A note on $n$-tuple colourings and circular colourings of planar graphs with large odd girth

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Note

A note on \(n\)-tuple colourings and circular colourings of planar graphs with large odd girth

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The existence of planar graphs with odd girth \(2k + 1\) and high girth that cannot be \((2k + 1, k)\)-coloured was left as an open question by Klostermeyer and Zhang. In this note we show that such graphs exist for arbitrarily large \(k\). We also show that these graphs have fractional chromatic number greater than \(2 + 1/k\).

Keywords: Circular colouring; Multicolouring; Fractional colouring; Planar graphs: Odd girth

C.R. Categories: G.2.2 Graph theory

1. Introduction

Let \(G\) be a simple, undirected graph. An \((n, k)\)-circular colouring of \(G\) is a mapping \(c: V(G) \rightarrow \{0, \ldots, n - 1\}\) such that \(k \leq |c(u) - c(v)| \leq n - k\) for every edge \(uv \in E(G)\). A graph having such a colouring is \((n, k)\)-colourable. An \((n, k)\)-circular colouring can also be viewed as a homomorphism to the graph \(G(k, n)\), where graph \(G(k, n)\) is a cycle on \(n\) vertices with additional edges between vertices \(u\) and \(v\) of \(C_n\) iff the distance \(d(u, v) \geq k\). An \((n, k)\)-multicolouring of \(G\) is a mapping \(C: V(G) \rightarrow \mathcal{P}_k(\mathbb{Z}_n)\), where \(\mathcal{P}_k(\mathbb{Z}_n)\) is the collection of all \(k\)-subsets of \(\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}\), such that \(C(u) \cap C(v) = \emptyset\) for \(uv \in E(G)\). An \((n, k)\)-multicolouring can also be viewed as a homomorphism to the Kneser graph \(G^n_k\). A graph having such a multicolouring is \((n, k)\)-multicolourable. The concept of \((n, k)\)-colouring and \((n, k)\)-multicolouring is a generalization of the conventional vertex colouring problem, in which \(k = 1\) in both cases.

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The circular chromatic number \( \chi_c(G) \) of \( G \) is the minimum of \( n/k \) over all pairs \( (n, k) \) such that \( G \) is \( (n, k) \)-colourable. Note that if \( \chi_c(G) = r \) this is equivalent to the existence of an \( r \)-circular colouring \( c : G \to C \), where \( C \) is a cycle of length \( r \), which assigns to each vertex \( v \) of \( G \) an open unit length arc \( c(v) \) of \( C \), such that \( c(u) \cap c(u) = \emptyset \) for every edge \( uv \in E(G) \). This is why the name circular chromatic number was introduced. Namely, in the original definition, given by Vince in 1988 [1], the name of \( \chi_c(G) \) was “the star-chromatic number”. For a more general definition of circular colourings of edge weighted graphs see [2]. The \( \chi_c(G) \) is often called a refined measure of colouring because \( \chi(G) - 1 < \chi_c(G) \leq \chi(G) \) for every infinite graph \( G \). For the proof and other results on circular chromatic number see [3]. The fractional chromatic number \( \chi_f(G) \) is the minimum of \( n/k \) over all pairs \( (n, k) \) such that \( G \) is \( (n, k) \)-multicolourable [4]. It is not difficult to see that \( \chi_f(G) \leq \chi_c(G) \) for any finite graph \( G \). Namely, if \( \chi_c(G) = n/k \) then, by definition, there exists an \( (n, k) \)-colouring \( c \) of \( G \) such that \( k \leq |c(u) - c(v)| \leq n - k \) for every edge \( uv \in E(G) \). One can easily see that by assigning colours \( c(v), c(v) + 1, \ldots, c(v) + k - 1 \) to every vertex \( v \) in \( G \), where the sum is taken by modulo \( n \), we obtain an \( (n, k) \)-multicolouring \( C \) of \( G \). Therefore \( \chi_f(G) \leq n/k = \chi_c(G) \).

Recently, a lot of work on the chromatic number of planar graphs with large odd girth has been done. The odd girth of a graph is the length of its shortest odd circuit. In [5] the authors proved that every planar graph \( G \) of odd girth at least \( 10k - 7 \) has a homomorphism to the Kneser graph \( G_{2k+1}^{2k+1} \), i.e. \( \chi_c(G) \leq 2 + 1/k \). Zhu [6] proved that every planar graph \( G \) of odd girth at least \( 8k - 3 \) has \( \chi_c(G) \leq 2 + 1/k \). The same result was obtained by G. Fijavž, M. Juvan, B. Mohar, R. Škrekovski: Circular colorings of planar graphs with prescribed girth (unpublished manuscript) at about the same time. Later, the authors of [7] proved that if a planar graph has girth at least \((20k - 2)/3\), then \( \chi_c(G) \leq 2 + 1/k \). To the best of our knowledge, this is the sharpest bound so far.

