Impersonal Failure Detection

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Abstract

What does it mean to solve a distributed task? In Paxos, Lamport proposed a definition of solvability in which every process is split into a proposer that submits commands to be executed, an acceptor that takes care of the command execution order, and a learner that receives the outcomes of executed commands. The resulting perspective of computation in which every proposed command can be executed, be its proposer correct or faulty, proved to be very useful when processes take steps on behalf of each other, i.e., in simulations.

Most interesting tasks cannot be solved asynchronously, and failure detectors were proposed to circumvent these impossibilities. Alas, when it comes to solving a task using a failure detector, we cannot leverage simulation-based techniques. A process cannot perform steps of failure detector-based computation on behalf of another process, since it cannot access the remote failure-detector module.

This paper proposes a new definition of solving a task with a failure detector in which processes that propose inputs and provide outputs are treated separately from synchronization processes that coordinate using a failure detector. In the resulting framework, any failure detector is shown to be equivalent to the availability of some $k$-set agreement. As a corollary, we obtain a complete classification of tasks, including ones that evaded comprehensible characterization so far, such as renaming.

1 Introduction

In the traditional perspective on distributed computing, it is the responsibility of every process to make its personal progress. For example, in solving a task, it is up to the process to keep taking steps until its personal output is computed. Respectively, a process does not benefit from the computation as long as it takes no steps.

In contrast, a “social-democratic” perspective \cite{16,17} assumes an intermingling of social and personal responsibilities. It distinguishes between processes, the responsibility of which is to declare its participation by publishing its input value, and their threads that use the published values of participating processes to provide them with outputs. As a result, a process holds no particular privileges to execute its thread.

For example, in the celebrated Paxos protocol \cite{27} for state-machine replication, separating proposers from acceptors and learners results in computation in which a command can be executed even when its proposer is slow or even crashed. More generally, the separation of computation from control leverages simulation-based computing in which processes cooperate trying bring all participating processes to their outputs. Therefore, even if a participating process has crashed, its thread may nevertheless be executed.
Simulations improve our understanding of distributed computing by establishing equivalence between seemingly different phenomena: wait-freedom and $t$-resilience \cite{4, 6, 10}, initial disagreement and effective number of failures \cite{21}, various adversarial models \cite{11, 20}, $k$-set agreement and $k$-concurrency \cite{17}, etc.

Simulations usually assume the \textit{asynchronous} read-write shared-memory model which maintains no bounds on relative processes’ speeds. However, many interesting tasks are impossible to solve asynchronously, and the \textit{failure detector} (FD) abstraction \cite{8, 9} was proposed to circumvent these impossibilities.

Consider an algorithm that in addition to reading and writing to the shared memory, also consults a failure detector. Recall that in a FD-based algorithm, each process periodically queries its \textit{personal} FD module to get hints about failures of other processes. Unfortunately, for such an algorithm, we cannot leverage simulation-based techniques, since it does not allow us to simulate a personal FD query. Indeed, even if a process announced its participating by publishing its input value, there is still no way to advance the process through simulation, because its failure detector module cannot be accessed remotely. But by enriching our model with a FD, we should expand our horizons rather than narrow them!

\textbf{Impersonal failure detection.} This paper changes the FD theory in a minute way. It proposes just a new definition of what does it \textit{mean} to solve a task with a FD, which allows the processes to simulate each other.

In our \textit{impersonal} failure-detector framework (IFD), the set of processes is partitioned into two classes: \textit{synchronization} processes and \textit{computation} processes. Computation processes solve tasks by receiving inputs and producing outputs. Synchronization processes help coordinating computation processes by using a FD. The FD provides each synchronization process with hints about the failures of other synchronization processes. The two classes of processes communicate by reading and writing in the shared memory.

If the synchronization processes behave as predicted by the FD, i.e., every synchronization process not hinted faulty by the FD takes infinitely many steps, then every live computation process should output. In contrast, a computation process is free to join or depart the computation whenever it feels like. Every computation process can be seen as an independent thread that executes regardless of where and when failures of other processes take place.

As a result, when it comes to solving tasks, our IFD framework demands from a FD more than the conventional FD model does. In IFD, a FD is supposed to help live computation processes output, regardless of the behavior of the synchronization processes. Thus, the conventional model is a special case of IFD where there is a bijective map between computation and synchronization processes, and a synchronization process crashes if and only if its computation counterpart does.

The IFD framework provides a natural way to model \textit{participation} in a distributed computation with FDs. A participating process is a computation process that published its input. It does not have to take steps after its output is computed. In contrast, in the conventional FD model, a process is expected to take steps unless it fails, even if it never proposes an input or has already obtained its output. Since computation processes do not have direct access to the FD, their steps can be simulated using asynchronous simulation techniques \cite{4, 6, 21, 17}, etc.

\textbf{Outcomes of IFD.} The idea of separating computation from control is not new. But applying it to distributed computing with FDs results in a surprisingly simple model which resolves a number of long-standing puzzles.

To warm up, consider the task of solving consensus among every two (computation) processes in a system of $n > 2$. Delporte et al. \cite{11} showed that any FD that allows for solving the task, also allows for solving consensus among \textit{all} $n$ processes. Does the phenomenon highlighted in \cite{11} only hold for 1-set agreement (consensus) or can it be generalized to any $k \geq 1$? Suppose that a FD provides enough synchrony to solve $k$-set agreement among any $k+1$ processes. A natural
conjecture would be that it also allows for solving $k$-set agreement among all. However, years of trying to prove the conjecture bear no fruits.

In IFD, we obtain the generalization of \cite{11} to any $k \geq 1$ almost for free, using simple induction. In fact, even a stronger result holds: if a FD solves $k$-set agreement among an arbitrary given set of $k+1$ processes, then it is strong enough to solve $k$-set agreement among all $n$ processes.

More generally, the use of IFD enables a complete characterization of distributed tasks. First we observe that the weakest FD to solve a task in IFD is at least as strong as the weakest FD in \cite{8}. Indeed, any FD that solves the task in IFD, also solves it in any its restriction, including the conventional FD model \cite{8}. But since IFD allows for a simulation of computation processes, the power of FDs to solve tasks can be completely characterized.

Consider any task $T$. As any $T$ can be solved 1-concurrently, i.e., assuming that at each moment of time there is at most one undecided participating process, there exists the maximal $k$ such that $T$ is solvable $k$-concurrently. We show that a FD $D$ can be used to solve $T$ if and only if $D$ can be used to solve $k$-set agreement. More precisely, we show that, in every environment, i.e., for all assumptions on where and when failures of synchronization processes may occur, any failure detector that solves $T$ is at least as strong as the anti-$\Omega$-$k$ FD \cite{28,29}, denoted $\neg\Omega_k$. Then we describe an algorithm that uses $\neg\Omega_k$ to solve $T$ (or any task that can be solved $k$-concurrently), in every environment.

The conclusion is that a task is completely characterized through the “level of concurrency” its solution can tolerate. All tasks that can be solved $k$-concurrently but not $(k+1)$-concurrently (e.g., $k$-set agreement) are equivalent in the sense that they require exactly the same amount of information about failures (captured by $\neg\Omega_k$). This characterization covers all tasks, including “colored” ones evading any characterization so far \cite{11,13,1}.

Consider, for example, the task of $(\ell, m)$-renaming in which $\ell$ processes come from a large set of potential participants and choose new names in a smaller name space $1, \ldots, m$, so that no two processes choose the same name. Surprisingly, in the conventional model, the renaming task itself can be formulated as a FD, so the question of the weakest FD for solving it results in a triviality. To avoid trivialities, additional assumptions on the scope of FDs are made \cite{1}.

In IFD, however, it is immediate to see that $(m, m)$-renaming (also called strong renaming) cannot be solved 2-concurrently and is thus equivalent to consensus. More generally, determining the weakest FD for $(\ell, m)$-renaming boils down to determining the maximal $k$ ($1 \leq k \leq \ell$) such that the task can be solved $k$-concurrently. We show finally that $(j, j+k-1)$-renaming can be solved $k$-concurrently, and, thus, using $\neg\Omega_k$.

Summary. In brief, this paper proposes a new way of thinking about failure detectors that enables generic simulation of FD-based algorithms and inherits all properties of asynchronous simulations. The new FD framework, which we call impersonal failure detection, sees a conventional process as two independent threads, a computation process and a synchronization process. A computation process receives application inputs and returns outputs, while a synchronization process queries its FD module to get hints about failures of other synchronization processes. The two classes of processes communicate via the same shared memory. The IFD framework enables simulation of participating computation processes, which implies a complete characterization of distributed tasks. Also, it sorts out some long-standing puzzles, such as the generalization of \cite{11} and the question of the weakest FD for solving renaming.

The paper is organized as follows. First we make the model and our new notion of task solvability with an FD precise. We then present a simple inductive proof of a generalization of \cite{11} to any $k > 1$. Then we extend the generalization even further by presenting a complete

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1 Informally, $D$ is the weakest FD to solve a task $T$ if it (1) solves $T$ and (2) can be deduced from any FD that solves $T$.

