GRAPHS CRITICALLY EMBEDDED ON RIEMANN SURFACES AND
IHARA-SELBERG FUNCTIONS: GENUS ONE CASE

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Abstract. Let $G = (V, E)$ be a bipartite graph embedded in the torus, each degree of $G$ being two or four, and let the embedding be critical. We define, using the geometric information on this embedding, $(|V| \times |V|)$ matrix $\Delta$. We show that the generating function of the even sets of edges of $G$ (i.e., the Ising partition function) is a linear combination of the square roots of 4 signed Ihara-Selberg functions determined by $\Delta$ and the spin structures of the torus.

1. Introduction

It is well understood (see e.g. [m], [k], and below) how to define the notion of critical embedding of a finite graph on an orientable surface, in a way so that one can derive, from the collection of the angles of $\theta$ of a conical singularity in $H$ of the embedding called discrete conformal structure and graphs $G$, the critical values of coupling constants (edge-weights) of the dimer problem and Ising problems on $G$.

It is an attractive task to see whether some properties associated with the notion of criticality may also be derived from the collection of data attached to critical embedding.

The aim of this paper is to propose a discrete analog of the claim made by Alvarez-Gaume et all (see [a]) that the partition function of free fermion on a closed Riemann surface of genus $g$ is a linear combination of $2^{2g}$ Pfaffians of Dirac operators.

1.1. Critical embedding. We follow [m] in this subsection. Let us consider a triple $(X, H, \varphi)$, where $X$ is a smooth closed Riemann surface, $H$ a graph and $\varphi : H \hookrightarrow X$ an embedding. It defines a CW decomposition of $X$ and $X \setminus \varphi(H)$ is a disjoint union of (open) faces. We fix an atlas $\{U_i, \varphi_i : U_i \to \mathbb{C} \mid \bigcup_{i} U_i = X\}$ on $X$.

We choose the realization of $X$ as follows. For a Riemann surface $X$ of genus $g$ and a finite set of points $\{P_j\}$ on $X$ there exists a flat metric with conical singularities at $\{P_j\}$ such that the cone angles $\theta_{P_i}$ fulfill the Gauss-Bonet formula $\chi(X) = \sum_j (1 - \frac{\theta_{P_j}}{2\pi})$. Notice that, due to the eveness of Euler characteristic $\chi(X)$, we can choose $\theta_{P_j}$ to be an odd multiple of $2\pi$ for each $j$. Recall that the local model for a neighborhood of a conical singularity in $P_j$ is the standard cone $C(\theta) := \{(r, \beta) | r \in \mathbb{R}^+, \beta \in \mathbb{R}/\theta\mathbb{R}\}/\{(0, \beta) \sim (0, \beta')\}$ with the metric $g_{C(\theta)} := (dr)^2 + r^2(d\beta)^2$.

An embedding of a graph $H$ on a surface $X$ defines the dual graph $H^*$ (of the embedding). Note that $H^*$ is an abstract graph and it has a natural embedding on $X$, so that each vertex of $H^*$ lies on the face of the embedding of $H$ it represents. A basic notion is that of the diamond graph: Given simultaneous embeddings of $H$ and $H^*$, the diamond graph $H'$ has the vertex-set equal to $V(H) \cup V(H^*)$ and the edges connecting the end-vertices of each dual pair of edges $e, e^*$ into a (facial) cycle $F(e)$ which is a 4–gon (see Figure 1). The diagonals of such a 4–gon are formed by the corresponding dual pair of edges. We are ready to define a critical embedding.

Definition 1.1. The embedding $\varphi$ of $H'$ is called critical embedding if each of its faces $F(e), e \in E(H)$ is rhombus, i.e., the following conditions hold true (for all $i$) with respect to the induced conformal class of metrics on $X$:

1. The diagonals of each rhombus $F(e), e \in E(H)$ (in $\text{Im}(\varphi_i \circ \varphi)$) are perpendicular,
2. The lengths of sides of all rhombi $F(e), e \in E(H)$ (in $\text{Im}(\varphi_i \circ \varphi)$) are the same.
In particular, the first property is independent of the choice of trivialization, i.e. of the index $i$ together with $\text{Im}(\phi_i \circ \phi)$. The reason is that the transition maps $\phi_i \circ \phi_j^{-1}$ are conformal and so angle preserving. The second condition already depends on the choice of a metric in a given conformal class of metrics $[g]$. In the realization of the Riemann surfaces we choose above, the critical embedding reduces to the so called isoradial embedding, i.e. the vertices of any face of $H$ lie on an ordinary circle in any local trivialization (complex plane). The same property holds for $H^*$ as well.

