cup t_3 \cup s_+ \cup s_- \), where \( s_\pm \) are the three dimensional vector subspaces given by

\[
s_\pm := \text{span}_\mathbb{R} \left\{ \left( \frac{1}{\lambda_1 + \lambda_4} E_3 \pm \frac{1}{\lambda_2 + \lambda_4} E_4 \right), \left( \frac{1}{\lambda_2 + \lambda_4} E_2 \pm \frac{1}{\lambda_1 + \lambda_4} E_5 \right), \left( \frac{1}{\lambda_3 + \lambda_4} E_3 \pm \frac{1}{\lambda_1 + \lambda_2} E_6 \right) \right\}.
\]

Further we remind the main results obtained in [2] on the nonlinear stability of the equilibrium states \( \mathcal{E} \cap \text{Orb}_{c_1; c_2} \) for the dynamics (2.8) on a generic adjoint orbit.

Since the system (2.8) on a generic adjoint orbit is completely integrable ([4], [6], [7], [8]), for the \( \mathfrak{so}(4) \) case we have a supplementary constant of motion. Using Mishchenko’s method ([7], [9]), we obtain the following additional constant of the motion for the equations (2.8) commuting with \( H \):

\[
I(M) = (\lambda_2^2 + \lambda_3^2)x_1^2 + (\lambda_1^2 + \lambda_3^2)x_2^2 + (\lambda_1^2 + \lambda_2^2)x_3^2 + (\lambda_1^2 + \lambda_4^2)y_1^2 + (\lambda_2^2 + \lambda_4^2)y_2^2 + (\lambda_3^2 + \lambda_4^2)y_3^2.
\]

4
Without loss of generality, we can choose an ordering for \( \lambda_i \)'s, namely
\[
\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4.
\]

The following results on nonlinear stability hold [2].

**Theorem 2.3.** [2] The equilibria \( M^1_{a,b}, M^3_{a,-b} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate of center-center type and therefore nonlinearly stable on the corresponding adjoint orbit and also nonlinearly stable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

We denote by \( \alpha_1, \alpha_2 \ (\alpha_1 < \alpha_2) \) the distinct real roots of the quadratic function
\[
f(t) = \bar{S}t^2 + \bar{T}t + \bar{U},
\]
where
\[
\bar{S} = (\lambda_1^2 - \lambda_3^2)^2 > 0; \quad \bar{U} = (\lambda_2^2 - \lambda_4^2)^2 > 0;
\]
\[
\bar{T} = -2((\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2) + (\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)) < 0.
\]

**Theorem 2.4.** [2] Under the hypothesis \( \lambda_1^2 + \lambda_2^2 \neq \lambda_2^2 + \lambda_3^2 \) the following hold:

(i) If \( \frac{\lambda_1^2}{\alpha_1} \in [0, \alpha_1) \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate unstable of saddle-saddle type on the adjoint orbit determined by \( a \) and \( b \) and hence are also unstable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

(ii) If \( \frac{\lambda_1^2}{\alpha_1} \in (\alpha_1, \alpha_2) \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate unstable of focus-focus type on the adjoint orbit determined by \( a \) and \( b \) and hence are also unstable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

(iii) If \( \frac{\lambda_1^2}{\alpha_1} \in (\alpha_2, 1) \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate stable of center-center type on the adjoint orbit determined by \( a \) and \( b \) and hence are also nonlinearly stable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

(iv) If \( \frac{\lambda_1^2}{\alpha_1} = \alpha_1 \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are degenerate and unstable on the adjoint orbit determined by \( a \) and \( b \) and hence are also unstable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

(v) If \( \frac{\lambda_1^2}{\alpha_1} = \alpha_2 \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are degenerate and the stability problem on the adjoint orbit determined by \( a \) and \( b \) remains open. However, these equilibria are unstable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

**Theorem 2.5.** [2] Under the hypothesis \( \lambda_1^2 + \lambda_2^2 = \lambda_2^2 + \lambda_3^2 \) the following holds:

(i) If \( \frac{\lambda_1^2}{\alpha_1} \in [0, \alpha_1) \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate unstable of saddle-saddle type.