2 For some values of $j$ and $k$, however, the question of the maximal tolerated concurrency of $(j, j+k-1)$-renaming is still open \cite{7}.
characterization of decision tasks, based on the level of concurrency they can tolerate. Then we derive the weakest failure detector for strong renaming and wrap up with obligatory concluding remarks. Proofs are partially delegated to the optional Appendix.

2 The model of impersonal failure detection

In this section we introduce our framework. Then we give a definition of what does it mean to solve a task using a failure detector in this framework and relate it to the conventional definition of \cite{8}. Parts of our model reuse elements of \cite{8, 9, 19, 24}.

2.1 Model for computation and synchronization

Intuitively, given a task to be solved, the computation part is made up of processes that get the input value and have to decide and the synchronization part is made up of processes that can help processes of the computation part. Processes in the computation part may or may not participate to the task whereas processes in the synchronization part are prone to failure and are equipped with failure detectors.

Processes. More precisely, we consider a read-write shared-memory system which consists of \(m\) \(C\)-processes, \(\Pi^C = \{p_1, \ldots, p_m\}\), and \(n\) \(S\)-processes, \(\Pi^S = \{q_1, \ldots, q_n\}\). We shall show shortly that the only “interesting” scenario to consider is when \(n = m\).

The \(C\)-processes are responsible for computation. The \(S\)-processes are responsible for synchronization and may be equipped with a failure detector module [9] that gives hints about failures of other \(S\)-processes. The processes in \(\Pi^C\) and \(\Pi^S\) communicate via reading and writing in the shared memory.

Failure patterns and failure detectors. Concerning process failure we are only interested here in failure of \(S\)-processes. Hence a failure pattern \(F\) is a function from the time range \(T = \mathbb{N}\) to \(2^{\Pi^S}\), where \(F(\tau)\) denotes the set of \(S\)-processes that have crashed by time \(\tau\). Once a process crashes, it does not recover, i.e., \(\forall \tau : F(\tau) \subseteq F(\tau + 1)\). \(\text{faulty}(F) = \cup_{\tau \in T} F(\tau)\) is the set of faulty processes in \(F\) and \(\text{correct}(F) = \Pi^S - \text{faulty}(F)\) is the set of correct processes in \(F\).

A failure detector history \(H\) with range \(R\) is a function from \(\Pi^S \times T\) to \(R\). \(H(q_i, \tau)\) is interpreted as the value output by the failure detector module of \(S\)-process \(q_i\) at time \(\tau\). A failure detector \(D\) with range \(R_D\) is a function that maps each failure pattern to a (non-empty) set of failure detector histories with range \(R_D\). \(D(F)\) denotes the set of possible failure detector histories permitted by \(D\) for failure pattern \(F\).

An environment \(E\) is a set of failure patterns that describes a set of conditions on when and where failures might occur. For example \(E_t\) is the environment that consists of all failure patterns \(F\) such that \(\text{correct}(F) \geq n - t\). We assume that for every failure pattern in the environments we consider, at least one \(S\)-process is correct.

Algorithms and runs. A distributed algorithm \(A\) using a failure detector \(D\) consists of two collections of deterministic automata, \(A^C_1, \ldots, A^C_m\), one automaton for each \(C\)-process, and \(A^S_1, \ldots, A^S_n\), one automaton for each \(S\)-process.

A state of \(A\) is defined as the state of each process (state of each process being identified with the state of its corresponding automaton) and each shared object in the system. An initial state \(I\) of \(A\) specifies an initial state for every process and every shared object.

A run of \(A\) using a failure detector \(D\) in an environment \(E\) is a tuple \(R = \langle F, H, I, \text{Sch}, T\rangle\) where \(F \in E\) is a failure pattern, \(H \in D(F)\) is a failure detector history, \(I\) is an initial state of \(A\), \(\text{Sch}\) is an infinite schedule, i.e., a sequence of processes in \(\Pi^C \cup \Pi^S\), \(T\) is a sequence of non-decreasing elements of \(T\). The \(k\)-th step of run \(R\) is a step of process \(\text{Sch}[k]\) determined by
the current state, the failure history \( H, T[k] \) and the algorithm \( \mathcal{A} \). If it is a step of a \( S \)-process, this process is alive (\( \text{Sch}[k] \notin F(T[k]) \)) and the value of the failure detector for this step is given by \( H(\text{Sch}[k], T[k]) \).

Let \( \inf^S(R) \) denote the set of processes in \( \Pi^S \) that appear infinitely often in \( \text{Sch} \). Respectively, \( \inf^C(R) \) denote the set of processes in \( \Pi^C \) that appear infinitely often in \( \text{Sch} \). We say that a run \( R = (F, H, I, \text{Sch}, T) \) is fair if \( \text{correct}(F) \) is equal to \( \inf^S(R) \), and \( \inf^C(R) \) is not empty. A finite run of \( \mathcal{A} \) is a prefix of a run of \( \mathcal{A} \).

**Tasks.** We focus on a class of problems called *tasks* that are defined uniquely through inputs and outputs.

A task \([24]\) is defined through a set \( \mathcal{I} \) of input vectors (one input value for each \( C \)-process), a set \( \mathcal{O} \) of output vectors (one output value for each \( C \)-process), and a total relation \( \Delta : \mathcal{I} \rightarrow 2^\mathcal{O} \) that associates each input vector with a set of possible output vectors. An input value equal to \( \bot \) denotes a *not participating* process and \( \bot \) output value denotes an *undecided* process.

A \( m \)-vector \( L' \) is a *prefix* of a \( m \)-vector \( L \) if \( L' \) contains at least one non-\( \bot \) item and for all \( i, 1 \leq i \leq m \), either \( L'[i] = \bot \) or \( L'[i] = L[i] \). A set \( \mathcal{L} \) of vectors is *prefix-closed* if for all \( L \) in \( \mathcal{L} \) every prefix of \( L \) is in \( \mathcal{L} \).

We assume that each element of \( \mathcal{I} \) and \( \mathcal{O} \) contains at least one non-\( \bot \) item and also that the sets \( \mathcal{I} \) and \( \mathcal{O} \) are prefix-closed. Moreover, we only consider tasks that have finite sets of input vectors \( \mathcal{I} \).

We stipulate that if \( (I, O) \in \Delta \), then (1) if, for some \( i, I[i] = \bot \), then \( O[i] = \bot \), (2) for each \( O' \), prefix of \( O \), \( (I, O') \in \Delta \) and, (3) for each \( I' \) such that \( I \) is a prefix of \( I' \), there exists some \( O' \) such that \( O \) is a prefix of \( O' \) and \( (I', O') \in \Delta \).

For example, in the task of \((U, k)\)-agreement, where \( U \subseteq \Pi^C \), input and output vectors are \( m \)-vectors, such that \( I[i] = \bot \) for all \( p_i \notin U \), input values are in \( \{\bot, 0, \ldots, k\} \), output values are in \( \{\bot, 0, \ldots, k\} \), and for each input vector \( I \) and output vector \( O \), \( (I, O) \in \Delta \) if the set of non-\( \bot \) values in \( O \) is a subset of values in \( I \) of size at most \( k \). \((\Pi^C, k)\)-agreement is the conventional \( k \)-set agreement task \([10]\) and \((\Pi^C, 1)\)-agreement is consensus \([12]\).

### 2.2 Solving a task in the IFD framework

Now we are ready to define what does it mean to solve a task in the impersonal failure detection framework.

**Input vector and output vector of a run.** First, we assume that each automaton \( \mathcal{A}^C_i \) (1) gets an input value \( \text{input}_i \) as part of its initial state, and (2) contains \( \text{decide} \) steps such that all the next steps of \( \mathcal{A}_i \) are null steps that do not affect the current state when they are executed and for each \( \text{decide} \) step is associated a decision value \( v_i \) (when such a decide step is executed we say that \( p_i \) decides or outputs \( v_i \)).

The first step of each \( C \)-process is to write its input value to shared memory. A process that wrote its input value is called *participating*. If a \( C \)-process executes a \( \text{decide} \) step with decision value \( v \), we say that the process decides \( v \).

Given a run \( R \), the *input vector* for the run is the \( m \)-vector \( I \) such that \( I(i) = \text{input}_i \) if \( p_i \) is a participating process and \( I(i) = \bot \) if \( p_i \) is a not participating process. In the same way, the *output vector* of the run is the \( m \)-vector \( O \) such that \( O(i) = v \) if \( p_i \) decides \( v \) in the run and \( O(i) = \bot \) if \( p_i \) does not decide in the run.

**Solving a task.** We say that a run \( R \) with input vector \( I \) and output vector \( O \) satisfies a task \( T = (\mathcal{I}, \mathcal{O}, \Delta) \) if (1) \( (I, O) \in \Delta \) and (2) \( O(i) = \bot \) only if \( p_i \) makes a finite number of steps \( (p_i \notin \inf^C(R)) \).
An algorithm $A$ *IFD-solves a task* $T = (I, O, \Delta)$ *using a failure detector* $D$ *in an environment* $E$ (in the rest we simply say “solves”) if every fair run of $A$ satisfies $T$. If such an algorithm exists for task $T$, $T$ is *solvable with failure detector* $D$ *in environment* $E$. By extension, a failure detector $D$ *solves a task* $T$ *in* $E$ if there is an algorithm $A$ that solves $T$ using $D$ in $E$.