1.2. Dimer and Ising partition functions. The theory of free fermion is considered to be closely related to the criticality of both the dimer and the Ising problems. Recently, the dimer problem and its determinant-type solution received considerable attention, see e.g. [K], [Q], mainly because the usual solution to the finite Ising problem goes along the lines to the reduction of dimer problem anyway. However, the results obtained so far for the determinant-type reduction of the critical dimer problem have strong limitations. This topic, briefly reviewed in the next section, is centered around the notion of Kasteleyn flatness.

In this paper we concentrate on the Ising partition function and propose to overcome these limitations of the determinant-type reasoning in replacing the determinants by so called Ihara-Selberg functions.

We will restrict to bipartite graphs $G = (W, B, E)$, where $W, B$ are the two edge-less classes of $G$ and $V = W \cup B$. We call the vertices in $W$ white and the vertices in $B$ black and assume that $G$ has at least one perfect matching. In particular, we restrict to the case of equal cardinalities $|W| = |B|$ from now on. We also associate a variable $x_e$ with each edge $e$ of $G$ and define the dimer partition function of $G$ by

$$\mathcal{P}(G, x) = \sum_{M \text{ perfect matching of } G} \prod_{e \in M} x_e,$$

where $x = (x_e)_{e \in E}$.

A subset $E' \subset E$ of edges is called even if the graph $(V, E')$ has each vertex of even degree.

The generating function of the even sets of edges of $G$ is defined by

$$\mathcal{E}(G, x) = \sum_{E' \subset E \text{ even}} \prod_{e \in E'} x_e.$$

It is well known that $\mathcal{E}(G, x)$ is equivalent to the Ising partition function on $G$.

1.3. Main results. As we already mentioned, in this paper we focus our attention to Ising partition function $\mathcal{E}(G, x)$ instead of the dimer partition function $\mathcal{P}(G, x)$, and the Ihara-Selberg function of a graph instead of the determinant.

We have been motivated by two older results. The first, appearing in the paper [RS], shows that in the case of the torus as an example of Riemann surface, the determinant of the Dirac operator is an object which may be viewed as a continuous analogue of the Ihara-Selberg function of a graph. Secondly, the paper [L] extends the Feynman’s and Sherman’s solution (see [S]) to the planar Ising problem and shows that for a
graph $G$ embedded on an orientable surface $X$ of genus $g$, and satisfying that each degree of $G$ is equal to 2 or 4, the Ising partition function $\mathcal{E}(G, x)$ is a linear combination of $4^g$ Feynman functions.

In the section 2 we start by introduction of the Ihara-Selberg function associated to $1$–graph, pass to the Feynman function and recall the result of \cite{[i]} for the case of Riemann surface of genus 1. Each Feynman function is defined by means of the rotation (denoted by $\text{rot}(p)$) of closed curves $p$ in the plane corresponding to certain closed walks called prime reduced cycles in the $1$–graph coming out of the toroidal embedding. In the case of unrestricted embedding we may need a vector $w$ of length $3|E|$ containing information about the embedding to write down terms $c(e)$, $e$ edge of $G$, so that for each prime reduced cycle $p$, $(-1)^{\text{rot}(p)}$ is a product of $c(e)$ over the edges $e$ traversed by $p$. However, in the case of critical embedding we show that the vector of length $|E|$, namely the discrete conformal structure, is sufficient. In particular we show in the final section that a critical embedding of a graph on the torus enables to turn the corresponding Feynman functions into the square roots of the Ihara-Selberg functions. As a corollary we obtain the main result:

**Theorem 1.** Let $G$ be a graph critically embedded on the torus and each degree of $G$ be 2 or 4. Then $\mathcal{E}(G, x)$ is a linear combination of the square roots of four Ihara-Selberg functions defined by the embedding. Moreover, these four Ihara-Selberg functions naturally correspond to the four spin-structures of the torus.

