GAUSSIAN INTERVAL QUADRATURE FORMULAE FOR TCHEBYCHEFF SYSTEMS∗
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Abstract. For any given system of continuously differentiable functions \( \{u_k\}_{k=1}^{2n} \) which constitute an Extended Tchebycheff system of order 2 on \([a, b]\) we prove the existence and uniqueness of the Gaussian interval quadrature formula based on \( n \) weighted integrals over non-overlapping subintervals of \([a, b]\) of preassigned lengths. This supplies an analogue of the result of Mark Krein about canonical representation of linear positive functionals.

Key words. Gaussian quadrature, Tchebycheff systems, interval analysis, polynomials

AMS subject classifications. 65D32, 65D30, 41A55

1. Introduction. The classical Gauss quadrature formula was extended over the years in various directions. The present paper concerns an interesting development of the subject, initiated by Mark Krein in [5]. He proved that for any given system \( U_{2n} := \{u_1, u_2, \ldots, u_{2n}\} \) of continuous functions, which form a Tchebycheff system on \([a, b]\), there exists a unique set of points \( x_1 < \cdots < x_n \) in \([a, b]\) such that the interpolatory quadrature formula

\[
\int_a^b f(t) \, dt \approx \sum_{k=1}^n C_k f(x_k)
\]

is exact for every generalized polynomial

\[
u(t) = c_1 u_1(t) + \cdots + c_{2n} u_{2n}(t)
\]

with real coefficients \( \{c_j\} \). Actually, his result is slightly stronger and covers the canonical representation of a general linear positive functional \( L[f] \) on \( C[a, b] \). Simplified proofs of Krein’s result can be found in [4] and [1].

A significant effort was expanded in the last few decades in extending the Gauss formula to other natural types of data, in addition to the standard one – of sampling function values. Recently, we proved in [2], [3] the existence and uniqueness of a formula of the form

\[
\int_a^b f(t) \, dt \approx \sum_{k=1}^n A_k \frac{1}{h_k} \int_{x_k}^{x_k+h_k} f(t) \, dt
\]

of highest degree of precision with respect to the class of algebraic polynomials, for any fixed system of lengths \( \{h_k\} \), \( h_1 + \cdots + h_n \leq b - a \). The problem stayed open for a quite long time (see [7], [9], [10], [11], [12] for previous results). In [8], Milovanović and Cvetković showed that the proof from [3] can be modified to cover the case of Jacobi weight function. The purpose of this paper is to go further and show that the result holds also for Tchebycheff systems and thus supplies an interval analogue of

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Krein’s result. In case all lengths \{h_k\} are supposed equal to zero, the quadrature reduces to Krein’s canonical representation.

Throughout this paper we assume that \(n\) is a natural number, \([a, b]\) is a fixed finite interval, and \(U_{2n} := \{u_1, u_2, \ldots, u_{2n}\}\) is a given system of continuously differentiable functions which constitute an Extended Tchebycheff (ET) system of order 2 on \([a, b]\). Recall that the system \(U_{2n}\) constitutes an ET system of order 2 on \([a, b]\) if any non-zero generalized polynomial \(u\) with respect to \(U_{2n}\) possesses no more than \(2n - 1\) zeros in \([a, b]\) (counting twice every common zero of \(u\) and \(u’\)) (see [4], Chapter II).

We shall denote by \(U_{2n}\) the linear space of all generalized polynomials with respect to the system \(U_{2n}\), that is,

\[ U_{2n} := \text{span} \{u_1, u_2, \ldots, u_{2n}\}. \]

Further, assume that \(\mu(t)\) is a given integrable function on \([a, b]\), which is continuous and strictly positive on \((a, b)\), and denote by \(L[f]\) the integral

\[ L[f] := \int_a^b \mu(t)f(t) \, dt. \]

We are going to give a canonical representation of \(L[f]\) in \(U_{2n}\) of the form

\[ L[f] = \sum_{k=1}^{n} a_k \frac{1}{I_k} I_k[f], \]

where

\[ I_k[g] := \int_{x_k}^{x_k+h_k} \mu(t)g(t) \, dt, \quad I_k := I_k[1]. \]

Here we assume that

\[ \Delta_k := [x_k, x_k + h_k] \]

are \(n\) non-overlapping subintervals of \([a, b]\) of preassigned lengths \(|\Delta_k| := h_k \geq 0\).