Note that we expect the algorithm to guarantee output to every $C$-process that takes sufficiently many steps, regardless of where and when $S$-processes fail. The algorithm only expects that every correct $S$-process in the current failure pattern takes infinitely many steps.

**Comparing failure detectors.** Failure detector reduction is defined as usual: failure detector $D'$ is *weaker than failure detector* $D$ *in environment* $E$ if $S$-processes can use $D$ to emulate $D'$ in $E$. More precisely, the automata of the $C$-processes of the distributed reduction algorithm $A$ are automata with only null steps and the emulation of $D'$ using $D$ is made by maintaining, at each $S$-process $q_i$, $D'$-output $i$ so that in any fair run with failure pattern $F$, the evolution of variables $\{D'$-output $i\}_{q_i \in I(I)}$ results in a history $H' \in D'(F)$. We say that two failure detectors are *equivalent* in $E$ if each is weaker than the other in $E$.

Note that this definition is equivalent to the original definition of [8].

As in the original definition, if failure detector $D'$ is weaker than failure detector $D$ in environment $E$, then every task solvable with $D'$ in $E$ can also be solved with $D$ in $E$. Now $D$ is the *weakest failure detector* to solve a task $T$ in $E$ if (i) $\exists D$ solves $T$ in $E$ and (ii) $D$ is weaker than any failure detector that solves $T$ in $E$. It is straightforward to extend the arguments of [25] to show that every task has a weakest failure detector.

**$k$-concurrency.** Consider the solvability of a task without the help of a failure detector. In this case the deterministic automata of the $S$-processes of the distributed algorithm $A$ are automata with only null steps. Such an algorithm will be called *restricted*.

It is clear that tasks that are solvable with a restricted algorithm are exactly tasks that are said *wait-free* solvable in the literature (e.g. in [23, 24]).

The notion of $k$-*concurrent* solvability, introduced in [17], is a weaker form of solvability: a task is solvable $k$-concurrently if it is solvable only when at most $k$ $C$-processes concurrently invoke the task. More precisely, a run of a distributed algorithm is $k$-*concurrent* if it is fair and at each time there is at most $k$ undecided participating $C$-processes. A task $T = (I, O, \Delta)$ is $k$-*concurrently* solvable if there is a restricted algorithm $A$ such that all $k$-concurrent runs $R$ of $A$ satisfy $T$. For example, a wait-free solvable task is $m$-concurrently solvable.

Note that for runs such that the number of $C$-processes that are participating but not yet decided is greater than $k$ it is possible that $O \notin \Delta(I)$. It is easy to prove that:

**Proposition 1** Every task is $1$-concurrently solvable.

**Restriction on the number of $C$-processes.** Trivially, if a task $T$ is solvable with a restricted algorithm then $T$ is also solvable with any number of $S$-processes and any failure detector. Reciprocally, consider an algorithm $A$ solving a task $T$ with a trivial failure detector$^3$ in environment $E_{n-1}$. If $n \geq m$ consider the following algorithm: each $C$-process $p_i$ executes alternatively steps of $A^C_{p_i}$ and steps of $A^S_{q_i}$ and each $S$-process executes only null steps. It is easy to verify that in this way we emulate runs of $A$ in failure pattern in which at least all $S$-processes $q_i$ with $i > m$ are crashed, and such runs satisfy task $T$. Hence we get:

**Proposition 2** If $n \geq m$, $T$ is solvable in $E_{n-1}$ with a trivial failure detector if and only if $T$ is solvable with a restricted algorithm.

$^3$A trivial failure detector is a failure detector that always outputs $\bot$. 
But if \( n < m \), the \( S \)-processes may help solving the task even if they do not use their failure detection capacities. For example, with \( n \) \( S \)-processes we can implement a \((\Pi^C, n)\)-set agreement in every environment. For this, each \( S \)-process waits until at least one \( C \)-process writes its input in shared memory, and then it writes this value to a shared variable \( V \). Each \( C \)-process waits until \( V \) has been written and outputs the read value. As at least one \( S \)-process is correct, eventually \( V \) will be written and as there are \( n \) \( S \)-processes at most \( n \) values may be output. In this way the \((\Pi^C, n)\)-set agreement is always solvable even without the help of any failure detector.

As we focus here on solvability where additional power of processes is only due to the failure detection, the only “interesting” scenario to consider is when the number of \( C \)-processes does not exceed the number of \( S \)-processes and more specifically the case where they are equal. Therefore, in the following we assume that the number of \( C \)-processes is equal to the number of \( S \)-processes, we denote this number by \( n \).

### 2.3 Conventional solvability

More conventional models of computation in which there is no separation between the computation and the synchronization part may be considered as a special case of the generalized model presented here. In conventional models, each process \( i \in \{1, \ldots, n\} \) can be seen as running two parallel threads: \( p_i \) corresponding to the computational part and \( q_i \) corresponding to the synchronization part. Moreover failure patterns correspond: \( i \) is correct in conventional systems if and only if \( q_i \) is correct in our setting. But, since in our model, computation and synchronization are separate, it is possible that \( p_i \) makes only a finite number of steps even if \( q_i \) is correct or vice-versa. Then we define **personified runs** of a distributed algorithm as being runs \( R \) that are fair and such that \( p_i \) crashes if and only if \( q_i \) crashes at the same time (as a result, \( \inf^C(R) \) is equal to \( \inf^S(R) \)). We say that algorithm \( A \) solves **classically** task \( T \) with failure detector \( D \) in environment \( E \) if every personified run \( R \) of \( A \) satisfied \( T \).

This definition corresponds exactly to the notion of solvability in a conventional setting as can be found in the literature \[8\].

As the set of personified runs of a distributed algorithm is a subset of the fair runs, we have:

**Proposition 3** If a failure detector \( D \) solves a task \( T \) in an environment \( E \) then \( D \) classically solves \( T \) in \( E \).

**Corollary 4** If \( D \) is the weakest failure detector to classically solve a task \( T \) in an environment \( E \), then \( D \) is weaker than the weakest failure detector to solve \( T \) in \( E \).

Note that the converse is not true. For example, consider the \(\{p_1, p_2\}, 1\)-agreement task (consensus among \( p_1 \) and \( p_2 \)), it is classically solvable in \( E_2 \) (assuming at most 2 failures) with the failure detector \( D \) that, for each \( S \)-process, outputs \( q_1 \) if \( q_1 \) is correct and outputs \( q_2 \) if \( q_1 \) is faulty. But this task is not solvable in \( E_2 \) with this failure detector (intuitively, otherwise, if \( q_1 \) is crashed we would be able to solve consensus between \( p_1 \) and \( p_2 \) without a failure detector).

However for **colorless tasks**\[4\], both notions of solvability coincide.

**Proposition 5** Let \( T \) be a colorless task, \( T \) is solvable with failure detector \( D \) in environment \( E \) if and only if \( T \) is classically solvable with \( D \) in \( E \). The weakest failure detector to solve \( T \) in \( E \) is the weakest failure detector to classically solve \( T \) in \( E \).

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\[4\] Informally, in a solution of a colorless task \[6\], a process is free to adopt the input or output value of any other participating process.
Failure detectors for $k$-set agreement. The failure detector $-\Omega_k$ outputs, at each $S$-process and each time, a set of $(n-k)$ $S$-processes. $-\Omega_k$ guarantees that there is a time after which some correct $S$-process is never output:

$$\forall F \in \mathcal{E}, \forall H \in -\Omega_k(F), \exists q_j \in \text{correct}(F), \tau \in T, \forall \tau' > \tau, \forall q_j \in \text{correct}(F) : q_i \notin H(q_j, \tau').$$

$-\Omega_1$ is equivalent to $\Omega$ \footnote{We could have used a “black-box” simulation of $\mathcal{A}_x$ using $(\Pi^C, x)$-set agreement objects presented in \cite{17}. To make the paper self-contained, we give a direct construction using $-\Omega_x$ in Appendix C.1} that outputs a $S$-process such that eventually the same correct $S$-process is permanently output at all correct processes.

From \cite{19}, we know that in every environment $\mathcal{E}$, $-\Omega_k$ is the weakest failure detector to classically solve $(\Pi^C, k)$-set agreement in $\mathcal{E}$. As $(\Pi^C, k)$-set agreement is a colorless task, from Proposition \footnote{We could have used a “black-box” simulation of $\mathcal{A}_x$ using $(\Pi^C, x)$-set agreement objects presented in \cite{17}. To make the paper self-contained, we give a direct construction using $-\Omega_x$ in Appendix C.1} we obtain:

**Proposition 6** In every environment $\mathcal{E}$, $-\Omega_k$ is the weakest failure detector to solve $(\Pi^C, k)$-set agreement in $\mathcal{E}$. 

3 Solving a puzzle

Let $U$ be a set of $k+1$ $C$-processes. Consider a failure detector $\mathcal{D}$ that solves $k$-set agreement among the processes in $U$. We show that $\mathcal{D}$ actually solves $k$-set agreement among all $n$ $C$-processes.