The proof strongly relies on our assumption that $G$ is bipartite, in particular on the orientation of $G$ where each vertex is a source or a sink. However, we believe that the restriction on the degree of $G$ being two or four should be possible to overcome. It also seems that the analogous result is true for general orientable surfaces; this is the subject of current investigation. The problem of infinite product expansions for theta functions beyond the genus 1 is to our best knowledge not completely clear and hence it might happen that the extension of Theorem \cite{i} to orientable surfaces $X$ of genus $g > 1$ need not enlighten the results of \cite{[i]}.

1.4. **Acknowledgement.** The first author would like to thank to Gregor Masbaum and David Cimasoni for enlightening discussions.

2. Dimers at criticality and determinants

In this section we briefly explain the existing results close to our line of reasoning for the dimer partition function.

2.1. **Kasteleyn curvature.** By a cycle in $G$ we mean a subset of edges $C \subset E(G)$ which form a cycle. Such $C$ can be decorated by one of the two possible orientations, i.e. one of the two possible ways of going around $C$. For example, if $C$ is bounding a face in a plane embedded $G$ these two orientations are called clockwise and anticlockwise. By oriented cycle we mean a cycle decorated with an orientation. We also denote by $D$ the orientation of $G$ in which each edge is oriented from its black vertex to its white vertex. The definition of Kasteleyn curvature follows \cite{[ku]}.

**Definition 2.1.** Let $G$ be given with the weights $w(e), e \in E(G)$. Let $C$ be an oriented cycle of $G$. We define

$$c(C) = (-1)^{|C|/2} \prod_{e \in C_+} w(e) \prod_{e \in C_-} w(e)$$

as the Kasteleyn curvature of $C$. Here $C_+$ denotes the subset of edges of $C$ whose orientation inherited from $C$ coincides with their orientation in $D$, and $C_- = C \setminus C_+$.

**Definition 2.2.** Let $G$ be a weighted graph, $w(e), e \in E(G)$, embedded in $X$. We say that $G$ is Kasteleyn flat if $c(F) = 1$ for each face $F$ of the embedding, arbitrarily oriented.

Let us fix a linear ordering on the set $B \cup W$ such that the elements of $B$ precede the elements of $W$. This allows to introduce $\Delta(w)$ as $(B \cup W) \times (B \cup W)$ skew-symmetric matrix defined by $\Delta(w)_{uv} = w(uv)x_e$ if $u \in B$ and $e = uv \in E(G)$ resp. $\Delta(w)_{uw} = -w(uw)x_e$ if $u \in W$ and $e = uv \in E(G)$, and $\Delta(w)_{uw} = 0$ otherwise. We further denote by $\Delta_B(w)$ the $(B \times W)$–block of $\Delta(w)$.

For $M$ a perfect matching of $G$ we denote $t(M)$ the coefficient of $\prod_{e \in M} x_e$ in $\det(\Delta_B(w))$. Note that $t(M) = \text{sign}(M) \prod_{e \in M} w(e)$, where $\text{sign}(M)$ is the sign of $M$ relative to the fixed linear ordering on the set of vertices $B \cup W$.

The significance of flatness of the Kasteleyn curvature rests on the following observation (see \cite{[ku]}).
Proposition 2.3. Let $C$ be a directed cycle of $G$ embedded on $X$ so that $C$ is a symmetric difference of two perfect matchings $M,N$ and the orientation of $C$ coincides with the orientation of the edges of $M$ in $D$. Then

$$\frac{t(M)}{t(N)} = c(C).$$

Proof. The proof is based on the fact that $\text{sign}(M) \neq \text{sign}(N)$ if and only if $C$ has length divisible by 4. □

This result leads to (see also [ku])

Proposition 2.4. If $G$ is embedded in the plane and its weight-function is Kasteleyn flat then $t(M)$ is constant.

We are able to prove a stronger statement. Let $G$ be embedded in a Riemann surface $X$ of genus $g$. We say that the weight-function $w$ is simple Kasteleyn flat if it is Kasteleyn flat and moreover $w(e) \in \{1,-1\}$ for each $e \in E$. The function $w$ may be viewed as assigning orientation $D(w)$ obtained from $D$ by reversing the orientation of all edges with negative weight. We note that such an orientation $D(w)$ is a Kasteleyn orientation, i.e. each face has an odd number of edges oriented clockwise.