(ii) If \( \frac{\lambda_1^2}{\alpha_1} \in (\alpha_1, 1) \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate unstable of focus-focus type.

(iii) If \( \frac{\lambda_1^2}{\alpha_1} = \alpha_1 \), then the equilibria \( M^1_{b,a}, M^3_{b,-a} \in T_1 \cap \text{Orb}_{c_1;c_2} \) are degenerate and unstable.

Thus these equilibria are also unstable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

**Theorem 2.6.** [2] All four equilibria in \( T_3 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate of center-center type and therefore nonlinearly stable on the corresponding adjoint orbit. These equilibria are also nonlinearly stable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).

**Theorem 2.7.** [2] All four equilibria in \( T_2 \cap \text{Orb}_{c_1;c_2} \) are non-degenerate, of center-saddle type and therefore unstable on the corresponding adjoint orbit. Thus these equilibria are also unstable for the Lie-Poisson dynamics on \( \mathfrak{so}(4) \).
3 Stability using energy methods

In the paper [2] we have chosen to use the Williamson-Bolsinov-Fomenko method [4] in approaching nonlinear stability for equilibria in $t_1 \cap \text{Orb}_{c_1;c_2}$ and in $t_3 \cap \text{Orb}_{c_1;c_2}$ because Arnold’s method (which is equivalent with the other energy methods [3]) proved to be ineffective for the equilibria in $t_3 \cap \text{Orb}_{c_1;c_2}$, when applied with the "canonical" constants of motion $H$ and/or $I$.

For the equilibria $M_{a,b}^1$ and $M_{-a,-b}^1$, in $t_1 \cap \text{Orb}_{c_1;c_2}$ the computations are more precisely as follows. Consider the smooth function $F_{mn} \in C^{\infty}(\mathfrak{so}(4),\mathbb{R})$, where $m, n$ are real numbers

$$F_{mn}(M) = I(M) + mC_1(M) + nC_2(M).$$

Choosing $m, n$ such that $dF_{mn}(M_{a,b}^1) = 0$ and taking into account that

$$W := \ker dC_1(M_{a,b}^1) \cap \ker dC_2(M_{a,b}^1) = \text{span}(E_2, E_3, E_5, E_6)$$

we obtain the determinants associated with all upper-left submatrices of the Hessian $d^2F_{mn}(M_{a,b}^1)|_{W \times W}$

$$D_1 = 2 \cdot \frac{a^2(\lambda_1^2 - \lambda_2^2) - b^2(\lambda_3^2 - \lambda_4^2)}{a^2 - b^2};$$
$$D_2 = 4 \cdot \frac{[a^2(\lambda_1^2 - \lambda_2^2) - b^2(\lambda_3^2 - \lambda_4^2)] \cdot [a^2(\lambda_1^2 - \lambda_3^2) - b^2(\lambda_2^2 - \lambda_4^2)]}{(a^2 - b^2)^2};$$
$$D_3 = -8 \cdot \frac{[a^2(\lambda_1^2 - \lambda_2^2) - b^2(\lambda_3^2 - \lambda_4^2)](\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2)}{a^2 - b^2};$$
$$D_4 = 16(\lambda_1^2 - \lambda_2^2)(\lambda_3^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_2^2).$$

For the definiteness of $d^2F_{mn}(M_{a,b}^1)|_{W \times W}$ it is necessary to have $D_1 \cdot D_3 > 0$ and $D_2 > 0$, which, from the above equalities, cannot hold simultaneously. Thus, $d^2F_{mn}(M_{a,b}^1)|_{W \times W}$ is indefinite and Arnold’s method shows its limitations when used with the constant of motion $I$.

Using Arnold’s method with the Hamiltonian of the system $H$, i.e. an energy function of the form $F_{mn}(M) = H(M) + mC_1(M) + nC_2(M)$, we obtain that equilibria $M_{a,b}^1$ and $M_{-a,-b}^1$ are nonlinear stable under the sufficient conditions on $\lambda_i$ and respectively on $a, b$:

$$\lambda_1 + \lambda_4 < \lambda_2 + \lambda_3$$

and

$$\frac{b^2}{a^2} > \frac{(\lambda_1 + \lambda_4)^2}{(\lambda_2 + \lambda_3)^2}.$$

These stability conditions were also obtained in [10]. This stability result is obviously weaker than Theorem 2.3.