**Theorem 7** Let $U$ be a set of $(k+1)$ $C$-processes, for some $1 \leq k < n$. For every environment $\mathcal{E}$, if a failure detector $\mathcal{D}$ solves $(U, k)$-set agreement in $\mathcal{E}$ then $\mathcal{D}$ solves $(\Pi^C, k)$-set agreement in $\mathcal{E}$.

**Proof sketch.** Without loss of generality, assume that $U = \{p_1, \ldots, p_{k+1}\}$. Let $\mathcal{A}$ be a distributed algorithm that solves the $(U, k)$-set agreement in $\mathcal{E}$ with $\mathcal{D}$.

Let $U_x$ denote $\{p_1, \ldots, p_x\}$, $x = k+1, \ldots, n$. We observe first that $\mathcal{D}$ can be used to solve $(U_x, x-1)$-set agreement as follows. Processes in $\{p_1, \ldots, p_{k+1}\}$ and $\{q_1, \ldots, q_n\}$ run $\mathcal{A}$ to solve $k$-set agreement and return the value returned by the algorithm, and processes in $\{p_{k+2}, \ldots, p_x\}$ simply return their own input values. In total, at most $x-1$ distinct input values are returned. Let $\mathcal{A}_x$ denote the resulting algorithm.

We proceed now by downward induction to show that for all $x = n$ down to $k$, $\mathcal{D}$ solves $(\Pi^C, x)$-set agreement.

The base case is immediate: $\{p_1, \ldots, p_n\}$ trivially solve $(\Pi^C, n)$-set agreement without any FD. Now suppose that $\mathcal{D}$ solves $(\Pi^C, x)$-set agreement for $x \geq k+1$. By Proposition \footnote{We could have used a “black-box” simulation of $\mathcal{A}_x$ using $(\Pi^C, x)$-set agreement objects presented in \cite{17}. To make the paper self-contained, we give a direct construction using $-\Omega_x$ in Appendix C.1} $\mathcal{D}$ can be used to implement $-\Omega_x$.

Using the generic simulation technique presented in Appendix C.1, the $C$-processes, $p_1, \ldots, p_n$, can use $-\Omega_x$ to simulate a run of the $C$-part of $\mathcal{A}_x$ on $p_1, \ldots, p_x$, so that at least one simulated process takes infinitely many steps. The $S$-part of $\mathcal{A}_x$ is executed by $S$-processes. In the simulation, each simulating process proposes its input value as an input value in the first step for each simulated process in $\{p_1, \ldots, p_x\}$.

Suppose that the current run is fair, i.e., every correct $S$-process takes infinitely many steps. Therefore, we simulate a fair run of $\mathcal{A}_x$ and thus eventually some simulated $C$-process in $\{p_1, \ldots, p_x\}$ decides on one of the input values of the $C$-processes. Once a simulator finds out that a simulated process decided, it returns the decided value. Thus, eventually, every correct simulator returns. Since all decided values come from a run of $\mathcal{A}_x$, at most $x-1$ distinct input values can be decided. Hence, $\mathcal{D}$ solves $(\Pi^C, x-1)$-set agreement.

Therefore, in our framework, we obtain a direct generalization of the fact that for a failure detector, it is as hard to solve consensus in a system of $n$ processes as to solve consensus among $n$ processes.
each pair of processes [11]. In fact, the separation between $C$-processes and $S$-processes, implies a stronger result: solving $k$-set agreement among one given set of $(k+1)$ processes is as hard (in the failure detector sense) as solving it among all $n$ processes.

4 Generalizing the puzzle

We showed in the previous section that solving $k$-set agreement among any given set of $k+1$ $C$-processes requires an amount of information about failures that is sufficient to solve $k$-set agreement among all $n$ $C$-processes. We show below that this statement can be extended to any task $T$ that cannot be solved $(k+1)$-concurrently. We present an explicit reduction algorithm that extracts $\neg \Omega_k$ from any failure detector that solves $T$. Conversely, we show that a task that is $k$-concurrently solvable can be solved with $\neg \Omega_k$ in any environment.

Finally, we derive a complete characterization of generic tasks: all tasks that can be solved $k$-concurrently but not $(k+1)$-concurrently are equivalent in the sense that they require the same information about failures to be solved ($\neg \Omega_k$).

4.1 Reduction to $\neg \Omega_k$

Let $T$ be any task that cannot be solved $(k+1)$-concurrently. Let $E$ by any environment. We show that every failure detector $D$ that solves $T$ in $E$ can be used to implement $\neg \Omega_k$ in $E$ as follows.

Let $A$ be the algorithm that solves $T$ using $D$ in $E$. Recall that $A$ consists of two parts: $A^C\in$ is run by the $C$-processes $p_1, \ldots, p_n$ and $A^S\in$ is run by the $S$-processes $q_1, \ldots, q_n$.

First, we construct a restricted algorithm $A_{\text{sim}}$. In $A_{\text{sim}}$, $C$-processes $p_1, \ldots, p_n$ perform two parallel tasks. In the first task, $C$-processes take steps on behalf of $A^C$. In the second task, they simulate a run of $A^S$ on $S$-processes using, instead of $D$, a directed acyclic graph (DAG) $G$. The DAG $G$ contains a sample of values output by $D$ in some run $R$ of $A$ [8, 29]. In $A_{\text{sim}}$, $S$-processes take null steps.

Informally, each run of $A_{\text{sim}}$ gives “turns” to the $S$-processes and if $G$ provides enough information about failures to simulate the next step of a $S$-process $q_j$, the step of $q_j$ appears in the simulated run of $A$. To simulate steps of $A^S$, $C$-processes employ BG-simulation [4, 5]. This simulation technique enables $k+1$ processes called simulators, to simulate a run of any asynchronous $n$-process protocol in which at least $(n-k)$ processes take infinitely many steps. Thus, if $k$ or less participating $C$-processes take a finite number of steps, the resulting run of $A_{\text{sim}}$ gives infinitely many turns to at least $n-k$ $S$-processes.

Let $F$ be the failure pattern of the run in which $G$ was constructed. $A_{\text{sim}}$ guarantees that (1) every finite run of $A_{\text{sim}}$ simulates a finite run of $A$, and (2) if every $S$-process that is correct in $F$ receives infinitely many turns to take steps, then the simulated run of $A$ is fair, and (3) if $k$ or less participating $C$-processes take only finitely many numbers of steps, then there are at most $k$ $S$-processes that receive only finitely many turns to take steps in the simulation.

Second we construct a reduction algorithm. In such an algorithm $C$-processes take null steps. Our reduction algorithm consists of two components (both are run exclusively by the $S$-processes). In the first component, every $S$-process $q_i$ queries $D$, exchanges the returned values with other $S$-processes and maintains a DAG $G_i$. In the second component, each $q_i$ locally simulates multiple $(k+1)$-concurrent runs of $A_{\text{sim}}$ using $G_i$, going over all combinations of inputs, exploring the runs in the depth-first manner. The simulation continues as long as some simulated $C$-process does not decide in the produced run of $A_{\text{sim}}$. Since $T$ cannot be solved $(k+1)$-concurrently, there must be a $(k+1)$-concurrent run of $A_{\text{sim}}$ in which some participating $C$-process that takes infinitely many steps never decides. The only reason for a $C$-process not to decide in a run of $A_{\text{sim}}$ is that some correct $S$-process receives only finitely many turns in the simulation. But in the simulation, at least $(n-k)$ $S$-processes receive infinitely many turns. Thus, by outputting
the identities of the \((n - k)\) \(S\)-processes that were last to receive turns in the current run we emulate the output of \(\neg \Omega_k\); we output sets of \(n - k\) \(S\)-processes that eventually never contain some correct process.

**Theorem 8** Let \(T\) be a task that cannot be solved \((k + 1)\)-concurrently. For every environment \(\mathcal{E}\), for every failure detector \(\mathcal{D}\) that solves \(T\) in \(\mathcal{E}\), \(\neg \Omega_k\) is weaker than \(\mathcal{D}\) in \(\mathcal{E}\).

### 4.2 Solving a \(k\)-concurrent task with \(\neg \Omega_k\)

In this section, instead of \(\neg \Omega_k\), we use an equivalent failure detector \(\overline{\Omega}_k\) \([29]\). Basically, \(\overline{\Omega}_k\) gives a \(k\)-vector of processes such that, eventually, at least one position of the vector stabilizes on the same correct process at all correct processes.

By definition if \(T\) is \(k\)-concurrently solvable, then there exists a restricted algorithm \(A\) that \(k\)-concurrently solves \(T\).

First, we define an abstract simulation technique that, with help of \(\overline{\Omega}_k\), allows us to simulate, in a system of \(n\) \(C\)-processes, runs of any restricted input-less algorithm on \(k\) \(C\)-processes (the set of non-\(\perp\) input values is a singleton). Moreover, in this simulation, if \(\ell\) simulators participate then at most \(\min(k, \ell)\) processes take infinitely many steps in the simulated execution. Basically, to perform a step for a simulated \(C\)-process \(p_i\), the \(C\)-processes and the \(S\)-processes execute an instance of a leader-based consensus algorithm \([9]\), using the item \(i\) of \(\overline{\Omega}_k\) as a leader. The property of \(\overline{\Omega}_k\) ensures that for some \(i\), infinitely many consensus instances terminate.

