RECONSTRUCTION OF NONUNIFORMLY SAMPLED BANDLIMITED SIGNALS USING DIGITAL FRACTIONAL DELAY FILTERS

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ABSTRACT
This paper considers the problem of reconstructing nonuniformly sampled bandlimited signals using a synthesis system composed of digital fractional delay filters. The overall system can be viewed as a generalization of time-interleaved ADC systems. By generalizing these systems, it is possible to eliminate the errors that are introduced in practice due to time-skew errors.

1. INTRODUCTION
In nonuniform sampling, the time between two consecutive sample instances is dependent on the sampling instances. In this paper, we deal with the situation where the samples can be separated into \( N \) subsequences \( x_k(m), k = 0, 1, \ldots, N-1 \), where \( x_k(m) \) are obtained by sampling an analog signal \( x_a(t) \) with the sampling rate \( 1/(MT) \) at \( t = mMT + t_k \), i.e., \( x_k(m) = x_a(mMT + t_k) \). This sampling scheme is illustrated in Fig. 1(b) for \( M = N = 2 \). Such nonuniformly sampled signals occur in, e.g., time-interleaved analog-to-digital converter (ADC) systems [1]. The question that arises is how to form a new sequence \( y(n) \) from \( x_k(m) \) such that \( y(n) \) is either exactly or approximately (in some sense) equal to \( x(n) = x_a(nT) \), i.e., \( x_a(t) \) sampled uniformly with the sampling period \( T \) [Fig. 1(a)]. For a time-interleaved ADC system, \( N = M \) and it is desired to have \( t_k = kT \) since \( y(n) = x(n) \) then is obtained by interleaving the \( x_k(m) \)'s. However, in practice, \( t_k = kT + \Delta t_k \), where \( \Delta t_k \) are time-skew errors. This results in aliasing components in \( Y(e^{j\omega T}) \) which means that \( y(n) \neq x(n) \). Thus, the information in \( y(n) \) is no longer the same as that in \( x(n) \). To retain the information, we propose in this paper the use of a synthesis system composed of \( N \) parallel digital fractional delay filters. The overall system (including sampling and reconstruction) can be viewed as a generalization of time-interleaved ADC systems, to which the former reduces as a special case. We show that the proposed system, with proper ideal digital filters, can achieve \( y(n) = x(n) \) (perfect reconstruction (PR)). We also consider the case in which \( y(n) \) and \( x(n) \) contain the same information in the lower frequency region [regionally perfect reconstruction (RPR)].

If the \( t_k \)'s are distinct, and \( x_k(t) \) bandlimited, then \( x_a(t) \) can be recovered from the \( x_k(m) \)'s through analog reconstruction functions [2]–[5]. However, it is very difficult to practically implement analog functions with high precision. It is therefore desired to recover \( x(n) \) by using reconstruction methods in the digital domain. One can then retain \( x_a(t) \) by using a conventional digital-analog converter (DAC). Methods for retaining \( x(n) \) from the \( x_k(m) \)'s by means of digital signal processing methods have been considered recently in [5], [6]. In [5], a digital filter bank with ideal noncausal filters are used. The issue of using practical causal filters approximating the ideal ones was not treated. The method in [6] employs causal interpolation functions but it is not clear how to select them so that RPR is approximated (in some sense) as close as desired. (In practice, one approximates RPR since a certain frequency region must be used for filter transition bands etc.) Using instead the proposed system, we can approximate RPR as close as desired by properly designing the digital fractional delay filters [7]. We finally mention another approach in the digital domain proposed in [8]. That is though a frequency domain approach which only recovers the spectrum at a finite number of frequencies. It may therefore be less suitable for real-time applications. The proposed system can on the other hand indeed be used for such applications. In particular, if properly implemented, the digital filters need not be redesigned in case the time skews \( t_k \) are changed. It suffices to adjust some multiplier coefficients values that are uniquely determined by the \( t_k \)'s [7].

2. UNIFORM SAMPLING AND HYBRID ANALOG/DIGITAL FILTER BANKS
In this section, we briefly recapitulate uniform sampling and hybrid analog/digital filter banks, the latter of which is convenient to use when analyzing nonuniformly sampled signals.

