Discrete Ramanujan-Fourier Transform of Even Functions (mod $r$)

Pentti Haukkanen
Department of Mathematics, Statistics and Philosophy
FI-33014 University of Tampere
Finland
mapehau@uta.fi


Abstract. An arithmetical function $f$ is said to be even (mod $r$) if $f(n) = f((n,r))$ for all $n \in \mathbb{Z}^+$, where $(n,r)$ is the greatest common divisor of $n$ and $r$. We adopt a linear algebraic approach to show that the Discrete Fourier Transform of an even function (mod $r$) can be written in terms of Ramanujan’s sum and may thus be referred to as the Discrete Ramanujan-Fourier Transform.

2000 Mathematics Subject Classification. 11A25, 11L03

Key words. Discrete Fourier Transform, arithmetical functions, periodic functions, even functions, Ramanujan sums, Cauchy product

1 Introduction

By an arithmetical function we mean a complex-valued function defined on the set of positive integers. For a positive integer $r$, an arithmetical function $f$ is said to be periodic (mod $r$) if $f(n + r) = f(n)$ for all $n \in \mathbb{Z}^+$. Every periodic function $f$ (mod $r$) can be written uniquely as

$$f(n) = r^{-1} \sum_{k=1}^{r} F_f(k) \epsilon_k(n), \quad (1)$$
where
\[ F_f(k) = \sum_{n=1}^{r} f(n)\epsilon_k(-n) \quad (2) \]
and \( \epsilon_k \) denotes the periodic function \( \text{(mod } r\text{)} \) defined as
\[ \epsilon_k(n) = \exp(2\pi i n / r). \]

The function \( F_f \) in (2) is referred to as the Discrete Fourier Transform (DFT) of \( f \), and (1) is the Inverse Discrete Fourier Transform (IDFT).

An arithmetical function \( f \) is said to be even \( \text{(mod } r\text{)} \) if
\[ f(n) = f((n, r)) \]
for all \( n \in \mathbb{Z}^+ \), where \( (n, r) \) is the greatest common divisor of \( n \) and \( r \). It is easy to see that every even function \( \text{(mod } r\text{)} \) is periodic \( \text{(mod } r\text{)} \).

Ramanujan’s sum \( C(n, r) \) is defined as
\[ C(n, r) = \sum_{k \text{ (mod } r\text{)}}^{(k, r) = 1} \exp(2\pi i n / r) \]
and is an example of an even function \( \text{(mod } r\text{)} \).

In this paper we show that the DFT (2) and IDFT (1) of an even function \( f \) \( \text{(mod } r\text{)} \) can be written in a concise form using Ramanujan’s sum \( C(n, r) \), see Section 3. We also review a proof of (1) and (2) for periodic functions \( \text{(mod } r\text{)} \), see Section 2, and review (1) and (2) for the Cauchy product of periodic functions \( \text{(mod } r\text{)} \), see Section 4. The Cauchy product of periodic functions \( f \) and \( g \) \( \text{(mod } r\text{)} \) is defined as
\[ (f \circ g)(n) = \sum_{a+b=n \text{ (mod } r\text{)}} f(a)g(b). \]

The results of this paper may be considered to be known. They have not been presented in exactly this form and we hope that this paper will provide a clear approach to the elementary theory of even functions \( \text{(mod } r\text{)} \).

The concept of an even function \( \text{(mod } r\text{)} \) originates from Cohen [2] and was further studied by Cohen in subsequent papers [3, 4, 5]. General accounts of even functions \( \text{(mod } r\text{)} \) can be found in the books by McCarthy [8] and Sivaramakrishnan [10]. For recent papers on even functions \( \text{(mod } r\text{)} \) we refer to [9, 11]. Material on periodic functions \( \text{(mod } r\text{)} \) can be found in the book by Apostol [11].
2 Proof of (1) and (2)

Let \( P_r \) denote the set of all periodic arithmetical functions (mod \( r \)). It is clear that \( P_r \) is a complex vector space under the usual addition and scalar multiplication. In fact, \( P_r \) is isomorphic to \( \mathbb{C}^r \). Further, \( P_r \) is a complex inner product space under the Euclidean inner product given as

\[
\langle f, g \rangle = \sum_{n=1}^{r} f(n) \overline{g(n)} = (f \circ \zeta)(r),
\]

where \( \zeta \) is the constant function 1. The set \( \{r^{-1/2} \epsilon_k : k = 1, 2, \ldots, r\} \) is an orthonormal basis of \( P_r \). Thus, every \( f \in P_r \) can be written uniquely as

\[
f(n) = \sum_{k=1}^{r} \langle f, r^{-1/2} \epsilon_k \rangle r^{-1/2} \epsilon_k(n),
\]

where

\[
\langle f, r^{-1/2} \epsilon_k \rangle = \sum_{n=1}^{r} f(n) r^{-1/2} \epsilon_k(n) = r^{-1/2} \sum_{n=1}^{r} f(n) \epsilon_k(-n).
\]

This proves (1) and (2).

