Legendre Spectral Projection Methods for Urysohn Integral Equations

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Abstract

In this paper, we consider the Legendre spectral Galerkin and Legendre spectral collocation methods to approximate the solution of Urysohn integral equation. We prove that the approximated solutions of the Legendre Galerkin and Legendre collocation methods converge to the exact solution with the same orders, \( \mathcal{O}(n^{-r}) \) in \( L^2 \)-norm and \( \mathcal{O}(n^{\frac{1}{2}-r}) \) in infinity norm, and the iterated Legendre Galerkin solution converges with the order \( \mathcal{O}(n^{-2r}) \) in both \( L^2 \)-norm and infinity norm, whereas iterated Legendre collocation solution converges with the order \( \mathcal{O}(n^{-r}) \) in both \( L^2 \)-norm and infinity norm, \( n \) being the highest degree of Legendre polynomial employed in the approximation and \( r \) being the smoothness of the kernel. We are able to obtain similar superconvergence rates for the iterated Galerkin solution for Urysohn integral equations with smooth kernel as in the case of piecewise polynomial basis functions.

Keywords: Urysohn integral equations, smooth kernels, Spectral method, Galerkin method, collocation method, Legendre polynomials, superconvergence rates

1. Introduction

In this section, we consider the following Urysohn integral equation

\[
x(t) - \int_{-1}^{1} k(t, s, x(s)) ds = f(t), -1 \leq t \leq 1,
\]

where \( k \) and \( f \) are known functions and \( x \) is the unknown solution to be found in a Banach space \( X \).

Several numerical methods for approximating the solution of Fredholm integral equation of type (1.1) are known. Numerical methods which use properties of a classical Schauder basis in the Banach space \( C([a, b] \times [a, b]) \), for approximation of the fixed point of Fredholm integral operator, are given in [17], [18]. Various spectral methods are available in literature to solve integral equations (see [13], [16], [19], [20], [21], [22], [24], [25]). The Galerkin, collocation,
Petrov-Galerkin, Nyström methods are most commonly used projection methods for finding numerical solutions of the equation of type (1.1) (see [3], [4], [5], [6]). In [3], [4] Atkinson developed general framework for Galerkin and collocation methods for the solution of Urysohn integral equations (1.1) using piecewise polynomial basis functions and obtain superconvergence results using the iterated version of the approximate solutions. Discretized versions of collocation and Galerkin methods for Urysohn integral equations were introduced in [5] and [6].

Let \( X_n \) be a sequence of piecewise polynomial subspaces of \( X \) of degree \( \leq r - 1 \) on \([-1, 1]\) and \( P_n \) be either orthogonal or interpolatory bounded projections from \( X \) onto \( X_n \). Then in Galerkin or in collocation method, the Urysohn integral equation (1.1) is approximated by

\[
x_n - P_n K(x_n) = P_n f,
\]

and its iterated solution is defined by \( \bar{x}_n = f + K(x_n) \). Under some suitable conditions on the kernel \( k \) and the right hand side function \( f \) of the equation (1.1), it is known that the orders of convergence for Galerkin and collocation solutions are \( O(h^r) \) and for the iterated Galerkin and iterated collocation solutions are \( O(h^{2r}) \), where \( h \) denotes the norm of the partition (see [3], [4]). However to obtain more accurate solutions using piecewise polynomial basis functions, one has to increase the number of partitioning points. Therefore a large system of nonlinear equations has to be solved, which is very much expensive computationally.

In this paper, we consider the Galerkin and collocation methods and their iterated versions to approximate the solutions of Urysohn integral equation (1.1) with a smooth kernel, using global polynomial basis functions. Use of global polynomials will imply smaller nonlinear systems, something which is highly desirable in practical computations. Hence we choose to use global polynomials rather than piecewise polynomial basis functions in this paper. In particular, we use Legendre polynomials, which can be generated recursively with ease and possess nice property of orthogonality. Further, these Legendre polynomials are less expensive computationally compared to piecewise polynomial basis functions. We obtain almost similar convergence rates using Legendre polynomial bases as in the case of piecewise polynomial bases.