2.1 Uniform Sampling
Let \( x(n) \) be a sequence obtained by sampling the analog input signal \( x_a(t) \) uniformly at the time instances \( nT \), for all integers \( n \), i.e.,

\[
x(n) = x_a(nT), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots
\]

where \( T \) is the sampling period and \( f_{\text{sample}} = 1/T \) is the sampling frequency. The Fourier transforms of \( x(n) \) and \( x_a(t) \) are related according to Poisson's summation formula as

\[
X(e^{j\omega T}) = \frac{1}{T} \sum_{r=-\infty}^{\infty} X_a(j\omega - \frac{2\pi p}{MT})
\]

(2)

2.2 Hybrid Analog/Digital Filter Banks
Consider the system in Fig. 2, which we refer to as a hybrid analog/digital filter bank or filter bank ADC. This system makes use of an analog analysis filter bank, uniform samplers and quantizers, upsamplers, and a digital synthesis filter bank. The sampling and quantization take place at the output of the analysis filters with a sampling frequency of \( 1/T_1 = f_{\text{sample}}/M \), since \( T_1 = MT \). In the filter bank ADC, both the sampling and quantizations are thus performed at the low sampling rate \( f_{\text{sample}}/M \). Ignoring the quantizations in the system of Fig. 2, the Fourier transform of the output sequence \( y(n) \) is easily obtained as [7]

\[
Y(e^{j\omega T}) = \frac{1}{T} \sum_{p=-\infty}^{\infty} V_p(j\omega)X_a\left(j\omega - \frac{2\pi p}{MT}\right)
\]

(3)

where

\[
V_p(j\omega) = \sum_{k=0}^{N-1} G_k(e^{j\omega T})H_k\left(j\omega - \frac{2\pi p}{MT}\right)
\]

(4)

Consider \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) as given by (2) and (3), respectively. Recall that the spectrum of a sampled signal always is periodic.
with a period of $2\pi$ ($2\pi$-periodic). Thus, both $X(e^{\text{i}\omega T})$ and $Y(e^{\text{i}\omega T})$ are apparently $2\pi$-periodic provided that $G_j(e^{\text{i}\omega T})$ are $2\pi$-periodic. Thus, it suffices to consider $X(e^{\text{i}\omega T})$ and $Y(e^{\text{i}\omega T})$ in the interval $-\pi \leq \omega T \leq \pi$. We treat two different types of reconstruction.

### 3. PERFECT AND REGIONALLY PERFECT RECONSTRUCTION SYSTEMS

**Perfect reconstruction:** The system in Fig. 2 is a perfect reconstruction (PR) system if

$$Y(e^{\text{i}\omega T}) = c e^{-\text{i}\omega T} X(e^{\text{i}\omega T}), \quad |\omega T| \leq \pi$$

(5)

for some non-zero constant $c$ and integer constant $d$. In the time-domain we have in the PR case $y(n) = cx(n-d)$. That is, with $c = 1$, $y(n)$ is simply a shifted version of $x(n)$. From (2), (3), and (5), we see that PR is obtained if

$$V_p(j\omega) = \begin{cases} c e^{-\text{i}\omega T}, & p = rM, \quad |\omega| \leq \pi/T \\ 0, & p \neq rM, \quad |\omega| \leq \pi/T \end{cases}$$

(6)

for all integers $r$.

**Regionally perfect reconstruction:** Let $x(n)$ and $y(n)$ be separated as

$$x(n) = x_{\text{low}}(n) + x_{\text{high}}(n), \quad y(n) = y_{\text{low}}(n) + y_{\text{high}}(n)$$

(7)

with

$$X(e^{\text{i}\omega T}) = X_{\text{low}}(e^{\text{i}\omega T}) + X_{\text{high}}(e^{\text{i}\omega T}), \quad Y(e^{\text{i}\omega T}) = Y_{\text{low}}(e^{\text{i}\omega T}) + Y_{\text{high}}(e^{\text{i}\omega T})$$