In [5] the existence of planar graphs with odd girth \( 2k + 1 \) and high girth that cannot be \((2k + 1, k)\)-multicoloured is left as an open question. In this note we show that such graphs exist for arbitrarily large \( k \) and even more, we show that the fractional chromatic number of these graphs is greater than \( 2 + 1/k \).

2. The main result

CONSTRUCTION Let \( n = 2k + 1 = 3l \) (note that \( l \) must be an odd number). Take four vertices \( o, a, b \) and \( c \). Connect each pair of vertices with a path of length \( l \). The graph obtained is shown in figure 1, where \( P_l \) denotes a path of length \( l \). Clearly, the girth of the resulting graph \( G_l \) is \( 3l = 2k + 1 \).

PROPOSITION For the constructed graph \( G_l \) it holds that \( \chi_f(G_l) > 2 + 1/k \).

![Figure 1. Graph \( G_l \).](image)
According to one of the referees Matt DeVos showed at a seminar in Vancouver (2000) that the sets used before. But, considering that exactly 3 vertex vi for each positive integer k must share exactly one colour, hence (again w.l.o.g.) we can assign 5 to A and C, and to B and C, respectively. By the observation each of the sets A, B and C must intersect O in exactly (l – 1)/2 colours. Furthermore, each pair of the sets A, B, and C must share exactly (l – 1)/2 colours. But this is not possible, because of a simple counting argument which follows.

The colours of O contribute exactly (l – 1)/2 = (k – 1)/3 to each of the sets A, B and C. Exactly 3(k – 1)/3 = k – 1 colours can be used twice reducing the total demand by 2(k – 1). The sets A, B and C thus need 3k – 3(k – 1)/3 – 2(k – 1) = 3 colours, which have not been used before. But, considering that a used k colours and a, b and c together used 2(k – 1), there are only 2k + 1 – k – (k – 1) = 2 colours available. Therefore, Gi is not (2k + 1, k)-multicolourable and thus \( \chi_f(G_i) > 2 + 1/k \).

\[ \square \]

Example The graph \( G_3 \) of figure 2 cannot be properly (9, 4)-coloured. Without loss of generality assign colours 1, 2, 3, 4 to O and let 1 \( \in \) A, 2 \( \in \) B and 3 \( \in \) C. The sets A, B, C must share exactly one colour, hence (again w.l.o.g.) we can assign 5 to A and to B, 6 to B and to C and 7 to C and to A. We need one unused colour for each of the sets A, B, and C, but the only colours available are 8 and 0.

3. Concluding remark

According to one of the referees Matt DeVos showed at a seminar in Vancouver (2000) that for each positive integer k, there is a family of planar graphs of girth \( 2k + 1 \), that have circular chromatic number greater than \( 2 + 1/k \). The graphs are constructed as follows: take any odd cycle \( C = (v_0, v_1, \ldots, v_{2m}) \) of length \( 2m + 1 \geq 2k + 1 \). Add a vertex u and connect u to each vertex \( v_i \) of C by a path of length k. Denote this graph by \( G(k, m) \), see figure 3.

It can be shown that also for graphs \( G(k, m) \) the inequality \( \chi_f(G(k, m)) > 2 + 1/k \) holds.

\[ \square \]

Case 1 \( k = 2t \) is even. In this case we need an observation different from the one in the note, but of the same flavour: Let \( f \) be a proper \((2k + 1, k)\)-multicolouring of \( C_{2k+1} \) and let \( u, v \) be two vertices of even distance l. Then \( |f(u) \cap f(v)| = k – l/2 \).
By this observation we have $|f(u) \cap f(v_i)| = k - k/2 = t$. Let $X_i = f(u) \cap f(v_i)$, which implies $X_i \cap X_{i+1} = \emptyset$ and $|X_i \cup X_{i+1}| = 2t = k$. Considering the last two equalities, we have $X_i \cup X_{i+1} = f(u)$. Therefore, $X_1 \cup X_2 = X_2 \cup X_3 = \cdots = X_{2m-1} \cup X_{2m}$. Hence $X_1 = X_3 = \cdots = X_{2m-1}$ and $X_0 = X_2 = \cdots = X_{2m}$. But this is a contradiction, as $f(v_0) \cap f(v_{2m})$ should be empty.

**Case 2** $k = 2t + 1$ is odd. In this case the same observation as the one in section 2 implies $|f(u) \cap f(v_i)| = (k - 1)/2 = t$. The same argument as the one in the Case 1 leads to the same contradiction.  

## 4. Future work

The result in this note constructs planar graphs of girth $2k + 1 = 3l$ for odd integers $l$. This takes care of only $1/3$ of the odd integers. It still remains open whether there exist planar graphs with odd girth $2k + 1 = 6l \pm 1$ for $l \in \mathbb{N}$ that cannot be $(2k + 1, k)$-multicoloured (for example, is there a planar graph of girth 7 which cannot be $(7, 3)$-multicoloured?).

Improvement of the bound given in [7] is another interesting avenue for future research.

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