As stated in [7], the system (2.1) admits a sequence $m_r \ (r = 1, 2, 3, \ldots)$ of integrals of motion, all of them depending quadratically on the angular momentum $M$. It is easy to compute an explicit form for these integrals as follows:

$$m_r(M) = \sum_{i<k}^4 \frac{\lambda_i^r - \lambda_k^r}{\lambda_i^2 - \lambda_k^2} m_{ik}^2.$$

It is to be noticed that $m_1(M) = 2H(M)$, $m_2(M) = 2C_1(M)$ and $m_4(M) = I(M)$.

But, as proved in our paper [5], there is a family of constants of motion for the dynamics of the free rigid body on $\mathfrak{so}(n)$, which have very simple and elegant expressions and which are generated by the Mishchenko’s constants of motion. For $n = 4$ these constants of motion have the following expressions:

$$G_1(M) = \frac{x_2^2}{\lambda_1^2 - \lambda_3^2} + \frac{x_3^2}{\lambda_1^2 - \lambda_2^2} + \frac{y_1^2}{\lambda_1^2 - \lambda_4^2};$$
\[ G_2(M) = \frac{x_1^2}{\lambda_2^2 - \lambda_3^2} + \frac{x_2^2}{\lambda_3^2 - \lambda_1^2} + \frac{y_1^2}{\lambda_1^2 - \lambda_2^2}; \]

\[ G_3(M) = \frac{x_1^2}{\lambda_2^2 - \lambda_1^2} + \frac{x_2^2}{\lambda_3^2 - \lambda_1^2} + \frac{y_1^2}{\lambda_1^2 - \lambda_3^2}; \]

\[ G_4(M) = \frac{y_1^2}{\lambda_1^2 - \lambda_3^2} + \frac{y_2^2}{\lambda_2^2 - \lambda_3^2} + \frac{y_3^2}{\lambda_3^2 - \lambda_1^2}. \]

More precisely, as proved in [5], the relation between these constants of motion and Mishchenko’s first integrals is given by the equations:

\[ m_r(M) = \sum_{i=1}^{4} \lambda_i G_i(M). \]

Even if these new constants of motion are obviously functionally dependent of \( H, I, C_1, C_2 \), they prove to make Arnold’s method work for all nonlinear stable equilibria for which the bifurcation phenomenon from Theorems 2.4 and 2.5 does not occur, that is equilibria \( M_{a,b}^1, M_{a,-b}^1 \in t_1 \cap \text{Orb}_{c_1;c_2} \) and all four equilibria in \( t_2 \cap \text{Orb}_{c_1;c_2} \). Also, they provide (under an additional assumption on \( \lambda_i \)) a sufficient condition of nonlinear stability for the equilibria \( M_{a,b}^1, M_{a,-b}^1 \in t_1 \cap \text{Orb}_{c_1;c_2} \).

More precisely, the results are the following. For the equilibrium \( M_{a,b}^1 \) in \( t_1 \cap \text{Orb}_{c_1;c_2} \) let us consider the Arnold function \( F_{mn} \in C^\infty(\mathfrak{so}(4), \mathbb{R}) \), where \( m, n \) are real numbers

\[ F_{mn}(M) = G_3(M) + mC_1(M) + nC_2(M). \]

Choosing \( m, n \) such that \( dF_{mn}(M_{a,b}^1) = 0 \) and taking into account that

\[ W := \ker dC_1(M_{a,b}^1) \cap \ker dC_2(M_{a,b}^1) = \text{span}(E_2, E_3, E_5, E_6) \]

we obtain determinants associated with all upper-left submatrices of the Hessian \( d^2F_{mn}(M_{a,b}^1)|W \times W \)

\[ D_1 = 2 \cdot \frac{a^2(\lambda_1^2 - \lambda_2^2) + b^2(\lambda_2^2 - \lambda_3^2)}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)} > 0; \]

\[ D_2 = 4a^2 \cdot \frac{a^2(\lambda_1^2 - \lambda_2^2) + b^2(\lambda_2^2 - \lambda_3^2)}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2} > 0; \]

\[ D_3 = 8a^4 \cdot \frac{\lambda_1^2 - \lambda_2^2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2} > 0; \]

\[ D_4 = 16a^4 \cdot \frac{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)^2(\lambda_3^2 - \lambda_1^2)^2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)^2(a^2 - b^2)^2} > 0. \]