Inspired by [cr], we say that two weight functions $w,w'$ are equivalent if $w$ may be obtained from $w'$ by executing operations of vertex multiplication. The operation of vertex multiplication at vertex $v$ of $G$ consists of multiplying the weight of each edge incident with $v$ by the same complex number. We note that the vertex multiplication cannot change the Kasteleyn curvature of an oriented cycle.

The following theorem is well known (see [21]).

Theorem 2. Let $G$ be a graph embedded in a Riemann surface $X$ of genus $g$ and $w$ be a simple Kasteleyn flat weight function. Then $P(G,x)$ is a linear combination of the determinants of $4^g$ signings $S_i$ of $\Delta_B(w)$.

This Theorem can be generalized in a rather simple way.

Proposition 2.5. Let $G$ be a graph embedded in a Riemann surface $X$ of genus $g$ and $w$ be a Kasteleyn flat weight function, satisfying in addition $c(C) \in \{1,-1\}$ for each cycle $C$. Then $w$ is equivalent to a simple Kasteleyn flat weight-function $w'$, i.e., to a Kasteleyn orientation.

Proof. We may assume that $G$ is connected. Let $T = (V(G),E')$, $E' \subset E(G)$ be a spanning tree of $G$. We can clearly perform a vertex multiplications (in $G$) in a way the resulting weight function $w'$ satisfies $w'(e) = 1$ for each $e \in E'$. Let $e \in E \setminus E'$. Then necessarily $w'(e) \in \{1,-1\}$ since $e$ makes a cycle (here denoted by $C$) containing $E'$ and $c(C)$ has values in $\{1,-1\}$. □

Proposition 2.5 along with Theorem 2 immediately implies

Corollary 2.6. Let $G$ be as in the previous Theorem and $w$ be a Kasteleyn flat weight function satisfying in addition $c(C) \in \{1,-1\}$ for each cycle $C$. Then $P(G,x)$ is a linear combination of $4^g$ signings of $\Delta_B(w)$.

2.2. Kasteleyn curvature and criticality. The computation of the dimer partition function $P(G,x)$ for a critically embedded graph $G$ is based on the following strategy (see [K], [Cr]): define the weight function $w$ using certain discrete geometric information contained in the data of critical embedding, prove that $w$ is Kasteleyn flat and insert it into Corollary 2.4. This approach works well for the planar graphs $G$ (see [K]), on the other hand the conditions of Corollary 2.4 are quite restrictive for non-planar surfaces.

As an illustration we construct such a weight function, from now on denoted by $K$. Note that analogous constructions presented in [K], [Cr] are different from ours. Let us assume that the dual graph $G^* = (W^*,B^*,E^*)$ (for the embedding of $G$) is bipartite as well. We recall that $G$ has the referential orientation $D$ where each edge is oriented from its black vertex to its white vertex. Analogously the dual graph $G^*$ has the referential orientation denoted by $D^*$: each dual edge is oriented from its vertex in $B^*$ to its vertex in $W^*$.

There are two ways how a dual pair of edges crosses in a critical embedding (see Figure 2). We call the edge $e$ of $G$ positive if its crossing is positive, and we call $e$ negative otherwise.

We assing the weight $K(e) = e^{i\alpha_e}$ to each positive edge $e$ of $G$, and similarly $K(e) = e^{i(\pi - \alpha_e)}$ for each negative edge $e$ of $G$. Clearly $\alpha_e = \arccotan(\rho_e)$ and $\rho_e = \frac{1}{d(e)}$, where $l$ denotes the length on $X$ (see Figure 2). This weight-function $K$ will be called critical.
Definition 2.7. The collection \( \{\alpha_e; e \in E\} \) introduced above is called the *discrete conformal structure*.

The weight function \( K \) has a special property.

**Proposition 2.8.** Let \( G \) be critically embedded in a Riemann surface \( X \) and \( K \) be the critical weight function on edges of \( G \). Then \( K \) is Kasteleyn flat.