(8)

where

$$X_{\text{low}}(e^{\text{i}\omega T}) = 0, \quad \omega_0 T < |\omega| \leq \pi$$

(9)

$$X_{\text{high}}(e^{\text{i}\omega T}) = 0, \quad |\omega| \leq \omega_0 T$$

$$Y_{\text{low}}(e^{\text{i}\omega T}) = 0, \quad \omega_0 T < |\omega| \leq \pi$$

(10)

$$Y_{\text{high}}(e^{\text{i}\omega T}) = 0, \quad |\omega| \leq \omega_0 T$$

The system in Fig. 2 is a regionally perfect reconstruction (RPR) system if

$$Y(e^{\text{i}\omega T}) = c e^{-\text{i}\omega T} X(e^{\text{i}\omega T}), \quad |\omega T| \leq \omega_0 T$$

(11)

or, equivalently,

$$Y_{\text{low}}(e^{\text{i}\omega T}) = c e^{-\text{i}\omega T} X_{\text{low}}(e^{\text{i}\omega T}), \quad |\omega T| \leq \pi$$

(12)

for some non-zero constant $c$ and integer constant $d$. In the time-domain we have in the RPR case $y_{\text{low}}(n) = c x_{\text{low}}(n-d)$. That is, with $c = 1$, $y_{\text{low}}(n)$ is simply a shifted version of $x_{\text{low}}(n)$. However, $y(n)$ is not a shifted version of $x(n)$, i.e., $y(n) = cx(n-d)$. From (2), (3), and (11), we see that RPR is obtained if

$$V_p(j\omega) = \begin{cases} c e^{-\text{i}\omega T}, & p = rM, \quad |\omega| \leq \omega_0 \\ 0, & p \neq rM, \quad |\omega| \leq \omega_0 \end{cases}$$

(13)

for all integers $r$. Regionally perfect reconstruction systems are of interest in oversampled systems where $x_{\text{high}}(n)$ carries the essential information, whereas $x_{\text{low}}(n)$ contains undesired components (e.g., noise) to be removed by digital and/or analog filters [9].

**PR and RPR Systems for Bandlimited Signals:** When $X_j(j\omega)$ is bandlimited, only a finite number of terms in the summations of (2) and (3) need to be handled in the interval $-\pi \leq \omega T \leq \pi$. We consider two different cases.

**Case A (PR case):** Let $x_d(t)$ be bandlimited according to

$$X_d(j\omega) = 0, \quad |\omega| \geq \pi/T$$

(14)

Then, PR is now obtained if

$$V_p(j\omega) = \begin{cases} c e^{-\text{i}\omega T}, & p = 0, \quad |\omega| < \pi/T \\ 0, & |\omega| \geq \pi/T \end{cases}$$

(15)

where $K_0$ is given by (14). In this case, $x_d(t)$ can thus be retained from $x(n)$ as well as $y(n)$ provided that the system in Fig. 2 is a PR system.

**Case B (RPR case):** Let $x_d(t)$ be bandlimited according to

$$X_d(j\omega) = 0, \quad |\omega| \geq \omega_1$$

(16)

and separated as

$$x_d(t) = x_{d,\text{low}}(t) + x_{d,\text{high}}(t)$$

(17)

with

$$X_d(j\omega) = X_{d,\text{low}}(j\omega) + X_{d,\text{high}}(j\omega)$$

(18)

where

$$X_{d,\text{low}}(j\omega) = 0, \quad |\omega| > \omega_0$$

(19)

$$X_{d,\text{high}}(j\omega) = 0, \quad |\omega| \leq \omega_0, \quad |\omega| \geq \omega_1$$

(20)

for all integers $r$. It is easy to verify that we only need to consider 2$K_0$$+1$ terms in (3), for $p = -K_0, -K_0 - 1, ... , K_0$, where

$$K_0 = \left[ M(\pi + \omega_0 T)/(2\pi) \right] - 1$$

(21)

where $\lfloor x \rfloor$ stands for the smallest integer larger than or equal to $x$. Further, in the range $-\omega_0 T \leq \omega T \leq \omega_0 T$, with $\omega_0$ being restricted as in (20), it is readily verified that we only need to consider 2$K_0$$+1$ terms in (3), for $p = -K_0, -K_0 - 1, ... , K_0$, where

$$K_0 = \left[ M(\omega_0 T + \omega_1 T)/(2\pi) \right] - 1$$

(22)

Obviously, RPR is now obtained if...
as well as

\[ y(t) \]

obtained. The delay an input by a fraction of the sample interval \[ [10] \].