It follows that \( d^2F_{mn}(M_{a,b}^1)|W \times W \) is positive definite and thus the equilibrium \( M_{a,b}^1 \) in \( t_1 \cap \text{Orb}_{c_1;c_2} \) is nonlinear stable. A similar computation proves that the equilibrium \( M_{1,a,-b}^1 \) in \( t_1 \cap \text{Orb}_{c_1;c_2} \) is also nonlinear stable.

It is immediately to be seen that for equilibria \( M_{a,b}^3, M_{a,-b}^3 \) in \( t_3 \cap \text{Orb}_{c_1;c_2} \) a convenient energy function for applying Arnold’s energy method is \( F_{mn}(M) = G_1(M) + mC_1(M) + nC_2(M) \), while for equilibria \( M_{b,a}^3, M_{b,-a}^3 \) in \( t_3 \cap \text{Orb}_{c_1;c_2} \) a convenient energy function for applying Arnold’s energy method is \( F_{mn}(M) = G_4(M) + mC_1(M) + nC_2(M) \).

In conclusion, for all stable equilibria \( M_{a,b}^1, M_{a,-b}^1 \) in \( t_1 \cap \text{Orb}_{c_1;c_2} \) and respectively \( M_{a,b}^3, M_{a,-b}^3, M_{b,a}^3, M_{b,-a}^3 \) in \( t_3 \cap \text{Orb}_{c_1;c_2} \) energy methods are effective. For \( M_{a,b}^1 \) and \( M_{a,-b}^1 \) energy methods are
Taking into account the expressions of the determinants it follows that
\[ S \]
where \( S \) appears, as described in Theorem 2.4 of the previous section, we make the following remarks.

Using Arnold’s method with the Hamiltonian of the system \( H \), i.e. an energy function of the form \( F_{mn}(M) = H(M) + mC_1(M) + nC_2(M) \), we obtain that equilibria \( M^1_{b,a} \) and \( M^2_{b,-a} \) are nonlinear stable under the sufficient conditions on \( \lambda_1 \) and respectively on \( a, b \):
\[
\lambda_1 + \lambda_4 > \lambda_2 + \lambda_3 \quad (3.1)
\]
and
\[
\frac{b^2}{a^2} > \frac{(\lambda_2 + \lambda_3)^2}{(\lambda_1 + \lambda_4)^2} \quad (3.2)
\]
These stability conditions were also obtained in [10] and they determine a subregion of the stability region given in Theorem 2.4.

Consider the energy function
\[
F_{mn}(M) = G_2(M) - G_4(M) + mC_1(M) + nC_2(M),
\]
where \( m, n \) are real numbers. Choosing \( m, n \) such that \( dF_{mn}(M^1_{b,a}) = 0 \) and taking into account that
\[
W := \ker dC_1(M^1_{b,a}) \cap \ker dC_2(M^1_{b,a}) = \text{span}(E_2, E_3, E_5, E_6)
\]
we obtain determinants associated with all upper-left submatrices of the Hessian \( d^2F_{mn}(M^1_{b,a})|_{W \times W} \)
\[
D_1 = -2 \cdot \frac{U_1}{(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)}; \\
D_2 = 4 \cdot \frac{U_1 \cdot U_2}{(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)^2(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2}; \\
D_3 = 8 \cdot \frac{U_2 \cdot U_3}{(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)^3(\lambda_2^2 - \lambda_3^2)^2(\lambda_2^2 - \lambda_3^2)(a^2 - b^2)^2}; \\
D_4 = 16 \cdot \frac{U_3^2(\lambda_2^2 - \lambda_3^2)}{(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)^4(\lambda_2^2 - \lambda_3^2)^4(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_4^2)(a^2 - b^2)^2};
\]
where \( U_1 = S_1 a^2 - T_1 b^2 \) with \( S_1 = \lambda_2^2 - \lambda_3^2 > 0 \) and \( T_1 = \lambda_2^2 - \lambda_4^2 > 0 \), \( U_2 = S_2 a^2 - T_2 b^2 \) with \( S_2 = (\lambda_2^2 - \lambda_3^2)(2\lambda_2^2 - \lambda_2^2 - \lambda_3^2) > 0 \) and \( T_2 = (\lambda_1^2 - \lambda_3^2)(\lambda_1^2 - \lambda_2^2) > 0 \), \( U_3 = S_3 a^2 - T_3 b^2 \) with \( S_3 = (\lambda_2^2 - \lambda_3^2)^3(2\lambda_2^2 - \lambda_2^2 - \lambda_3^2) > 0 \) and \( T_3 = (\lambda_1^2 - \lambda_3^2)(\lambda_2^2 + \lambda_2^2 - 2\lambda_3^2) \).