We organize this paper as follows. In Section 2, we discuss the Legendre spectral Galerkin and Legendre spectral collocation methods to obtain superconvergence results. In Section 3, numerical results are given to illustrate the theoretical results. Throughout this paper, we assume that \( c \) is a generic constant.

2. Legendre Spectral Galerkin and Collocation Methods: Urysohn Integral Equations with Smooth Kernel

In this section, we describe the Galerkin and collocation methods for solving Urysohn integral equations using Legendre polynomial basis functions.

Let \( X = L^2[-1, 1] \) or \( C[-1, 1] \) and consider the following Urysohn integral equation

\[
x(t) - \int_{-1}^{1} k(t, s, x(s)) \, ds = f(t), \quad -1 \leq t \leq 1,
\]

(2.1)
where \( k(.,.,.) \) and \( f \) are known functions and \( x \) is the unknown function to be determined. Let

\[
\mathcal{K}(x)(t) = \int_{-1}^{1} k(t, s, x(s)) \, ds, \quad x \in \mathcal{X}.
\]

Then the equation (2.1) can be written as

\[
x - \mathcal{K}(x) = f.
\]

The Frechet derivative \( \mathcal{K}'(x) \) is the linear integral operator defined by

\[
(\mathcal{K}'(x)h)(t) = \int_{-1}^{1} \frac{\partial}{\partial u} k(t, s, x(s)) h(s) \, ds = \int_{-1}^{1} k_u(t, s, x(s)) h(s) \, ds.
\]

Throughout the paper, the following assumptions are made on \( f \) and \( k(t, s, u) \):

1. \( f \in C[-1, 1] \),
2. \( M = \sup_{t,s \in [-1,1]} |k_u(t, s, x(s))| < \infty \),
3. the kernel \( k(t, s, u), k_u(t, s, u) \) and \( \frac{\partial}{\partial t} k_u(t, s, u) \), satisfies Lipschitz conditions in the third variable \( u \), i.e., for any \( u_1, u_2 \in \mathbb{R} \), \( \exists c_1, c_2, c_3 > 0 \) such that

\[
|k(t, s, u_1) - k(t, s, u_2)| \leq c_1 |u_1 - u_2|,
\]

\[
|k_u(t, s, u_1) - k_u(t, s, u_2)| \leq c_2 |u_1 - u_2|,
\]

and

\[
\left| \frac{\partial}{\partial t} k_u(t, s, u_1) - \frac{\partial}{\partial t} k_u(t, s, u_2) \right| \leq c_3 |u_1 - u_2|.
\]

Next, we define the operator \( \mathcal{T} \) on \( \mathcal{X} \) by

\[
\mathcal{T} u := \mathcal{K}(u) + f, \quad u \in \mathcal{X},
\]

then the equation (2.2) can be written as

\[
x = \mathcal{T} x.
\] (2.3)

The following theorem gives the condition for the existence of unique solution of the equation (2.3) in \( \mathcal{X} \).

**Theorem 2.1.** Let \( \mathcal{X} = L^2[-1, 1] \) or \( C[-1, 1] \) and \( f \in \mathcal{X} \). Assume \( k(.,.,.) \in C([-1, 1] \times [-1, 1] \times \mathbb{R}) \) satisfies the Lipschitz condition in the third variable, i.e.,

\[
|k(t, s, u_1) - k(t, s, u_2)| \leq c_1 |u_1 - u_2|, \quad u_1, u_2 \in \mathbb{R},
\]

with \( 2c_1 < 1 \). Then the operator equation \( x = \mathcal{T} x \) has a unique solution \( x_0 \in \mathcal{X} \), i.e., we have \( x_0 = \mathcal{T} x_0 \).
Proof of the above theorem can be easily done using similar technique given in Theorem 2.4 of [15].