In case \(h_k = 0\) the quantity \(I_k[f]/I_k\) is defined by continuity, that is,

\[ \frac{1}{I_k} I_k[f] \bigg|_{h_k=0} := \lim_{h_k \to 0} \frac{1}{I_k} I_k[f] = f(x_k). \]

Keeping the notation from [3], we introduce the set \(H\) of admissible lengths \(h = (h_1, \ldots, h_n)\),

\[ H := \{h \in \mathbb{R}^n : h_k \geq 0, k = 1, \ldots, n, \sum_{k=1}^{n} h_k < b - a\}, \]

the associated set

\[ D = D(h) := \{x \in \mathbb{R}^n : a < x_1 \leq x_1 + h_1 < \cdots < x_n \leq x_n + h_n < b\} \]

of admissible nodes, and its closure

\[ \bar{D} = \bar{D}(h) := \{x \in \mathbb{R}^n : a \leq x_1 \leq x_1 + h_1 \leq \cdots \leq x_n \leq x_n + h_n \leq b\}. \]
Let us denote by $\partial D$ the boundary of $D$. We prove the following.

**Theorem 1.** Let $U_{2n} = \{u_1, \ldots, u_{2n}\}$ be any ET system of order 2 of continuously differentiable functions on $[a, b]$ and let $\mu$ be an integrable function on $[a, b]$ which is continuous and positive on $(a, b)$. Then, for every given set of numbers $h \in H$ there exists a unique set of nodes $x \in D(h)$ such that

\[
(1) \quad L[f] = \sum_{k=1}^{n} a_k \frac{1}{I_k} \int_{x_k}^{x_k+h_k} \mu(t)f(t)\,dt
\]

for every $f$ from the space $U_{2n}$.

We shall call (1) a Gaussian formula.

Note that Theorem 1 holds also in the trivial case $h_1 + \cdots + h_n = b - a$ since the uniqueness of the best coefficients $a_k = I_k$, $k = 1, \ldots, n$, can be easily verified.

The proof of Theorem 1 is given in Section 3 and it is based on some auxiliary results contained in Section 2.

**2. Auxiliary results.** The lemmas in this section hold under weakened conditions on the space $U_{2n}$ and the weight $\mu$. It is enough to assume that the Tchebycheff system $U_{2n}$ consists of continuous functions on $[a, b]$ and $\mu(t)$ is any given continuous weight function on $(a, b)$ such that $\int_a^b \mu(t)\,dt > 0$ for every $a \leq \alpha < \beta \leq b$.

We begin with two lemmas concerning interpolation properties of the Tchebycheff space $U_{2n}$.

**Lemma 1.** Let

\[
a \leq t_1 \leq t_1 + h_1 \leq \cdots \leq t_{2n} \leq t_{2n} + h_{2n} \leq b,
\]

be given points such that $t_i < t_{i+1}$ in case $h_i = h_{i+1} = 0$. Then the interpolation problem

\[
(2) \quad \left( \int_{t_k}^{t_k+h_k} \mu(t)\,dt \right)^{-1} \cdot \int_{t_k}^{t_k+h_k} \mu(t)u(t)\,dt = f_k, \quad k = 1, \ldots, 2n,
\]

is uniquely solvable in $U_{2n}$, for any values $f_k \in \mathbb{R}$.

**Proof.** It is sufficient to show that the corresponding homogeneous interpolation problem admits only the zero solution $u \equiv 0$. In order to do this, suppose that $u \in U_{2n}$ satisfies conditions (2) with $f_k = 0$. Note that in case $h_k = 0$ the corresponding homogeneous condition reduces to $u(t_k) = 0$, and if $h_k > 0$ it implies that $u(t)$ must change sign on $(t_k, t_k + h_k)$. Then $u$ must have at least one zero in every interval $[t_k, t_k + h_k]$, i.e., at least $2n$ zeros in total. But this means that $u \equiv 0$, since $U_{2n}$ is a Tchebycheff space, and the proof is complete.