In order for \( d^2F_{mn}(M^1_{b,a})|_{W \times W} \) to be definite one needs that \( D_1 \cdot D_3 > 0, D_2 > 0 \) and \( D_4 > 0 \). Taking into account the expressions of the determinants it follows that \( S_3 a^2 - T_3 b^2 < 0 \), which implies that
\[
\lambda_2^2 + \lambda_3^2 - 2\lambda_4^2 > 0,
\]
and which is then equivalent to
\[
\frac{b^2}{a^2} > \frac{S_3}{T_3}.
\]

If \( d^2F_{mn}(M^1_{b,a})|_{W \times W} \) is negative definite, then additionally inequalities \( \frac{b^2}{a^2} < \frac{S_1}{T_1} \) and \( \frac{b^2}{a^2} < \frac{S_2}{T_2} \) must hold. But in this case \( \frac{S_1}{T_1} > \frac{S_2}{T_2} \) should hold. An easy computation shows us that
\[
\frac{S_2}{T_2} - \frac{S_3}{T_3} = -\frac{(\lambda_2^2 - \lambda_3^2)(2\lambda_2^2 - \lambda_2^2 - \lambda_3^2)(\lambda_1^2 - \lambda_3^2) + \lambda_2^2 - \lambda_3^2 - \lambda_2^2)}{(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)^2(\lambda_2^2 + \lambda_2^2 - 2\lambda_3^2)} < 0,
\]
which implies that \( \frac{S_3}{T_3} > \frac{S_2}{T_2} \), thus \( d^2F_{mn}(M^1_{b,a})|_{W \times W} \) cannot be negative definite.
If $d^2 F_{mn}(M^1_{b,a})_{|W\times W}$ is positive definite, then additionally inequalities $\frac{b^2}{a^2} > \frac{S_1}{T_1}$ and $\frac{b^2}{a^2} > \frac{S_2}{T_2}$ must hold. An easy computation shows us that

$$\frac{S_1}{T_1} - \frac{S_2}{T_2} = - \frac{(\lambda^3_2 - \lambda^3_1)(\lambda^2_1 - \lambda^2_2 + \lambda^2_3 - \lambda^2_4)}{(\lambda^2_1 - \lambda^2_2)(\lambda^2_1 - \lambda^2_2)} < 0$$

and then $\frac{S_1}{T_1} < \frac{S_2}{T_2}$, which proves that a condition for $d^2 F_{mn}(M^1_{b,a})_{|W\times W}$ to be (positive) definite is

$$\frac{b^2}{a^2} > \frac{S_3}{T_3},$$

but under the provision that this represents a non-empty set, i.e. $S_3 < T_3$, since $b < a$.