**Proof.** Let \( F \) be a face of \( G \). Note that the positive and negative edges along \( F \) alternate. Hence we may assume, without loss of generality, that the set of the positive edges of \( F \) equals \( F_+ \) (see Definition 2.1). We have

\[
c(F) = (-1)^{|C|/2+1} \prod_{e \in C_+} e^{i\alpha_e} = (-1)^{|C|/2+1} \prod_{e \in C_-} e^{i(-\alpha_e)} = (-1) \prod_{e \in C} e^{i\alpha_e} = 1.
\]

In the last equality we used the flatness condition for the dual vertex corresponding to \( F \).

The condition \( c(C) \in \{1, -1\} \) for each cycle \( C \) in Corollary 2.6 is disappointingly strong. It is, by Proposition 2.5, equivalent to \( w(e) \in \{1, -1\} \) for each edge \( e \in E \) and so the details for the critical embedding seem to be lost. Moreover, a discouraging example by David Cimasoni shows that this approach cannot go beyond Corollary 2.6.

**Example 2.9.** Let \( G \) be the toroidal \( 2n \times 2n \) square grid with the set of vertices \( i, j, i, j = 1, \ldots, 2n \), \( i_j \) black, and the set of edges \( \{i, j_{i+1}\}, \{i, (i+1)j\} \). Here the addition is in residue classes modulo 2n. Let \( e_t = (i_1, i_2) \).

We define the partial weight function \( w' \) by \( w'(e_{2l}) = \gamma \) and \( w'(e_{2l+1}) = \gamma^{-1} \), \( l = 1, \ldots, n \), \( \gamma \neq \pm 1 \). It is a simple fact that \( w' \) may be extended to a Kasteleyn flat weight function \( w' \) satisfying \( w(e) \in \{1, -1\} \) for each \( e \notin \{e_t; \ i = 1, \ldots, n\} \).

Consequently, we get:

- Let \( F \) be a face of \( G \). Then \( c(F) = 1 \) since \( w \) is Kasteleyn flat.
- Let \( H = C^1 \cup C^3 \), where \( C^i \) is the cycle with the vertices \( i_j; j = 1, \ldots, 2n \). Let us orient \( C^i, i = 1, 3 \) according to the orientation of edge \( e_i; i = 1, 3 \). Then \( c(H) = \gamma^2 \).

Clearly both \( F \) and \( H \) are \( \mathbb{Z}_2 \)-homologous to 0, but \( c(F) \neq c(H) \); this means that one cannot expect the Corollary 2.6 holds true.

### 3. Criticality and the Ihara-Selberg function of a graph

In this section we suggest to consider the Ising partition function \( \mathcal{E}(G, x) \) instead of the dimer partition function \( \mathcal{P}(G, x) \), and the Ihara-Selberg function of a graph instead of the determinant.

**3.1. Ihara-Selberg function.** Let \( G = (V, E) \) be a graph. For \( e \in E \) we denote by \( a_e \) an orientation of \( e \), and \( a_e^{-1} \) the reversed directed edge to \( a_e \). As above, let \( x = (x_e)_{e \in E} \) be formal variables associated with edges of \( G \). If \( a \) is any orientation of the edge \( e \), we associate new variable \( x_{ae} \) with it but always let \( x_a = x_e \).

A circular sequence \( p = v_1, a_1, v_2, a_2, \ldots, a_n, v_n+1 \) and \( v_{n+1} = v_1 \) is called a *prime reduced cycle* if the following conditions are satisfied: \( a_i \in \{a_e, a_e^{-1}; e \in E\} \), \( a_i \neq a_{i+1}^{-1} \) and \( (a_1, \ldots, a_n) \neq Z^m \) for some sequence \( Z \) and natural number \( m > 1 \).
Definition 3.1. Let $G = (V, E)$ be a graph and assume that the vertex set $V$ is linearly ordered. Let $B$ be $V \times V$ matrix with entries $b(i, j)$, where we think of $b(i, j)$ as the weight of directed edge $(i, j)$ of $G$. We denote the set of prime reduced cycles of $G$ by $G$. The Ihara-Selberg function associated to $G$ is

$$I(G, x) = \prod_{\gamma \in G} (1 - B(\gamma))$$

where the infinite product is defined by the formal power series:

$$\prod_{\gamma \in G} (1 - B(\gamma)) = \sum_{\Sigma} (-1)^{|\Sigma|} \prod_{\gamma \in \Sigma} \prod_{e \in \gamma} b(e)x_e,$$

and the sum is over all finite subsets $\Sigma$ of $G$.