**Lemma 2.** Let $h \in H$ be fixed. Then for every $x \in \partial D$ there exists a generalized polynomial $u \in U_{2n}$ which is positive on $[a, b] \setminus \bigcup_{k=1}^{n} \Delta_k$ and

\[
(3) \quad \frac{f_k[u]}{I_k} = 0, \quad \text{for } k = 1, \ldots, n.
\]

**Proof.** We consider only the case $x_j + h_j = x_{j+1}$ for some $j \in \{1, \ldots, n - 1\}$, $h_j > 0$, $h_{j+1} > 0$, $x_k + h_k < x_{k+1}$ for all $k \neq j$. The other cases follow similarly.
Any generalized polynomial \( w \) from \( U_{2n} \),
\[
w = c_1 u_1 + \cdots + c_{2n} u_{2n},
\]
is completely determined by its coefficients \( \{c_i\} \). We shall look for a generalized polynomial \( w \) satisfying the conditions (3) and, in addition,
\[
w(x_k + h_k) = 0, \quad \text{for every} \quad k \neq j, j + 1.
\]
If some \( h_k = 0 \), we interpret the conditions \( I_k[w]/I_k = 0 \), \( w(x_k + h_k) = 0 \) as "\( x_k \) is a double zero of \( w \)" (see [4], Theorem 5.1 and Theorem 5.2 in Chapter I, for handling "double zeros" of \( T \)-systems). Thus, we have imposed \( 2n - 2 \) zero conditions on \( w \).

Now, if some of the end points, say \( a \), is not among the prescribed nodes \( x_1, \ldots, x_n \), we define one more condition \( w(a) = 0 \). If \( a = x_1 \) and \( b = x_n + h_n \), then we delete the condition \( w(x_1 + h_1) = 0 \), choose two distinct points \( \xi_1 \) and \( \xi_2 \) outside any of the subintervals \( \Delta_k \), \( k = 1, \ldots, n \) and impose the conditions: "\( w \) has a double zero at \( \xi_1 \)". In this way, we have a system of \( 2n \) linear equations with respect to the coefficients of \( w \). It is easy to see that the corresponding homogeneous system admits only the zero solution since any of the \( 2n \) homogeneous equations leads to a zero of \( w \). Thus the system of conditions imposed on \( u \) determine it uniquely. All zeros of \( w \) lie in the intervals \( \{\Delta_i\}_{i=1}^n \) and also at \( \xi_1 \). Constructing similarly another polynomial \( v \in U_{2n} \) with different point \( \xi_1 \), we conclude that \( u = w + v \) is positive outside \( \{\Delta_i\}_{i=1}^n \) and satisfies (3). The lemma is proved.

The next observation is a simple consequence from known properties of the Tchebycheff systems.

**Lemma 3.** For all \( h \in H \), the coefficients \( a_k \) of the Gaussian formula (1) are uniformly bounded.

**Proof.** It is a well-known fact (see, for example, [6], Chapter II, Theorem 1.4) that every Tchebycheff space contains a strictly positive function. Let \( \tilde{u} \in U_{2n} \) and \( \tilde{u}(t) > 0 \) on \( [a, b] \). Since (1) holds for \( \tilde{u} \) and the coefficients \( a_k \) are strictly positive (see [2], Theorem 1), then
\[
a_k < \int_a^b \mu(x) \tilde{u}(x) \frac{dx}{\min_{x \in [a, b]} \tilde{u}(x)}, \quad k = 1, \ldots, n.
\]

Further, we prove a lemma similar to Lemma 2 from [3], which asserts that the node subintervals of any Gaussian formula are uniformly distant from each other and from the end points of the interval \( [a, b] \).