For the rest of the paper we assume that the kernel $k(., .) \in C^r([-1,1] \times [-1,1] \times \mathbb{R})$. For $j = 0, 1, 2, ..., r$, we have

$$||[K'(x_0)x]^j(t)|| = \left| \int_{-1}^{1} \frac{\partial^j}{\partial t^j} k_u(t, s, x_0(s)) x(s) ds \right| \leq \sup_{t, s \in [-1,1]} \left| \frac{\partial^j}{\partial t^j} k_u(t, s, x_0(s)) \right| \int_{-1}^{1} |x(s)| ds \leq \sqrt{2} ||k||_{j, \infty} ||x||_{L^2} \leq 2 ||k||_{j, \infty} ||x||_{\infty}$$

where $||k||_{r, \infty} = \max_{0 \leq i, j \leq r, t, s \in [-1,1]} \left| \frac{\partial^{i+j}}{\partial t^i \partial s^j} k_u(t, s, x_0(s)) \right|$. Hence for $j = 0, 1, 2, ..., r$, we have

$$||[K'(x_0)x]^j||_{\infty} \leq \sqrt{2} ||k||_{j, \infty} ||x||_{L^2} \leq 2 ||k||_{j, \infty} ||x||_{\infty} \quad (2.4)$$

and

$$||[K'(x_0)x]^j||_{L^2} \leq \sqrt{2} ||[K'(x_0)x]^j||_{\infty} \leq 2 ||k||_{j, \infty} ||x||_{L^2}. \quad (2.5)$$

Next we prove the following lemma, which we need in our convergence analysis.

**Lemma 2.1.** For any $x, y \in L^2[-1,1]$ or $C[-1,1]$, the following hold

$$||\mathcal{K}(x_0) - \mathcal{K}(x)||_\infty \leq c_1 ||x_0 - x||_{L^2} ||y||_{L^2},$$

$$||\mathcal{K}'(x_0) - \mathcal{K}'(x)||_\infty \leq c_2 ||x_0 - x||_{L^2} ||y||_{L^2}.$$

**Proof.** Using Lipschitz’s continuity of $k(t, s, u)$ and Cauchy-Schwarz inequality, we have

$$||\mathcal{K}(x_0) - \mathcal{K}(x)||_\infty = \max_{t \in [-1,1]} |(\mathcal{K}(x_0) - \mathcal{K}(x)) y(t)|$$

$$= \max_{t \in [-1,1]} \left| \int_{-1}^{1} [k(t, s, x_0(s)) - k(t, s, x(s))] y(s) ds \right| \leq c_1 \int_{-1}^{1} |x_0(s) - x(s)||y(s)| ds \leq c_1 ||x_0 - x||_{L^2} ||y||_{L^2}. \quad (2.6)$$

On the similar lines, using Lipschitz’s continuity of $k_u(t, s, u)$ and Cauchy-Schwarz inequality, we obtain

$$||(\mathcal{K}'(x_0) - \mathcal{K}'(x)) y||_\infty \leq c_2 ||x_0 - x||_{L^2} ||y||_{L^2}. \quad (2.7)$$

Hence the proof follows. □

Next we will apply Legendre Galerkin and Legendre collocation methods to the equation (2.1). To do this, we let $\mathbb{X}_n$ be the sequence of polynomial subspaces of $\mathbb{X}$ of degree $\leq n$ and
we choose Legendre polynomials \( \{ \phi_0, \phi_1, \phi_2, \ldots, \phi_n \} \) as an orthonormal basis for the subspace \( X_n \), where,

\[
\phi_0(x) = 1, \phi_1(x) = x, \quad x \in [-1, 1],
\]

and for \( i = 1, 2, \ldots, n - 1 \)

\[
(i + 1)\phi_{i+1}(x) = (2i + 1)x\phi_i(x) - i\phi_{i-1}(x), \quad x \in [-1, 1]. \tag{2.8}
\]

**Orthogonal projection operator:** Let \( X = L^2[-1, 1] \) or \( C[-1, 1] \) and let the operator \( \mathcal{P}_n^G : X \to X_n \) be the orthogonal projection defined by

\[
\mathcal{P}_n^G x = \sum_{j=0}^{n} \langle x, \phi_j \rangle \phi_j, \quad x \in X, \tag{2.9}
\]

where \( \langle x, \phi_j \rangle = \int_{-1}^{1} x(t) \phi_j(t) dt \).

We quote the following lemmas which follows from (Canuto et al. [8], pp 283-287).

**Lemma 2.2.** Let \( \mathcal{P}_n^G : X \to X_n \) denote the orthogonal projection defined by (2.9). Then the projection \( \mathcal{P}_n^G \) satisfies the following properties.