Because

$$T_3 - S_3 = (\lambda^3_1 - \lambda^3_2 + \lambda^3_3 - \lambda^3_4) [\lambda^2_1 \lambda^2_2 + \lambda^2_1 \lambda^2_3 - \lambda^2_1 - (2 \lambda^2_3 - \lambda^2_2 - \lambda^2_4) \lambda^2_1]$$

we obtain a second condition on $\lambda_1$, which is

$$\frac{\lambda^2_1 \lambda^2_2 + \lambda^2_1 \lambda^2_3 - \lambda^2_1 - (2 \lambda^2_3 - \lambda^2_2 - \lambda^2_4) \lambda^2_1}{2 \lambda^2_1 - \lambda^2_2 - \lambda^2_4} > \lambda^3_3. \tag{3.3}$$

But since

$$\frac{\lambda^2_1 \lambda^2_2 + \lambda^2_1 \lambda^2_3 - \lambda^2_1 - (2 \lambda^2_3 - \lambda^2_2 - \lambda^2_4) \lambda^2_1}{2 \lambda^2_1 - \lambda^2_2 - \lambda^2_4} = \frac{1}{2} \frac{(\lambda^3_2 - \lambda^3_1)^2}{2 \lambda^2_1 - \lambda^2_2 - \lambda^2_4} < 0$$

it follows that condition (3.3) is sufficient, that if $\frac{b^2}{a^2} > \frac{S_3}{T_3}$, then $d^2 F_{mn}(M^1_{b,a})_{|W\times W}$ is positive definite and hence the equilibrium $M^1_{b,a}$ (and similarly $M^1_{b,-a}$) in $t_1 \cap \text{Orb}_{c_1;G_2}$ is nonlinear stable.

Thus, using energy methods, we proved the following result:

**Theorem 3.1.** Assume that the condition

$$\frac{\lambda^2_1 \lambda^2_2 + \lambda^2_1 \lambda^2_3 - \lambda^2_1 - (2 \lambda^2_3 - \lambda^2_2 - \lambda^2_4) \lambda^2_1}{2 \lambda^2_1 - \lambda^2_2 - \lambda^2_4} > \lambda^3_3$$

is fulfilled.

If

$$\frac{b^2}{a^2} \in \left(\frac{S_3}{T_3}; \frac{S_3}{T_3} + 1\right), \tag{3.4}$$

then the equilibria $M^1_{b,a}$ and $M^1_{b,-a} \in t_1 \cap \text{Orb}_{c_1;G_2}$ are nonlinear stable.

It is important to notice that $\frac{b^2}{a^2} \in \left(\frac{S_3}{T_3}; \frac{S_3}{T_3} + 1\right)$ represents a subset of the stability region where equilibria $M^1_{b,a}$ and $M^1_{b,-a} \in t_1 \cap \text{Orb}_{c_1;G_2}$ were proved in [2] to be stable using Williamson-Bolsinov-Fomenko method.

In order to show that, we firstly notice that

$$\tilde{f} \left(\frac{S_4}{T_4}\right) = \frac{(\lambda^3 - \lambda^3)^2(-4 \lambda_3^2 \lambda_3^2 + 3 \lambda_2^2 \lambda_3^2 + \lambda_1^2 \lambda_3^2 + 3 \lambda_2^2 \lambda_3^2 + 3 \lambda_1^2 \lambda_3^2 - \lambda_2^4 - 2 \lambda_2^2 \lambda_4^2 - \lambda_4^2)}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 + \lambda_3^2 - 2 \lambda_4^2)^2} \geq 0$$

and cannot equal to 0, otherwise

$$\lambda^3_3 = \frac{3 \lambda_1^2 \lambda_2^2 - \lambda_2^4 + \lambda_1^2 \lambda_2^2 - 2 \lambda_2^2 \lambda_4^2 - \lambda_4^4}{4 \lambda_1^2 - 3 \lambda_2^2 - \lambda_4^2},$$

which from (3.3) would imply

$$\frac{\lambda^2_1 \lambda^2_2 + \lambda^2_1 \lambda^2_3 - \lambda^2_1 - (2 \lambda^2_3 - \lambda^2_2 - \lambda^2_4) \lambda^2_1}{2 \lambda^2_1 - \lambda^2_2 - \lambda^2_4} - \frac{3 \lambda_1^2 \lambda_2^2 - \lambda_2^4 + \lambda_1^2 \lambda_2^2 - 2 \lambda_2^2 \lambda_4^2 - \lambda_4^4}{4 \lambda_1^2 - 3 \lambda_2^2 - \lambda_4^2} > 0.$$
But the difference above proves to be equal with
\[-2 \cdot \frac{(\lambda_2^3 - \lambda_1^3) (\lambda_2^3 - \lambda_4^3)^2}{(2 \lambda_1^2 - \lambda_2^2 - \lambda_4^2) (4 \lambda_1^2 - 3 \lambda_2^2 - \lambda_4^2)} < 0.\]

The considerations above prove that the value $\frac{\bar{S}_3}{T_3}$ is not between the real positive distinct values $\alpha_1 < \alpha_2$.