For each \( 0 < \epsilon < b - a \) we define
\[
H_\epsilon := \{ h \in H : \sum_{k=1}^n h_k \leq b - a - \epsilon \}.
\]

**Lemma 4.** Let \( 0 < \epsilon < b - a \). Then there exists an \( \epsilon_0 \in (0, \epsilon) \) such that all \( h \in H_\epsilon \) and the corresponding nodes \( x \in D(h) \) which define a Gaussian formula (1), satisfy the conditions:
\[
(4) \quad x_1 - a > \epsilon_0, \quad x_{j+1} - x_j - h_j > \epsilon_0, \quad j = 1, \ldots, n - 1, \quad b - x_n - h_n > \epsilon_0.
\]
Let \( I \) be a fixed vector. First we shall find a unique pair of generalized polynomials \( p_i, q_i \) from \( \mathcal{U}_{2n} \) such that

\[
L[u] = \sum_{k=1}^{n} a_k \frac{1}{I_k} I_k[u]
\]

(with \( I_k(u) \) and \( I_k[u] \), defined by \( x_k(u) \) and \( h_k(u) \)). In addition, since \( x^{(0)} \in \partial D(h^{(0)}) \), at least one of the equalities holds:

\[
x_1^{(0)} = a, \quad x_j^{(0)} + h_j^{(0)} = b, \quad j = 1, \ldots, n - 1, \quad x_n^{(0)} + h_n^{(0)} = b.
\]

Using Lemma 2, in any of the above cases we can construct a generalized polynomial \( u \in \mathcal{U}_{2n} \) for which the right hand side of (5) is zero but the integral in the left hand side is positive. This leads to a contradiction and the proof is complete.

**3. Proof of Theorem 1.** Let \( h \in \mathcal{H}_n \) be a fixed vector. First we shall find a system of equations for the nodes \( x \in D(h) \) ensuring the Gaussian property of formula (1). Lemma 4 implies that the nodes of any Gaussian formula (1) belong to the set \( D_{\varepsilon_0}(h) \) defined by (4). Let us choose arbitrary points

\[
a < s_1 < \cdots < s_n < a + \varepsilon_0/2 \quad \text{and} \quad b - \varepsilon_0/2 < t_1 < \cdots < t_n < b.
\]

According to Lemma 1, for each \( i \in \{1, \ldots, n\} \), there exists a unique pair of generalized polynomials \( p_i, q_i \) from \( \mathcal{U}_{2n} \) such that

\[
p_i(s_k) = 0, \quad \frac{1}{I_k} I_k[p_i] = \delta_{ik}, \quad i, k = 1, \ldots, n,
\]

and

\[
q_i(t_k) = 0, \quad \frac{1}{I_k} I_k[q_i] = \delta_{ik}, \quad i, k = 1, \ldots, n.
\]

It is easily seen that \( \{p_i, q_i\}_{i=1}^{n} \) are linearly independent and thus, form a basis in \( \mathcal{U}_{2n} \). Thus, the interval formula (1) is Gaussian if and only if

\[
a_i = L[p_i] = L[q_i], \quad i = 1, \ldots, n,
\]

and the latter is equivalent to the system

\[
\psi_i(x) := \psi_i(x, h) := L[p_i - q_i] = 0, \quad i = 1, \ldots, n.
\]

Let

\[
p_i(t) = \sum_{m=1}^{2n} \alpha_{in} u_m(t), \quad q_i(t) = \sum_{m=1}^{2n} \beta_{in} u_m(t).
\]

We have to show that the system (8) possesses a unique solution in \( D_{\varepsilon_0}(h) \). To this aim, we prove first that the Jacobian of (8) is distinct from zero and has a constant sign.
at any solution of (8). In order to compute the elements of the Jacobian matrix, we note that, in view of Lemma 1, the matrix $D$ (which does not depend on $i$) of the linear system (6) with respect to the coefficients $\{\alpha_{im}\}$ possesses a non-zero determinant. Moreover, by the implicit function theorem, the coefficients are differentiable functions of $x_j$ and

$$\frac{\partial \alpha_{im}}{\partial x_j} = -C_{ij} \frac{\det D_{jm}}{\det D}$$

where $D_{jm}$ is the matrix obtained from $D$ by replacing the $(n+j)$th element of its $m$th column by 1 and the other elements of the column by zero. The constant $C_{ij}$ is given by

$$C_{ij} = \frac{\partial}{\partial x_j} \left\{ \frac{I_j[p_i]}{I_j} \right\}.$$