Second, we define a restricted algorithm \(B\) for \(k\) \(C\)-processes that simulates a \(k\)-concurrent run of \(A\), using the BG-simulation techniques \([3, 6]\). Applying the abstract simulation technique to \(B\), we obtain an algorithm in which every run \(R\) simulates a run \(R_{\text{sim}}\) of \(A\) such that: (1) \(R_{\text{sim}}\) contains only steps of participating processes of \(R\), (2) the inputs of the participating processes are the same in \(R\) and \(R_{\text{sim}}\), (2) \(R_{\text{sim}}\) is \(k\)-concurrent, and (3) every \(C\)-process that takes infinitely many steps in \(R\) takes also infinitely many steps in \(R_{\text{sim}}\). So if \(T\) is \(k\)-concurrent solvable with \(A\), \(R_{\text{sim}}\) satisfies \(T\), and, consequently, \(R\) satisfies \(T\).

To sum up, we have constructed an algorithm that solves \(T\) with \(\neg \Omega_k\): with the help of \(S\)-processes and \(\neg \Omega_k\), \(p_1, \ldots, p_n\) simulate \(C\)-processes \(p'_1, \ldots, p'_k\) that in turn simulate \(C\)-processes \(p''_1, \ldots, p''_n\) that execute \(k\)-concurrently \(A\).

**Theorem 9** Let \(T\) be any \(k\)-concurrently solvable task. For every environment \(\mathcal{E}\), \(\neg \Omega_k\) solves \(T\) in \(\mathcal{E}\).

### 4.3 Task hierarchy

From Theorems 8 and 9 we deduce:

**Theorem 10** Let \(T\) be a task that can be solved \(k\)-concurrently but not \((k + 1)\)-concurrently. In every environment \(\mathcal{E}\), \(\neg \Omega_k\) is the weakest failure detector to solve \(T\) in \(\mathcal{E}\).

As a corollary, all tasks that can be solved \(k\)-concurrently but not \((k + 1)\)-concurrently (e.g., \(k\)-set agreement) are equivalent in the sense that they require exactly the same amount of information about failures (captured by \(\neg \Omega_k\)).

### 5 Characterizing the task of strong renaming

To illustrate the utility of our framework, we consider the task of \((j, \ell)\)-renaming \([2]\). The task is defined on \(n\) \((n > j)\) processes and assumes that in every run at most \(j\) processes participate (at least \(n - j\) elements of each vector \(I \in \mathcal{I}\) are \(\perp\)). As an output, every participant obtains a
unique name in the range \( \{1, \ldots, \ell\} \) (every non-\( \perp \) element in each \( O \in \mathcal{O} \) is a distinct value in \( \{1, \ldots, \ell\} \)).

In this section, we first focus on \((j, j)\)-renaming (also called strong \(j\)-renaming). Using Theorem 10, we show that the weakest failure detector for strong \(j\)-renaming is \(\Omega\) (for each \(1 < j < n\)). In other words, strong renaming is equivalent to consensus.

Note that in strong 2-renaming at most 2 \(C\)-processes concurrently execute steps of the algorithm. So the impossibility to achieve strong 2-renaming is equivalent to the impossibility of solving strong 2-renaming 2-concurrently. By a simple reduction to the impossibility of wait-free 2-processes consensus, we show (Appendix D):

**Lemma 11** Strong 2-renaming cannot be solved 2-concurrently.

By reducing to the impossibility of Lemma 11 we get a more general result:

**Theorem 12** For all \(1 < j < n\), strong \(j\)-renaming cannot be solved 2-concurrently.

Proposition 1, Theorem 10, and Theorem 12 imply:

**Corollary 13** For all \(j \ (1 < j < n)\), in every environment \(E\), \(\Omega\) is the weakest failure detector for solving strong \(j\)-renaming in \(E\).

In fact, there exists a generic algorithm (Appendix D.2) that, for all \(k = 1, \ldots, j\), solves \((j, j + k - 1)\)-renaming in all \(k\)-concurrent runs, and thus \((j, j + k - 1)\)-renaming can be solved using \(\neg\Omega_k\). For some values of \(k\) and \(j\), \((j, j + k - 1)\)-renaming can be shown to be impossible to solve \((k + 1)\)-concurrently, for others determining the maximal level of concurrency of \((j, j + k - 1)\)-renaming is still an open question [7].

6 Conclusions

This paper introduces a new model of distributed computing with failure detectors that allows processes to cooperate. A process in this model is able to advance the computation of other processes in the way used previously only in asynchronous simulations [4, 6, 16, 17]. The resulting model implies a natural and complete classification of distributed tasks: class \(k\) \((1, \ldots, n)\) consists of tasks that can be solved at most \(k\)-concurrently, and all tasks in the class are equivalent to \(k\)-set agreement.

Our IFD formalism is perfectly suitable for tasks, since the correctness of outputs in a task solution is determined solely by the participating set. An interesting open question is how to extend it to more general classes of distributed computing problems in which failures of participating processes may affect correctness, such as NBAC [22, 12], FTME [13], etc.

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References


A Proof for 1-concurrent solvable (section 2.2)

Proposition 1. Every task is 1-concurrent solvable.

Proof. Each C-process \( p_i \) executes the following code (1) writes its input, (2) reads the other inputs already written getting a vector \( I \) such that \( I[i] = \bot \), and (3) reads all the other outputs already written getting a vector \( O \). If \( O \) is only composed with \( \bot \) then \( p_i \) is the first process, it chooses an output according to its input and \( \Delta \). Otherwise let \( I' \) obtained from \( I \) by replacing the \( i \)-th item with the input value of \( p_i \). By definition of tasks, if \( (I, O) \in \Delta \), there exists a vector \( O' \) obtained from \( O \) by replacing the \( i \)-th item by a non \( \bot \) value such that \( (I', O') \in \Delta \). Then \( p_i \) decides and outputs value \( O'[i] \). Let \( R \) be a 1-concurrent run, by an easy induction on the number of participating processes we prove that \( R \) satisfies \( T \).

B Proof for the reduction to \( -\Omega_k \) (Section 4.1)

The algorithm sketched in Figure 1 describes the steps to be taken by \( S \)-processes \( q_1, \ldots, q_n \) to emulate \( -\Omega_k \). First we describe the asynchronous algorithm \( A_{\text{sim}} \) used by the \( C \)-processes to simulate runs of \( A \), given a sample of the output of \( D \). Then we describe how the \( S \)-processes use multiple simulated runs of \( A_{\text{sim}} \) to emulate the output of \( -\Omega_k \).

Asynchronous simulation of \( A \). Following the technique of Chandra et al. [8], we represent a sample of the failure-detector output in the form of a directed acyclic graph (DAG). The DAG is constructed by the \( S \)-processes by periodically querying \( D \) and collecting the output values: every vertex of the DAG has the form \( [q_i, d, k] \) which conveys that the \( k \)-th query of \( D \) performed by process \( q_i \) returned value \( d \). An edge between vertexes \( [q_i, d, k] \) and \( [q_j, d', k'] \) conveys that the \( k \)-th query of \( D \) performed by \( q_i \) causally precedes \( q_j \) the \( k' \)-th query of \( D \) performed by \( q_j \).

As in [29, 19], any such DAG \( G \) can be used to construct a restricted algorithm \( A_{\text{sim}} \).

In \( A_{\text{sim}} \) the \( C \)-processes \( p_1, \ldots, p_n \) simulates runs of \( A \). The \( C \)-processes obtain input values for \( T \) and perform two parallel tasks. First, the \( C \)-processes take steps on behalf of \( A^C \). Second, they use BG-simulation [5, 6] to simulate a run of \( A^S \) on \( q_1, \ldots, q_n \) [17]. But to simulate step of \( S \)-process instead of \( D \) they use the information provided by \( G \). More precisely, in the simulation, every \( S \)-process \( q_i \) takes steps as prescribed by \( A^S \), except that when \( q_i \) is about to query \( D \), it chooses the next vertex \( [q_i, d, k] \) causally succeeding the latest simulated steps of \( A^S \) of all \( S \)-processes seen by \( q_i \) so far. If \( G \) was constructed in a run of \( A \) with failure pattern \( F \), it is guaranteed that (1) every finite run simulated by \( A_{\text{sim}} \) is a run of \( A \) with failure pattern \( F \), and (2) if the run of \( A_{\text{sim}} \) contains infinitely many simulated steps of processes in correct(\( F \)) then the simulated run is a fair run of \( A^S \) with failure pattern \( F \) [29, 18].

\( A^S \) does not have inputs. Therefore, the simulation tries to promote all \( n \) \( S \)-process (but succeed to take step for a \( S \)-process \( q_i \) if there is enough value for \( q_i \) in \( G \)).

If the simulated run of \( A \) generates an output value for \( p_i \), \( p_i \) outputs this value and leaves the computation. Note that since \( T \) cannot be solved \( (k + 1) \)-concurrently, and all run of \( A \) are safe, there must be a \( (k + 1) \)-concurrent (simulated) run of \( A \) in which some participating process takes infinitely many steps without outputting a value.