A well known theorem of Bass (see [b]) reads

Theorem 3. (Bass) For any graph

$$I(G, x) = \det(I - T),$$

where $T$ is the matrix of transitions between directed edges defined as follows: Let $a, a' \in \{a_e, a_e^{-1} : e \in E\}$. If the terminal vertex of $a$ is the initial vertex of $a'$ and $a' \neq a^{-1}$ then $T_{a, a'} = b(a)x_a$, otherwise $T_{a, a'} = 0$.

3.2. Torus. Here we recall for our purposes more suitable realization (then the one outlined in the introductory part) of Riemann surfaces of genus one. We regard the torus $T$ as a rectangle $R$ in the plane with the pairs of its parallel both vertical and horizontal sides identified. We denote $h_1$, $h_2$ the horizontal sides of $R$ and $v_1$, $v_2$ the vertical sides of $R$. We declare the graph $G$ is critically embedded in $T$ if no vertex is embedded on $h_1 \cup h_2 \cup v_1 \cup v_2$ and the periodic extension of the embedding to the plane is critical as defined in [1.1]. Given such a critical embedding of graph $G$ on the torus, we construct its 1−graph $G'$ obtained by projecting the edges which cross a side of $R$ to the plane outside $R$ in a way indicated by Figure 3. We note that $G'$ is a drawing of $G$ in the plane with edge-crossings.

3.3. Feynman functions. Let $G$ be a graph with each degree equal to 2 or 4 and critically embedded on the torus and let $G'$ be its 1−graph. We note that the embedding of each edge of $G'$ is piece-wise linear. Let $p$ be a prime reduced cycle of $G'$. We need to introduce the rotation of $p$. In analogy with the usual definition of the rotation of a regular closed curve in the plane we define it by setting $\text{rot}(p) = \sum_t y(t)$, where the sum is over all the transitions of the linear parts of the edges of $p$. If the transition $t$ consists in passing from directed segment $e$ to directed segment $e'$ then $y(t) = z(t)(2\pi)^{-1}$ for $z(t)$ the angle of the transition. The angle $z(t)$ is negative if the transition is clockwise, and $z(t)$ is positive if the transition is anti-clockwise (see Figure 4).

We denote by $p^{-1}$ the prime reduced cycle which is the inverse of $p$. It clearly satisfies $(-1)^{\text{rot}(p)} = (-1)^{\text{rot}(p^{-1})}$. 

Figure 3. Projections of edges crossing the horizontal sides and the vertical sides of $R$. 

Figure 4.
We introduce equivalence relation on the set of prime reduced cycles of $G'$: we say that $p$ is equivalent to $p'$ if $p' = p^{-1}$. The set of the equivalence classes of this equivalence relation is denoted by $[G]$.

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{figure4.png}
\end{array}
\]

**Figure 4.** Angle of the transition.

For each edge $e$ of $G'$ we denote by $v(e)$ resp. $h(e)$ the number of times $e$ crosses the pair of vertical resp. horizontal sides of the rectangle $R$. We define in complete analogy with Definition 3.1

\[
\prod(G, x) = \sum_{\Sigma} (-1)^{|\Sigma|} \prod_{[\gamma] \in \Sigma} (-1)^{\text{rot}(\gamma)} \prod_{e \in \gamma} x_e,
\]

where the sum is over all finite subsets $\Sigma$ of $[G]$. We also set

\[
\prod_{\emptyset}(G, x) = \sum_{\Sigma} (-1)^{|\Sigma|} \prod_{[\gamma] \in \Sigma} (-1)^{\text{rot}(\gamma)} \prod_{e \in \gamma} x_e,
\]

\[
\prod_{v}(G, x) = \sum_{\Sigma} (-1)^{|\Sigma|} \prod_{[\gamma] \in \Sigma} (-1)^{\text{rot}(\gamma)} \prod_{e \in \gamma} v(e) x_e,
\]

\[
\prod_{h}(G, x) = \sum_{\Sigma} (-1)^{|\Sigma|} \prod_{[\gamma] \in \Sigma} (-1)^{\text{rot}(\gamma)} \prod_{e \in \gamma} h(e) x_e,
\]

and

\[
\prod_{vh}(G, x) = \sum_{\Sigma} (-1)^{|\Sigma|} \prod_{[\gamma] \in \Sigma} (-1)^{\text{rot}(\gamma)} \prod_{e \in \gamma} (v(e) + h(e)) x_e.
\]

We will call the functions

\[
\prod_{\emptyset}(G, x), \prod_{v}(G, x), \prod_{h}(G, x), \prod_{vh}(G, x)
\]

the Feynman functions of $G$. The following theorem appears in more general form in [1].