But, by Cramer’s rule,

$$\alpha_{im} = \frac{\det D_{im}}{\det D}.$$

Therefore

$$\frac{\partial \alpha_{im}}{\partial x_j} = -\frac{\partial}{\partial x_j} \left\{ \frac{I_j[p_i]}{I_j} \right\} \alpha_{jm}.$$

This implies the relation

$$\frac{\partial p_i(t)}{\partial x_j} = -\frac{\partial}{\partial x_j} \left\{ \frac{I_j[p_i]}{I_j} \right\} p_j(t).$$

Similarly we obtain

$$\frac{\partial q_i(t)}{\partial x_j} = -\frac{\partial}{\partial x_j} \left\{ \frac{I_j[q_i]}{I_j} \right\} q_j(t).$$

Using the last relations and the equality

$$L[p_i] = L[q_i] = a_i,$$

we easily compute the elements of the Jacobian matrix

$$J := J(x, h) := \left\{ \frac{\partial \Psi_i}{\partial x_j} \right\}_{i,j=1}^n.$$

We have

$$\frac{\partial \Psi_i}{\partial x_j} = \frac{\partial}{\partial x_j} L[p_i(t) - q_i(t)]$$

$$= L \left[ \frac{\partial}{\partial x_j} (p_i(t) - q_i(t)) \right]$$

$$= -\frac{\partial}{\partial x_j} \left\{ \frac{I_j[p_i]}{I_j} \right\} L[p_j] + \frac{\partial}{\partial x_j} \left\{ \frac{I_j[q_i]}{I_j} \right\} L[q_j]$$

$$= a_j d_j [r_i],$$
where \( r_i := q_i - p_i \) and

\[
d_j[g] := \frac{\partial}{\partial x_j} \left\{ \frac{I_j[g]}{I_j} \right\}.
\]

Thus,

\[
\det J(x, h) = a_1 \cdots a_n \det J_1(x, h),
\]

\( J_1 \) being the matrix

\[
J_1(x, h) := \begin{bmatrix}
    d_1[r_1] & d_2[r_1] & \cdots & d_n[r_1] \\
    d_1[r_2] & d_2[r_2] & \cdots & d_n[r_2] \\
    \vdots & \vdots & \ddots & \vdots \\
    d_1[r_n] & d_2[r_n] & \cdots & d_n[r_n]
\end{bmatrix}.
\]

Besides, the coefficients of any Gaussian formula are positive (see [2]). Thus \( a_j > 0 \) at any solution \( x \) of (8). Next we examine the sign of \( J_1(y, v) \) in the set

\[
E_{\varepsilon_0} := \{(y, v) : v \in H, y \in D_{\varepsilon_0}(v)\}.
\]

\( E_{\varepsilon_0} \) is a bounded connected set with non-empty interior which contains all the points \((x, h)\) corresponding to any Gaussian formula (1).

Now we show that \( \det J_1(y, v) \neq 0 \) at every point \((y, v) \in E_{\varepsilon_0}\). Assume the contrary, i.e., \( \det J_1(x, h) = 0 \) for some \((x, h) \in E_{\varepsilon_0}\). Then there exists a linear dependence between the rows of \( J_1(x, h) \) and thus there exists a non-zero vector \((b_1, \ldots, b_n)\) such that

\[
d_j[r] = \frac{\partial}{\partial x_j} \left\{ \frac{I_j[r]}{I_j} \right\} = 0, \quad j = 1, \ldots, n,
\]

where \( r := \sum_{i=1}^{n} b_i r_i \). In case \( h_j = 0 \) the last condition reduces to \( r'(x_j) = 0 \). Besides, by (6) and (7), \( r(x_j) = 0 \) and thus \( r \) has a double zero at \( x_j \). If \( h_j > 0 \), then \( I_j > 0 \) and (9) leads to

\[
\left( \frac{\partial}{\partial x_j} I_j[r] \right) I_j - I_j[r] \frac{\partial}{\partial x_j} I_j = 0.
\]