Extracting \( -\Omega_k \). Now to derive \( -\Omega_k \), each \( S \)-process in \( i \in \{1, \ldots, k\} \) collects the output of \( D \) in \( G \) and simulates locally multiple \( (k + 1) \)-concurrent runs of \( A_{\text{sim}} \). The runs are simulated in the corridor-based depth-first manner [18] that works as follows.

We assume a total order on the subsets \( P \subseteq \Pi^C \) so that if \( P \subset P' \) then \( P \) precedes \( P' \) in the order. Each initial state \( I \) and each schedule \( \sigma \), a sequence specifying the order in which
for all \( I_0 \), input vectors of \( T \) (in some order) do
/* All possible inputs for \( p_1, \ldots, p_n \) */

for all \( \pi_0 \), permutations of \( p_1, \ldots, p_n \) (in some order) do
/* All possible "arrival orders" */

\( P_0 := \) the set of first \( k + 1 \) \( C \)-processes in \( \pi_0 \)

```
function explore(I, \sigma, P, \pi)

否定Ω⁹-output := \( n - k \) \( S \)-processes that appear the latest in \( \alpha_i(I, \sigma) \)
(\( n - k \) \( S \)-processes if not possible)

if \( \exists q_j \in \Pi^S_1: \forall \sigma' \in \text{dom}(\alpha_i), \exists \sigma'', \text{a prefix of } \sigma': \alpha_j(I, \sigma'') \) is deciding then
/* If all schedules explored so far were found deciding by \( q_j \) */

else
\( N := \) the set of undecided processes in \( (I, \sigma) \)

for all \( p_j \in P - N \) do /* For each decided process in \( P \) */

\( P := P - \{ p_j \} \)

\( P := P \cup \{ \text{the first process in } \pi \text{ that does not appear in } \sigma \} \)
/* Replace \( p_j \) with the next non-participant in \( \pi \) */

for all \( P' \subseteq P \) (in some order consistent with \( \subseteq \)) do /* For all "sub-corridors" */

for all \( p_j \in P' \) (in \( \pi \)) do
explore(I, \sigma \cdot p_j, P', \pi)
```

Figure 1: Deriving \( \neg \Omega_k \): code for each \( S \)-process \( q_i \).

\( p_1, \ldots, p_n \) take steps of \( A_{sim} \), determine a unique run of \( A_{sim}^S \) simulated at process \( q_i \), denoted \( \alpha_i(I, \sigma) \).

For a given input vector \( I \) and a given permutation \( \pi \) of \( p_1, \ldots, p_n \), that describes the order in which the \( C \)-processes “arrive” at the computation. Initially, we select a set \( P \) of the first \( k + 1 \) processes in \( \pi \) as the participating set. Subsets \( P' \subseteq P \) are then explored as “corridors” (line 10), in the deterministic order, from the narrowest (sole) corridors to wider and wider ones. Recursively, we go through simulating all runs in which only \( C \)-processes in \( P' \) take steps. In the course of simulation, if a participating \( C \)-process \( p_j \) decides, we replace it with a process that has not yet taken steps in the current computation (line 13). Since we only replace a decided process with a “fresh” non-participant, the participating set keeps the size of \( k + 1 \) or less processes. This procedure is repeated until every \( C \)-process decides. Thus, every simulated run is \((k + 1)\)-concurrent. Once the exploration of the current corridor is complete (the call of explore in line 10 returns), we proceed to the next corridor, etc.

If, at some point, \( q_i \) finds out that another \( S \)-process \( q_j \) made more progress in the simulation (simulated more runs than \( q_i \)), then \( q_i \) “adopts” the simulation of \( q_i \) (line 8) by adopting \( q_j \)’s version of the DAG and the map \( \alpha_j \) [18].

The output of \( \neg \Omega_k \) is evaluated as the set of the ids of the latest \( n - k \) processes in \( q_1, \ldots, q_n \) that appear in the run of \( A_{sim}^S \) in the currently simulated run of \( A_{sim} \) (line 3).

Recall that \( T \) cannot be solved \((k + 1)\)-concurrently and thus there must exist a \((k + 1)\)-concurrent run of \( A_{sim} \) in which some participating live process never decides. Since the only reason for the run of \( A_{sim} \) not to decide is the absence of some correct process in the simulated \( k \)-resilient run of \( A_{sim}^S \), eventually, and the emulated output eventually never contains some contain some correct process—thus, \( \neg \Omega_k \) is emulated. Thus:

**Theorem** [8] Let \( T \) be a task that cannot be solved \((k + 1)\)-concurrently. For every environment \( \mathcal{E} \), for every failure detector \( \mathcal{D} \) that solves \( T \) in \( \mathcal{E} \), \( \neg \Omega_k \) is weaker than \( \mathcal{D} \) in \( \mathcal{E} \).

**Proof sketch.** Our reduction algorithm works as follows. Every \( S \)-process \( q_i \) runs two parallel
tasks. First, it periodically queries its module of $D$ and maintains its directed acyclic graph $G_i$, as in $[8, 13]$. Second, it uses $G_i$ to locally simulate multiple runs of $A_{\text{sim}}$ and emulates the output of $-\Omega_k$. Consider any run of the reduction algorithm. Let $F$ be the failure pattern of that run.

First we observe that every simulated run of $A_{\text{sim}}$ is $(k+1)$-concurrent. Indeed, initially, exactly $(k+1)$ C-processes participate and a new participant joins only after some participating C-process decides and departs.

Then we show that the correct S-processes eventually perform the same infinite sequence of recursive invocations of explore: explore$(I, \bot, P_0, \pi)$ invokes explore$(I, \sigma_1, P_1, \pi)$, which in turn invokes explore$(I, \sigma_2, P_2, \pi)$, etc. (line 14). Indeed, all S-processes perform the simulations in the same order and since, the task is not $(k+1)$-concurrently solvable, there must be a never deciding $(k+1)$-concurrent run of $A_{\text{sim}}$. Since all these $P_\ell$ are non-empty, there exists $\ell^*$ and $P^*$ such that $\forall \ell \geq \ell^*, P_\ell = P^*$. Since we proceed from narrower corridors to wider ones, $P^*$ is the set of live C-processes that never decide in the “first” never deciding $(k+1)$-concurrent simulated run with a schedule $\sigma^*$.

Now we observe that all simulated runs eventually always extend a prefix $\bar{\sigma}^*$ of $\sigma^*$ in which some simulated processes not in $P^*$ already took all their steps in $\sigma^*$. Moreover, there is a time after all explored extensions of $\bar{\sigma}^*$ only contain steps of processes in $P^*$. By the properties of BG-simulation $[14, 9]$, every S-process that appears only finitely often in the run of $A_{\text{sim}}$ simulated by $\sigma^*$ (we called these processes blocked by $\sigma^*$) eventually never appears in all simulated run of $A$. Let $U$ be the set of S-processes blocked by $\sigma^*$. Since the run of $A_{\text{sim}}$ simulated by $\sigma^*$ is $(k+1)$-concurrent, processes in $U$ eventually never appear among the last $n-k$ processes in $\alpha(I, \sigma)$ (line 8).

Now we observe that $U$ must contain a correct (in $F$) S-process. If it is not the case, i.e., $U$ doesn’t contain a correct S-process, then the simulated run of $A$ is fair and thus the simulated run of $A$ must be deciding.

Thus, eventually some correct S-processes never appears in $-\Omega_k$-output, at every correct S-process $q_i$—$\Omega_k$ is emulated.

C Proof for solving a $k$-concurrent solvable task with $-\Omega_k$ (section 2.2)

This section presents a distributed algorithm that uses $-\Omega_k$ to solve, in any environment, any task that can be solved $k$-concurrently. The result could have been obtained from the simulation of $k$-concurrency using (black-box) $k$-set agreement objects $[17]$. But for the sake of self-containment, we present a (simpler) direct construction of a $k$-concurrent run using $-\Omega_k$.

First we describe an abstract simulation technique that uses $\overrightarrow{\Omega}_k$ (equivalent to $-\Omega_k$ $[29]$) to simulate, in a system of $n$ C-processes, a run of an arbitrary asynchronous algorithm $B$ on $k$ C-processes.

Then we apply this technique to show that, in every environment, we can use $\overrightarrow{\Omega}_k$ to simulate a $R_{\text{sim}}$ of any given $n$-C-process protocol $A$. If $R$ is the current run, we have the following properties: (1) $R_{\text{sim}}$ only contains steps of participating processes of $R$, (2) $R_{\text{sim}}$ is $k$-concurrent, and (3) every participating C-process of $R$ that take infinitely many steps is given enough steps in $R_{\text{sim}}$ to decide.

\footnote{The construction is similar to the one presented in the technical report version of $[18]$ for the actively $k$-resilient case.}
C.1 Simulating $k$ codes using $-\Omega_k$

Suppose we are given a read-write algorithm $\mathcal{B}$ on $k$ processes, $p'_1, \ldots, p'_\ell$. Assuming that $\Omega_k$ is available, the algorithm in Figure 2 describes how $n$ simulators, $C$-processes $p_1, \ldots, p_n$ can simulate an infinite run of $\mathcal{B}$.