### 4.1 Proposed System of Fractional Delay Filters for PR and RPR

For PR it is required that \( V_{p}(j\omega) \) can be obtained by sampling the output signals from \( H_{k}(s) \) in the system of Fig. 2 if these filters are selected according to (22) and \( A_{p}(j\omega) \) is some arbitrary complex-valued functions. From (26) and (30) we obtain

\[ V_{p}(j\omega) = \frac{1}{M} e^{-j\omega T} \sum_{k=0}^{N-1} a_{k} e^{\frac{2\pi p}{MT} j} \]

where \( A_{k}(e^{j\omega T}) \) are some arbitrary complex-valued functions. From (26) and (30) we obtain

\[ V_{p}(j\omega) = \frac{1}{M} e^{-j\omega T} \sum_{k=0}^{N-1} a_{k} e^{\frac{2\pi p}{MT} j} \]

where

\[ A_{p}(j\omega) = \frac{1}{M} \sum_{k=0}^{N-1} a_{k} A_{k}(e^{j\omega T}) e^{\frac{2\pi p}{MT} j} \]

For RPR it is required that \( V_{p}(j\omega) \) as given by (31) fulfills (23). That is, RPR is obtained if, again, (29) is satisfied.

### 4.2 Computing the \( a_{k} \)'s

For PR, (29) must be fulfilled. This equation can be written in matrix form

\[ Ba = c \]

where \( B \) is a \((2K_{p}+1) \times N \) matrix according to

\[ B = \begin{bmatrix}
  u_{0} & u_{-K_{0}} & \ldots & u_{-N_{-1}} \\
  u_{1} & u_{-K_{1}} & \ldots & u_{-N_{-1}} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & u_{K_{0}} & \ldots & u_{K_{N_{-1}}} \\
\end{bmatrix} \]

where

\[ u_{k} = e^{\frac{2\pi k}{MT}} \]

Further, \( a \) is a column vector with \( N \) elements, and \( c \) is a column vector with \( 2K_{p}+1 \) elements according to

\[ a = \begin{bmatrix} a_{0} & a_{1} & \ldots & a_{N_{-1}} \end{bmatrix}^{T}, \quad c = \begin{bmatrix} c_{0} & c_{1} & \ldots & c_{2K_{0}} \end{bmatrix}^{T} \]

where \( T \) stands for the transpose (without complex conjugate). The \( a_{k} \)'s are the unknowns whereas the \( c_{q} \)'s are

\[ c_{q} = \begin{cases}
  M, & q = K_{0} \\
  0, & q = 0, 1, \ldots, 2K_{p}, \quad q \neq K_{0}
\end{cases} \]

Equation (33) is a system of \( 2K_{p}+1 \) linear equations with \( N \) unknown parameters \( a_{k} \). We distinguish two different cases.

**Case I:** \( 2K_{p}+1 = N \). In this case, the number of unknowns equals the number of equations. The \( a_{k} \)'s can in this case be uniquely determined under the conditions stated by the following theorem. The proof is given in [7].

**Theorem 1:** If \( B \) and \( c \) are as given by (34) and (36), respectively, \( 2K_{p}+1 = N \), and \( t_{k}, t_{p}, M, T, k, m, r \in Z \), then there exists a unique \( a \) satisfying (33), and thereby also unique \( a_{k} \)'s satisfying (29). Further, all the \( a_{k} \)'s in \( a \) are real-valued constants and obtained from (33) as \( a = B^{-1}c \).
then there exists a unique $M \leq N$ such that $K_0 \leq \frac{\omega}{T}$.