(i) \( \{ \mathcal{P}_n^G : n \in \mathbb{N} \} \) is uniformly bounded in \( L^2 \)-norm.

(ii) There exists a constant \( c > 0 \) such that for any \( n \in \mathbb{N} \) and \( u \in X \),

\[
\| \mathcal{P}_n^G u - u \|_{L^2} \leq c \inf_{\phi \in X_n} \| u - \phi \|_{L^2}.
\]

**Lemma 2.3.** Let \( \mathcal{P}_n^G \) be the orthogonal projection defined by (2.9). Then for any \( u \in C^r[-1, 1] \), there hold

\[
\| u - \mathcal{P}_n^G u \|_{L^2} \leq cn^{-r} \| u^{(r)} \|_{L^2}, \tag{2.10}
\]
\[
\| u - \mathcal{P}_n^G u \|_{\infty} \leq cn^{\frac{3}{2}-r} \| u^{(r)} \|_{L^2}, \tag{2.11}
\]
\[
\| u - \mathcal{P}_n^G u \|_{\infty} \leq cn^{1-\frac{r}{2}} V(u^{(r)}), \tag{2.12}
\]

where \( c \) is a constant independent of \( n \). and \( V(u^{(r)}) \) denotes the total variation of \( u^{(r)} \).

**Interpolatory projection operator:** Let \( \{ \tau_0, \tau_1, \ldots, \tau_n \} \) be the zeros of the Legendre polynomial of degree \( n + 1 \) and define interpolatory projection \( \mathcal{P}_n^C : X \to X_n \) by

\[
\mathcal{P}_n^C u \in X_n, \quad \mathcal{P}_n^C u(\tau_i) = u(\tau_i), \quad i = 0, 1, \ldots, n, \quad u \in X. \tag{2.13}
\]

According to the analysis of (Canuto et al. [8], pp 289), \( \mathcal{P}_n^C \) satisfies the following lemmas.

**Lemma 2.4.** Let \( \mathcal{P}_n^C : X \to X_n \) be the interpolatory projection defined by (2.13). Then there hold

(i) \( \{ \mathcal{P}_n^C : n \in \mathbb{N} \} \) is uniformly bounded in \( L^2 \)-norm.
(ii) There exists a constant \( c > 0 \) such that for any \( n \in \mathbb{N} \) and \( u \in X \),
\[
\|P_n^C u - u\|_{L^2} \leq c \inf_{\phi \in X_n} \|u - \phi\|_{L^2} \to 0, \quad n \to \infty.
\]

**Lemma 2.5.** Let \( P_n^C : X \to X_n \) be the interpolatory projection defined by (2.13). Then for any \( u \in C^r[-1, 1] \), there exists a constant \( c \) independent of \( n \) such that
\[
\|u - P_n^C u\|_{L^2} \leq c n^{-r} \|u^{(r)}\|_{L^2}. \tag{2.14}
\]

Noting that
\[
\|P_n^C\|_{\infty} = 1 + \frac{9^{3/2}}{\sqrt{\pi}} n^{1/2} + B_0 + O(n^{-1/2}),
\]
where \( B_0 \) is a bounded constant (see Tang et al. [22]), we have
\[
\|(I - P_n^C)u\|_{\infty} \leq (1 + \|P_n^C\|_{\infty}) \inf_{\chi \in X_n} \|u - \chi\|_{\infty}
\]
\[
\leq c n^{1/2} n^{-r} \|u^{(r)}\|_{\infty} \leq c n^{1/2} \|u^{(r)}\|_{\infty}. \tag{2.15}
\]

Throughout this paper, we assume that the projection operator \( P_n : X \to X_n \) is either orthogonal projection \( P_n^G \) defined by (2.9) or interpolatory projection operator \( P_n^C \) defined by (2.13).