The simulation is similar in spirit to BG-simulation [4, 6]. Every simulator $p_i$ first registers its participation in the shared memory and then tries to advance simulated $C$-processes $p'_1, \ldots, p'_{\min(k,m)}$, where $m$ is the number of simulators that $p_i$ has witnessed participating.

To simulate a step of $p'_j$, simulators agree on the view of the $C$-process after performing the step. However, instead of the BG-agreement protocol of [4, 6], we use here a leader-based consensus algorithm [8]. In the algorithm, a process periodically (in every round of computation $r$, queries the current leader to get an estimate of the decision.

Since in our algorithm both $C$-processes and $S$-processes can be elected leaders, we modify the algorithm of [8] as follows. When a process wants to get an estimate of the decision (say in round $r$), it publishes a query $(\text{query}, \text{est}', r)$ in the shared memory (proposing its current estimate $\text{est}'$), waits until the current leader publishes a response $(\text{est}, r)$, and adopts the estimate. For simplicity, we assume that every process (be it a $C$-process or a $S$-process) periodically scans the memory to find new queries of the kind $(\text{query}, \text{est}', r)$ and responds to them by publishing one of the proposed proposed estimates. Furthermore, we assume that each $S$-process periodically updates the shared array $\Omega_k-S[1, \ldots, k]$ with the output of its module of $\Omega_k$. Recall that eventually some position $\Omega_k-S[j]$ ($j \in \{1, \ldots, k\}$) stabilizes on the identity of some correct $S$-process.

The resulting algorithm terminates under the condition that all $C$-processes eventually agree on the same correct leader. The instance of the consensus algorithm used to simulate $\ell$-th step of $C$-process $p'_j$ is denoted by $\text{cons}_{j,\ell}$.

The rule to elect the leader is the following. As long as the number of participating simulators is $k$ or less, the participating simulator with the $j$-th smallest identity acts as a leader for simulating steps of $p'_j$. When the number of participating simulators exceeds $k$, the leader for simulating steps of $p'_j$ is given by $\Omega_k-S[j]$.

In both cases, at least one simulated $C$-process is eventually associated with the same correct leader. Thus, at least one simulated $C$-process makes progress in the simulation.

The algorithm also assumes that a simulator $p_i$ may decide to leave the simulation if the simulated run produced a desired output (line 28). We use this option in the next section.

Theorem 14 In every environment, the protocol in Figure 2 simulates an infinite run of any $k$-process algorithm $\mathcal{B}$ (as long as there is at least one not decided participating simulated process). Moreover, if $\ell$ simulators participate, i.e., $|\text{pars}| = \ell$, then at most $\min(k, \ell)$ processes participate in the simulated run.

Proof. Consider an infinite run of the algorithm. Since every next state of each simulated process $p'_j$ is decided using a consensus algorithm, every simulator observes exactly the same evolution of states for every simulated process. Thus, the simulated schedule indeed belongs to a run of $\mathcal{B}$.

Now consider the construction of variables $\text{Leader}_1$, ..., $\text{Leader}_k$ used by the consensus algorithms $\text{cons}_{1,\ell}$, ..., $\text{cons}_{k,\ell}$ (lines 34-36). Let $\ell$ be the number of participating simulators.

If $\ell \leq k$, the simulator with the $j$-th smallest identity in $\text{pars}$ is assigned to be the leader of exactly one simulated process $p'_j$. Since at least one simulator is correct, there exists $p'_j$ ($j = 1, \ldots, |\text{pars}|$) such that all instances $\text{cons}_{j,\ell}$ using $\text{Leader}_j$ eventually terminate. Thus, $p'_j$ accepts infinitely many steps in the simulated run.

If $\ell > k$, at least one $\text{Leader}_j$ ($j = 1, \ldots, k$) eventually stabilizes on some correct process identity, as guaranteed by the properties of $\Omega_k$. Again, $p'_j$ takes infinitely many steps in the simulated run.

In both cases, at most $\min(\ell, k)$ simulated processes appear in the produced run of $\mathcal{B}$, and at least one simulated process takes infinitely many steps. \hfill $\square$
Shared variables:
- $R_j$, $j = 1,\ldots,m$, initially $\perp$
- $V_j$, $j = 1,\ldots,k$, initially the initial state of $p'_j$
- $\Omega_k-S[j]$, $j = 1,\ldots,n$, initially $q_1$

Local variables:
- $Leader_j$, $j = 1,\ldots,k$, initially $p_1$
- $\ell_j$, $j = 1,\ldots,k$, initially 1
- $v_j$, $j = 1,\ldots,k$, initially $\perp$

**Task 1:**

17 $R_i := 1$
18 undecided := true
19 for $j = 1,\ldots,k$ do $v_j := \{V_1,\ldots,V_k\}$
20 while undecided do
21 for $j = 1,\ldots,\min(|pars|, k)$ do
22 perform one more step of $cons_j,\ell_j(v_j)$ using $Leader_j$ as a leader
23 if $cons_j,\ell_j(v_j)$ returns $v$ then
24 $V_j := v$ { The next state of $p'_j$ is decided }
25 simulate the next step of $p'_j$ in $B$
26 if $v$ allows $p_i$ to decide then
27 undecided := false { The simulator can depart }
28 $R_i := 1$
29 $v_j := \{V_1,\ldots,V_k\}$ { Evaluate the next state of $p'_j$ }
30 $\ell_j := \ell_j + 1$

**Task 2:**

31 while true do
32 pars := $\{p_j, R_j \neq \perp\}$
33 if $|pars| \leq k$ then
34 for $j = 1,\ldots,|pars|$ do $Leader_j :=$ the $j$-th smallest process in $pars$
35 else
36 for $j = 1,\ldots,k$ do $Leader_j := \Omega_k-S[j]$

Figure 2: Simulating $k$ codes using vector-$\Omega_k$: the program code for simulator $p_i$
C.2 Solving a \( k \)-concurrent task with \( \neg \Omega_k \)

**Theorem 9** Let \( T \) be any \( k \)-concurrently solvable task. In every environment \( \mathcal{E} \), \( \neg \Omega_k \) solves \( T \) in \( \mathcal{E} \).

**Proof.** Let \( \mathcal{A} \) be the algorithm that solves \( T \) \( k \)-concurrently. We simply employ the simulation protocol in Figure 2 (Theorem 14), and suppose that the simulated algorithm \( \mathcal{B} \) is Extended BG-simulation \([16]\) for \( \mathcal{A} \). More precisely, \( \mathcal{B} \) simulates with \( k \) \( C \)-processes the algorithm \( \mathcal{A} \) with \( n \) \( C \)-processes.

Thus, the double simulation is built as follows. Every process \( p_i \) writes its input value of \( T \) to the shared memory and starts the simulation of \( k \) processes \( p'_1, \ldots, p'_k \) using the algorithm in Figure 2. The simulated processes \( p'_1, \ldots, p'_k \) run, in turn, BG-simulation of \( \mathcal{A} \) on \( n \) processes \( p''_1, \ldots, p''_n \).

Each simulated process \( p''_j \) is simulated only if the corresponding \( p_j \) has written its input of \( T \) in the shared memory and \( p'_j \) has not yet obtained an output in the simulated run. Moreover, to make sure that the simulation indeed produces a \( k \)-concurrent run, at any point of the simulation, each simulator in \( p'_j \in \{p'_1, \ldots, p'_k\} \) tries to advance the participating and not yet decided process with the smallest id. If the currently simulated process is found blocked \([4, 6]\), i.e., the process cannot advance because another simulator started simulating a steps of it but has not yet finished, \( p'_j \) proceeds to the next smallest undecided participating process in \( \{p''_1, \ldots, p''_n\} \). Since there are at most \( k \) simulators, at most \( k - 1 \) undecided participating processes can be found blocked and thus there are at most \( k \) undecided participating processes at a time—the resulting simulated run is \( k \)-concurrent.

When \( p''_j \) obtains an output, the corresponding simulator \( p_j \) considers itself “decided” (line 20), writes \( \bot \) in \( R_i \) (line 28) and departs.

If \( p'_i \) cannot make progress because each code it tries to simulate is blocked and there are no more codes to add, it “aborts” all blocked agreements \([16]\) and resumes the simulation. Since, at each point of time, the number of simulated codes does not get below the number of simulators that take steps, the simulation keeps making progress.