In order to prove that $\frac{\bar{S}_3}{T_3} > \alpha_2$ we have to show that $\frac{\bar{S}_3}{T_3} + \frac{\bar{T}}{2S} > 0$.

We write
\[\frac{S_3}{T_3} + \frac{\bar{T}}{2S} = (\frac{S_3}{T_3} - \frac{S_1}{T_1}) + (\frac{S_1}{T_1} + \frac{\bar{T}}{2S}).\]

Further, we notice that
\[\frac{S_3}{T_3} - \frac{S_1}{T_1} = \frac{(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_4^2)(\lambda_2^3 - \lambda_3^3 + \lambda_4^3 - \lambda_4^3)}{(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2 - (\lambda_4^2 - \lambda_4^2))}\]
and
\[\frac{S_1}{T_1} + \frac{\bar{T}}{2S} = -2 \cdot \frac{(\lambda_2^2 - \lambda_1^2)(\lambda_2^3 - \lambda_4^3)}{(\lambda_1^2 - \lambda_4^2)^2}.\]

Thus we have to prove that
\[\frac{(\lambda_2^2 - \lambda_1^2)(\lambda_2^3 - \lambda_4^3)(\lambda_2^3 - \lambda_3^3 + \lambda_4^3 - \lambda_4^3)}{\lambda_2^2 - \lambda_3^2 - (\lambda_4^2 - \lambda_4^2)} > 2(\lambda_2^3 - \lambda_4^3)(\lambda_3^3 - \lambda_4^3).\]

Denoting $A := \lambda_2^3 - \lambda_1^3 > 0$, $B := \lambda_2^3 - \lambda_4^3 > 0$, $C := \lambda_2^2 - \lambda_3^2 > 0$, $D := \lambda_3^2 - \lambda_4^2 > 0$ it reduces to
\[\frac{BC(A + D)}{B - D} > 2AD.\]

Noticing that $B = C - D$ the above inequality is equivalent with
\[A(C^2 + 4D^2 - 3CD) + BCD > 0,\]
which is obviously true. Thus, we have proved that $\frac{\bar{S}_3}{T_3} > \alpha_2$ and consequently our remark above.

Let us also remark that hypothesis $\lambda_1^3 + \lambda_4^3 = \lambda_2^3 + \lambda_3^3$ from Theorem 2.5 (where a stability region does not appear) cannot hold simultaneously with condition (5.3), since
\[\frac{\lambda_2^3 \lambda_3^3 + \lambda_1^3 \lambda_3^3 - \lambda_4^3 - \lambda_4^3}{2\lambda_1^2 - \lambda_2^2 - \lambda_4^2} = (\lambda_1^2 + \lambda_2^2 - \lambda_4^2) = -2 \cdot \frac{\lambda_2^3 - \lambda_4^3}{2\lambda_1^2 - \lambda_2^2 - \lambda_4^2} < 0.\]

We notice that, assuming conditions (5.1) and (5.3) on $\lambda_i$ simultaneously fulfilled, it is impossible to generally decide which of the two regions designated by (5.2) and (5.4) is larger.

A similar result can be obtained using as energy function $F_{mn}(M) = G_1(M) - G_3(M) + nC_1(M) + nC_2(M)$.

**Acknowledgments.** The authors dedicate this work to the memory of our dear Professor Mircea Puta. The authors thank Professors Tudor Rațiu and Daisuke Tarama for useful discussions on the topics of this paper.
P. Birtea has been supported by PN 2 ID 1081 and PN 2 ID 131 grants.
References