We prove the following theorem. The proof is given in [7].

Theorem 1: If $\hat{M}$ is odd and $2N_0 \leq N$, then $\hat{M}_0 \leq \frac{N}{2}$. If $\hat{M}$ is even and $2N_0 \leq N$, then $\hat{M}_0 \leq \frac{N}{2} - 1$.

5. RELATION TO TIME-INTERLEAVED ADC SYSTEMS

This section points out the relation between the overall system (i.e., nonuniform sampling with the proposed synthesis system) and time-interleaved ADC systems. As we shall see, the former can be viewed as a generalization of the latter. Let $\hat{M}$ be given by

$$\hat{M} = \left\lfloor \frac{N}{2} \right\rfloor + 1$$

for all $N$. Thus, PR is obtained. We have here a time-interleaved ADC system [1]. The output sequence $y(n)$ is obtained by interleaving the $x_k(m)$’s. In practice, $\Delta_n$ will however no longer be exactly zero. In time-interleaved ADC systems, this introduces aliasing components in the output sequence. If $\Delta_n$ are known, aliasing in the band $-\omega_0 T < \omega < \omega_0 T$ is avoided by employing the proposed system with $G_k(e^{j\omega T})$ taking the form of (30). In this case, PR can not be achieved since $N = M$ and PR requires that $K_0 \leq \frac{N}{2}$ [see (14)]. Thus, neither $2K_0 + 1$ nor $2K_0 + 1 < N$ can be fulfilled. With $N = M$, the proposed synthesis system can thus only achieve RPR. Given $N = M$, $K_0$, and $\omega_0$, the maximum value of $\omega_0 T$ we can allow and still obtain RPR is

$$\omega_0 T \leq 2\pi(K_0 + 1)/M - \omega_0 T$$

Next consider the case with $N = M$ and where $G_k(e^{j\omega T})$ are given by (27) with $\omega_0 = \pi/N$, $k = 0, \ldots, N-1$, $c = d = 0$. In this case, $V_p(j\omega)$ in (26) becomes

$$V_p(j\omega) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi k\omega / N} = \begin{cases} 1, & p = 0 \\ 0, & p \neq 0 \end{cases}$$

for $|\omega| < \pi/T$, and $V_p(j\pi T) = 0$. Thus, PR is obtained.

Again, $\Delta_n$ will in practice no longer be exactly zero. If $\Delta_n$ are known, the $a_k$’s can be computed as outlined in the previous section, i.e., from (33) if $N$ is odd and $2K_0 + 1 = N$, or (38) if $2K_0 + 1 < N$. As opposed to the $M$-channel case, we can in the $N$-channel case achieve both PR and RPR by selecting $K_0$ according to (14) and (22), respectively, and of course choosing $N$ so that $2K_0 + 1 \leq N$. In the PR and RPR cases, $G_k(e^{j\omega T})$ are selected according to (27) and (30), respectively. To achieve RPR, for given $M$, $K_0$, and $\omega_0$, $\omega_0 T$ must again satisfy (44). If $2K_0 + 1 = N$ we get

$$\omega_0 T \leq \pi(N + 1)/M - \omega_0 T$$

Hence, by increasing the number of channels we obtain RPR over a wider frequency region.

6. CONCLUSION

A synthesis system of fractional delay filters for reconstructing nonuniformly sampled bandlimited signals has been proposed. The overall system can be viewed as a generalization of time-interleaved ADC systems, to which the former reduces as a special case. By generalizing these systems, it is possible to eliminate the errors at the output that are introduced in practice due to time-skew errors. Both perfect reconstruction (PR) and regionally perfect reconstruction (RPR) systems were considered, and it was shown how to obtain such systems by selecting the ideal digital filters properly. The proposed system is suitable for real-time applications. In particular, if properly implemented, the digital filters need not be redesigned in case the time skews $\tau_r$ are changed. It suffices to adjust some multiplier coefficient values that are uniquely determined by the $\tau_r$’s. More details about this, as well as error analysis (e.g., the effects of using practical filters approximating the ideal ones and quantization noise) are given in [7].

7. REFERENCES