Thus, as long as \( \ell \) processes \( \{p_{j_1}, \ldots, p_{j_\ell}\} \) participate, only \( \min(k, \ell) \) processes in \( \{p''_1, \ldots, p''_n\} \) take steps, which results in a \( k \)-concurrent simulated run of \( \mathcal{A} \). Every process \( p''_j \) that takes steps eventually decides in a \( k \)-concurrent run of \( \mathcal{A} \) and the corresponding simulator \( p_j \) departs. As soon as the decided process \( p_i \) departs by writing \( \bot \) to \( R_i \), we have one simulator \( p_i \) and one simulated process \( p''_j \) less. Therefore, as long as there is a simulator take steps and the run is fair, the simulated run makes progress, i.e., more and more participants decide. Thus, we obtain an algorithm that, in every environment, solves \( T \). \( \square \)

**D Proof for characterizing the task of renaming (section 5)**

To illustrate the utility of our framework, we consider the task of \((j, \ell)\)-renaming \([2]\). The task is defined on \( n \) \((n > j)\) processes and assumes that in every run at most \( j \) processes participate (at least \( n - j \) elements of each vector \( I \in \mathcal{I} \) are \( \bot \)). As an output, every participant obtains a unique name in the range \( \{1, \ldots, \ell\} \) (every non-\( \bot \) element in each \( O \in \mathcal{O} \) is a distinct value in \( \{1, \ldots, \ell\} \)).

We show first that \((j, j)\)-renaming (also called strong \( j \)-renaming) is not \( 2 \)-concurrently solvable. Then we present a generic algorithm that, for all \( k = 1, \ldots, j \), solves \((j, j + k - 1)\)-renaming in all \( k \)-concurrent run, and thus \((j, j + k - 1)\)-renaming can be solved (in IFD) using \( \neg \Omega_k \).
Shared variables: $R_l$, $l = 1, \ldots, n$, initially $\perp$.

```plaintext
37 \( R_i := 1 \) /* register participation */
38 repeat
39 \( S := \{ p_l \mid R_l \neq \perp \} \) /* get the current participating set */
40 \( S' := \{ p_l \mid R_l = 1 \} \) /* get the set of not yet decided participants */
41 \( \min_1 := \min(S') \)
42 if \( |S'| = 1 \) then \( \min_2 := \min_1 \) else \( \min_2 := \min(S' - \min(S')) \)
43 if \( |S| = j \) and \( (p_i = \min_1 \text{ or } p_i = \min_2) \) or \( (|S| = j - 1 \text{ and } p_i = \min_1) \) then
44 take one more step of $A$ /* if among two not decided with smallest ids */
45 until decided
46 \( R_i := 0 \)
47 return the name decided in $A$
```

Figure 3: A $1$-resilient strong $j$-renaming algorithm: code for each $C$-process $p_i$.

### D.1 Impossibility of 2-concurrent strong 2-renaming

**Lemma 11** Strong 2-renaming cannot be solved 2-concurrently.

**Proof.** We start with showing that for the special case of $j = 2$, strong renaming cannot be solved 2-concurrently. Suppose, by contradiction, that there exists a (restricted) algorithm $A$ that solves (2,2)-renaming 2-concurrently. Since we assumed $j < m$, we have at least 3 processes in the system. By the pigeon-hole principle, there exist two processes that decide on the same name $v \in \{1,2\}$ in their solo runs of $A$. Without loss of generality, let these processes be $p_1$ and $p_2$ and let $v = 1$.

Now $p_1$ and $p_2$ can wait-free solve 2-processes consensus as follows. Each process publishes its input and then runs $A$ until it obtains a name. If the name is 1, the process decides on its input, otherwise it decides on the input of the other process. Since a process in $\{p_1,p_2\}$ obtains 1 as a name in a solo run of $A$, if 1 is not obtained, then the other process participates in the run of $A$ and, thus, has previously written its input. Therefore, every decided value was previously proposed. Since every obtained name is distinct, the two processes cannot decide on different values. This conclude the proof that strong 2-renaming cannot be 2-concurrently solvable. □

**Theorem 12** For all $1 < j < n$, strong $j$-renaming cannot be solved 2-concurrently.

**Proof.** By Lemma 11 we have already the result for $j = 2$. Suppose, by contradiction, that for some $2 < j < n$, there exists an (restricted) algorithm $A$ solving strong $j$-renaming 2-concurrently. As we deal here with 2-concurrent solvability, we are only interested by the $C$-processes and their algorithms. We use $A$ to solve strong $j$-renaming in all 1-resilient runs, i.e., runs in which at least $j - 1$ $C$-processes participate and take infinitely many steps. Recall that at most $j$ $C$-processes participate in every run, so either $j - 1$ or $j$ processes take infinitely many steps. In the algorithm (Figure 3), every process registers its participation (line 37) and then periodically checks the current set of participants (line 39). If it finds out that it is among 2 processes with the smallest identities among $j$ participating but not yet processes (line 43), then it starts taking steps of $A$ until the algorithm provides $p_i$ with a new name. Then $p_i$ declares that it has decided (line 46) and departs.

Note that the resulting run of $A$ is 2-concurrent: either the participating set is of size $j - 1$ and only the not yet decided participant with the smallest identity is allowed to take steps of $A$ solo, or exactly $j$ processes participate and the two not yet decided processes with the smallest identity are allowed to take step concurrently.

Now we observe that the run of $A$ continues as long as there is at least one not yet decided participant that take steps. Indeed, either the participating set is of size $j - 1$ and every participant...
Shared variables:
\[ R, \ell = 1, \ldots, n, \text{ initially } \perp. \]

\[ \begin{aligned} &s := 1 \\
&\text{repeat forever} \\
&R_i := (i, s, \text{true}) \quad /* \text{register new name}*/ \\
&S := \{p_i \mid R_i \neq \perp\} \quad /* \text{collect suggested names}*/ \\
&\text{if } \exists (\ell, s, b) \in S: i \neq \ell \text{ and } s = s_{\ell} \text{ then} \\
&r := \text{the rank of } i \text{ in } \{\ell \mid (\ell, s, b) \in S, b = \text{true}\} \quad /* \text{rank among not yet decided participants}*/ \\
&s := \text{the } r\text{th integer not in } \{s_{\ell} \mid (\ell, s_{\ell}, b) \in S, i \neq \ell\} \quad /* \text{suggest a new name among not yet suggested}*/ \\
&\text{else} \\
&R_i := (i, s, \text{false}) \\
&\text{return } s \end{aligned} \]

Figure 4: A \( k \)-concurrent \((j, j + k - 1)\)-renaming algorithm: code for each process \( p_i \).

takes an infinity number of steps (including the not yet decided one with the smallest identity) or exactly \( j \) \( C \)-processes participate and at least one of the not yet decided processes with the two smallest identity takes an infinity number of steps. Thus, every \( C \)-process that keeps taking steps of \( A \) in the resulting 2-concurrent run eventually decides and departs. The set of undecided participants gets smaller by one, and the next \( C \)-process with the smallest identity joins the 2-concurrent run of \( A \).

But it is shown in [16] that if all 1-resilient runs of a restricted algorithm \( A \) satisfy strong \( j \)-renaming then there is a restricted algorithm to solve strong 2-renaming 2-concurrently—a contradiction with Lemma 11.

\[ \square \]

D.2 Solving renaming

The distributed algorithm used to solve \((j, j + k - 1)\)-renaming \( k \)-concurrently essentially mimics the algorithm of [2, 3] for wait-free \((j, 2j - 1)\)-renaming.

**Theorem 15** For all \( 1 < k \leq j < m \), \((j, j + k - 1)\)-renaming can be solved \( k \)-concurrently.

**Proof.** Our algorithm, described in Figure 4, essentially mimics the algorithm of [2, 3] for wait-free \((j, 2j - 1)\)-renaming.

In the algorithm, every process periodically selects a new name according to the set of the names not yet suggested by other processes and its rank among the set of currently not yet decided participants (lines \[ 53 \] and \[ 54 \]).

Note that since at most \( j \) process participate in every run, \( p_i \) can observe at most \( j - 1 \) names suggested by other processes in line \[ D.2 \]. Furthermore, since in a \( k \)-concurrent run, \( p_i \) can observe at most \( k \) not yet decided participants, its rank can be at most \( k \). Therefore, the highest name \( p_i \) can suggest in line \[ 54 \] is \( j + k - 1 \).

Now we show that no two processes output the same name. Suppose, by contradiction, that \( p_i \) and \( p_j \) output the same name \( s \). Thus, both \( p_i \) and \( p_j \) previously suggested \( s \) in line \[ 54 \]. But since after than both processes read each oher’s registers after that, at least one of them would see that \( s \) has been suggested by another process and thus would not decide—a contradiction.

Finally, we show that every correct process eventually decides. Consider, by contradiction, an run \( R \) in which a set of correct processes \( \{p_{j_1}, \ldots, p_{j_t}\} \) (ordered by their ids) never decide. We call these processes trying. We establish a contradiction by showing that \( p_{j_1} \) must eventually decide. Indeed, consider \( R' \), a prefix of \( R \), in which only trying processes take steps, and let \( S \)
be the set of names suggested by the processes not in \( \{ p_j, \ldots, p_{j_t} \} \) (note that this set does not change in \( R \)). Since, \( p_{j_1} \) has the smallest rank among the trying processes (let us denote it by \( r \)), eventually no trying process will ever suggest the \( r \)th name not in \( S \). Thus, \( p_{j_1} \) eventually finds itself to be the only process to suggest the name and decides—a contradiction. \( \square \)

From this result and Theorem 9 we can conclude:

**Theorem 16** For all \( 1 < k \leq j < m \), \((j, j + k - 1)\)-renaming can be solved with \( \neg \Omega_k \).