**Theorem 4.** Let $G$ be a graph with each degree equal to 2 or 4 critically embedded on the torus. Then

\[
E(G, x) =
\]

\[
1/2 \left( \prod_{\emptyset}(G, x) + \prod_{v}(G, x) + \prod_{h}(G, x) - \prod_{vh}(G, x) \right).
\]

3.4. **Proof of Theorem** [1] It is straightforward to observe that

\[
|\prod_{\emptyset}(G, x)|^2 = \sum_{\Sigma'} (-1)^{|\Sigma'|} \prod_{[\gamma] \in \Sigma'} (-1)^{\text{rot}(\gamma)} \prod_{e \in \gamma} x_e
\]

where the sum is over all finite subsets $\Sigma'$ of the prime reduced cycles $\gamma$ of $G'$. Moreover, the analogous statement holds for the remaining three Feynman functions.

Comparing this with Definition 3.1 it remains to find $|V| \times |V|$ matrix $B$ so that for each prime reduced cycle $p$ of $G'$, $(-1)^{\text{rot}(p)} = \prod_{e \in p} b(e)$. This is done using the details of the critical embedding of $G$ in the rest of the paper.

We recall that we fixed orientation $D$ of $G$ so that each edge is directed from its black vertex to its white vertex. For each black vertex $u$ of $G$ we fix an edge $r(u)$ starting in $u$ and for each white vertex $u$ of $G$ we fix an edge $r(u)$ ending in $u$.

Let $e = (jk)$ be a directed edge of $D$. We let $b(j; k) = e^{iz(j; k)/2}$ with $z(j, k)$ denoting the slope of the directed edge $(j, k)$, where the edge $r(j)$ is considered to be the positive real direction (see Figure 5). We let $b(j; k) = e^{-i(\pi + z(j, k))/2}$. 

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We also let \( b(k; j, j) = e^{iz'(k, j)/2} \) for the slope \( z'(k, j) \) of the directed edge \( (k, j) = (j, k)^{-1} \) if the edge \( r(k)^{-1} \) is considered to be the positive real direction (see Figure 5). We let \( b(k; j, k) = e^{-i(\pi+z'(k, j))/2} \).

![Figure 5. Slopes \( z(j, k) \) and \( z'(k, j) \).](image)

Finally, for two vertices \( m, n \) which form the edge \( e \) of \( G \) we form \( b(m, n) = (-1)^{v(e)+h(e)}b(m; m, n)b(n; m, n) \). It is straightforward to realize that \( b(m, n), m, n \in V \) are determined by discrete conformal structure of \( G \) (see Definition 2.7). Finally, we have

**Theorem 5.** For each prime reduced cycle \( p \) of \( G' \), \((-1)^{\text{rot}(p)} = \prod_{e \in p} b(e) \).

**Proof.** Each edge \( e \) of \( p \) contributes by \((-1)^{v(e)+h(e)} \) to \((-1)^{\text{rot}(p)} \), see Figure 3. Let us focus on transition between edges and follow the transition \( k_1j_2k_2 \), see Figures 4, 5. Its contribution to \((-1)^{\text{rot}(p)} \) (with the notation introduced in the previous paragraphs of this subsection) is

\[
(-1)^{-\pi z'(j_1, j_2) + \pi z'(j_2, k_2)} = \\
\left( e^{\pi z'(j_1, j_2) - \pi z'(j_2, k_2)} \right) = \\
\left( e^{\pi z'(j_2, k_2) - \pi z'(j_1, j_2)} \right) = b(j; k_1, j)b(j; j, k_2).
\]

The transitions at a white vertex are treated analogously. 

This finishes the proof of Theorem 5.

The treatment of the general orientable surface of genus \( g \geq 1 \) appears to be analogous and is just currently under investigation.

**References**