On the other hand, (6) and (7) yield

\[
I_j[r] = 0, \quad j = 1, \ldots, n.
\]

Therefore, if \( h_j > 0 \), (9) reduces to

\[
\frac{\partial}{\partial x_j} I_j[r] = 0.
\]

Performing the differentiation we obtain,

\[
\frac{\partial}{\partial x_j} I_j[r] = \mu(x_j + h_j) r(x_j + h_j) - \mu(x_j) r(x_j) = 0.
\]

Since \( I_j[r] = 0 \) and \( \mu(t) > 0 \), the function \( r(t) \) must have at least two zeros in \([x_j, x_j + h_j]\) (counting the multiplicities up to 2). And this holds for every \( j = 1, \ldots, n \).
Therefore \( r \) has at least \( 2n \) zeros in \((a, b)\) and hence \( r \equiv 0 \). This was the point, where we have used that \( \mathcal{U}_{2n} \) is an ET space of order 2. Then
\[
\sum_{i=1}^{n} b_i (q_i - p_i) = 0
\]
and thus, \( \sum_{i=1}^{n} b_i p_i \) must vanish at \( s_1, \ldots, s_n, t_1, \ldots, t_n \). This means that \( b_1 = \cdots = b_n = 0 \), which is a contradiction. Therefore \( \det J_1(y, v) \) does not vanish in \( E_{c_0} \) and consequently \( \det J(x, h) \neq 0 \) (and even has a constant sign) at any solution \( x \) of the system (8).

Now, we are ready to complete the proof of Theorem 1. The existence of (1) was proved in [2] in a more general case. It can be derived easily also from the implicit function theorem. Indeed, for any given \( h \in \mathcal{H} \), consider the family of lengths \( \mathcal{H} \), parameterized by \( \alpha \), \( 0 \leq \alpha \leq 1 \). According to Krein’s theorem (see [5]), the Gaussian quadrature exists for \( \alpha = 0 \). We shall use this fact to extend the solution \( (x(\alpha), h) \) to any \( 0 \leq \alpha \leq 1 \). And this can be done by the implicit function theorem since the Jacobian \( J \) is different from zero at any solution \( (x(\alpha), h) \) of (8) (since \( h \in \mathcal{H} \) for \( 0 \leq \alpha \leq 1 \)).

It remains to prove that the Gaussian quadrature is unique. Assume the contrary, that is, assume that the system (8) has two distinct solutions \( x \) and \( y \). Consider the unique extensions \( x(\alpha), y(\alpha) \) of these solutions for \( \alpha \) going back from 1 to 0. We have \( x(1) = x, y(1) = y \) and \( x(1) \neq y(1) \). On the other hand, by Krein’s theorem \( x(0) = y(0) \). Let us set
\[
\alpha_0 := \max\{\alpha \in [0, 1) : x(\alpha) = y(\alpha)\}.
\]
Then, there must be two different extensions of \( x(\alpha_0) = y(\alpha_0) \) in a neighborhood of \( \alpha_0 \) which is a contradiction to the implicit function theorem. This ends the proof.

It is worth mentioning explicitly the important particular case of interval quadrature formula of Gauss-Christoffel type.

**Corollary 1.** Let \( \mu \) be any integrable function on \([a, b]\) which is continuous and positive on \((a, b)\). Then, for every given set of non-negative numbers \( h \) satisfying the condition
\[
h_1 + \cdots + h_n < b - a,
\]
there exists a unique set of nodes \( x \in D(h) \) such that
\[
\int_a^b \mu(t) f(t) \, dt = \sum_{k=1}^{n} a_k \int_{x_k}^{x_k + h_k} \mu(t) f(t) \, dt
\]
for every algebraic polynomial of degree less than or equal to \( 2n - 1 \).

**Acknowledgments.** This work was accomplished while the second named author was visiting the Linköping University. He would like to thank Professor Lars-Erik Andersson for the invitation and the Swedish Institute for the scholarship.
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