3 DFT and IDFT for even functions (mod \( r \))

Let \( E_r \) denote the set of all even functions (mod \( r \)). The set \( E_r \) forms a complex vector space under the usual addition and scalar multiplication. In fact, \( E_r \) is a subspace of \( P_r \). Thus (1) and (2) hold for \( f \in E_r \). We can also present (1) and (2) for \( f \in E_r \) in terms of Ramanujan’s sum as is shown below.

Note that Ramanujan’s sum \( C(n, r) \) is an integer for all \( n \) and can be evaluated by addition and subtraction of integers. In fact, \( C(n, r) \) can be written as \( C(n, r) = \sum_{d|\tau(n,r)} d \mu(r/d) \), where \( \mu \) is the Möbius function.

An arithmetical function \( f \in E_r \) is completely determined by its values \( f(d) \) with \( d|r \). Thus \( E_r \) is isomorphic to \( \mathbb{C}^{\tau(r)} \), where \( \tau(r) \) is the number of divisors of \( r \). The inner product (3) in \( P_r \) can be written in \( E_r \) in terms of the Dirichlet convolution. In fact, we have

\[
\sum_{k=1}^{r} \frac{1}{(k, r) = d} = \sum_{j=1}^{r/d} 1 = \varphi(r/d),
\]

where \( \varphi \) is Euler’s totient function. This proves (1) and (2) for \( f \in E_r \).
where $\phi$ is Euler’s totient function, and thus (3) can be written for $f, g \in E_r$
as
$$\langle f, g \rangle = \sum_{d | r} f(d)g(d)\phi(r/d) = (fg\ast\phi)(r),$$
where $\ast$ is the Dirichlet convolution.

**Theorem 3.1.** The set

$$\{(r\phi(d))^{-\frac{1}{2}}C(\cdot, d) : d \mid r\}$$

is an orthonormal basis of the inner product space $E_r$.

**Proof** As the dimension of the inner product space $E_r$ is $\tau(r)$ and the number of elements in the set (5) is $\tau(r)$, it suffices to show the set (5) is an orthonormal subset of $E_r$. This follows easily from the relation

$$\sum_{e \mid r} C(r/e, d_1)C(r/e, d_2)\phi(e) = \begin{cases} r\phi(d_1) & \text{if } d_1 = d_2, \\ 0 & \text{otherwise,} \end{cases}$$

where $d_1 \mid r$ ja $d_2 \mid r$ (see [8, p. 79]). □

We now present (1) and (2) for $f \in E_r$.

**Theorem 3.2.** Every $f \in E_r$ can be written uniquely as

$$f(n) = r^{-1} \sum_{d \mid r} R_f(d)C(n, d),$$

where

$$R_f(d) = \phi(d)^{-1} \sum_{n=1}^{r} f(n)C(n, d).$$

**Proof** On the basis of Theorem 3.1,

$$f(n) = \sum_{d \mid r} \langle f, (r\phi(d))^{-\frac{1}{2}}C(\cdot, d) \rangle (r\phi(d))^{-\frac{1}{2}}C(n, d).$$

Applying (3) to (8) we obtain (6) and (7). □

The function $R_f$ in (7) may be referred to as the Discrete Ramanujan-Fourier Transform of $f$, and (6) may be referred to as the Inverse Discrete Ramanujan-Fourier Transform. Cf. [8].

Another expression of (7) can be obtained easily. Namely, applying (4) to (5) and then applying

$$\phi(e)C(r/e, d) = \phi(d)C(r/d, e)$$
(see [8, p. 93]) we obtain

\[ R_f(d) = \sum_{e|d} f(r/e)C(r/d,e). \] (9)

Note that (6) can also be derived from (1). In fact, if \( f \in E_r \), then (2) can be written as

\[
F_f(k) = \sum_{n=1}^{r} f(n) \exp(-2\pi i k n/r) \\
= \sum_{e|r} \sum_{\substack{n=1 \\ (n,r)=e}}^{r} f(e) \exp(-2\pi i k n/r) \\
= \sum_{e|r} f(e) \sum_{\substack{m=1 \\ (m,r/e)=1}}^{r/e} \exp(-2\pi i k m/(r/e)) \\
= \sum_{e|r} f(e)C(k,r/e).
\]

A similar argument shows (6) with \( R_f(d) = F_f(r/d) \). We omit the details.

4 The Cauchy product

It is well known that if \( h \) is the Cauchy product of \( f \in P_r \) and \( g \in P_r \), then \( F_h = F_f F_g \). This follows from the property

\[
\sum_{a+b\equiv n \pmod{r}} \epsilon_k(a)\epsilon_j(b) = \begin{cases} 
re_k(n) & \text{if } k \equiv j \pmod{r}, \\
0 & \text{otherwise}.
\end{cases}
\]

Analogously, if \( h \) is the Cauchy product of \( f \in E_r \) and \( g \in E_r \), then \( R_h = R_f R_g \). This follows from the property

\[
\sum_{a+b\equiv n \pmod{r}} C(a,d_1)C(b,d_2) = \begin{cases} 
rC(a,d_1) & \text{if } d_1 = d_2, \\
0 & \text{otherwise},
\end{cases}
\]

where \( d_1 \mid r \) ja \( d_2 \mid r \) (see [10, p. 333]).

References