By Lemma 2.2 and 2.4, we have that \( \|P_n\|_{L^2} \) is uniformly bounded. We denote, \( \|P_n\|_{L^2} \leq p, \) for all \( n \in \mathbb{N} \) and \( \|P_n x\|_{L^2} \leq p_1 \|x\|_{\infty}, \) where \( p \) and \( p_1 \) are constants independent of \( n \). Further, we have from Lemma 2.3, 2.5 and estimate (2.15) that
\[
\|u - P_n u\|_{L^2} \leq c n^{-r} \|u^{(r)}\|_{L^2}, \tag{2.16}
\]
\[
\|u - P_n u\|_{\infty} \leq c n^{\beta - r} \|u^{(r)}\|_{\infty}, \quad 0 < \beta < 1, \quad \text{and} \quad r = 0, 1, 2, \ldots \tag{2.17}
\]
where \( c \) is a constant independent of \( n \), \( \beta = \frac{3}{4} \) for orthogonal projection operators and \( \beta = \frac{1}{2} \) for interpolatory projections. Note that \( \|P_n u - u\|_{\infty} \to 0 \), as \( n \to \infty \) for any \( u \in C[-1, 1] \).

The projection method for equation (2.2) is seeking an approximate solution \( x_n \in X_n \) such that
\[
x_n - P_n K(x_n) = P_n f. \tag{2.18}
\]

If \( P_n \) is chosen to be \( P_n^G \), the above scheme (2.18) leads to Legendre Galerkin method, whereas if \( P_n \) is replaced by \( P_n^C \) we get the Legendre collocation method.

Let \( T_n \) be the operator defined by
\[
T_n u := P_n K(u) + P_n f. \tag{2.19}
\]

Then the equation (2.18) can be written as
\[
x_n = T_n x_n. \tag{2.20}
\]
In order to obtain more accurate approximated solution, we further consider the iterated projection method for (2.2). To this end, we define the iterated solution as

$$\tilde{x}_n = f + \mathcal{K}(x_n).$$  \hspace{1cm} (2.21)

Applying $\mathcal{P}_n$ on both sides of the equation (2.21), we obtain

$$\mathcal{P}_n\tilde{x}_n = \mathcal{P}_n f + \mathcal{P}_n \mathcal{K}(x_n).$$  \hspace{1cm} (2.22)

From equations (2.18) and (2.22), it follows that

$$\mathcal{P}_n\tilde{x}_n = x_n.$$  \hspace{1cm} (2.23)

Using this, we see that the iterated solution $\tilde{x}_n$ satisfies the following equation

$$\tilde{x}_n - \mathcal{K}(\mathcal{P}_n\tilde{x}_n) = f.$$  \hspace{1cm} (2.23)

Letting $\tilde{T}_n(u) := f + \mathcal{K}(\mathcal{P}_n u)$, $u \in \mathcal{X}$, the equation (2.23) can be written as $\tilde{x}_n = \tilde{T}_n\tilde{x}_n$.

We quote the following theorem from Vainikko [23] which gives us the conditions under which the solvability of one equation leads to the solvability of other equation.

**Theorem 2.2.** Let $\hat{T}$ and $\tilde{T}$ be continuous operators over an open set $\Omega$ in a Banach space $\mathcal{X}$. Let the equation $x = \hat{T}x$ has an isolated solution $\hat{x}_0 \in \Omega$ and let the following conditions be satisfied.

(a) The operator $\hat{T}$ is Frechet differentiable in some neighbourhood of the point $\hat{x}_0$, while the linear operator $I - \hat{T}'(\hat{x}_0)$ is continuously invertible.

(b) Suppose that for some $\delta > 0$ and $0 < q < 1$, the following inequalities are valid (the number $\delta$ is assumed to be so small that the sphere $\|x - \hat{x}_0\| \leq \delta$ is contained within $\Omega$).

$$\sup_{\|x - \hat{x}_0\| \leq \delta} \| (I - \hat{T}'(\hat{x}_0))^{-1}(\hat{T}'(x) - \hat{T}'(\hat{x}_0)) \| \leq q,$$  \hspace{1cm} (2.24)

$$\alpha = \| (I - \tilde{T}'(\hat{x}_0))^{-1}(\tilde{T}(x) - \tilde{T}(\hat{x}_0)) \| \leq \delta(1 - q).$$  \hspace{1cm} (2.25)

Then the equation $x = \tilde{T}x$ has a unique solution $\hat{x}_0$ in the sphere $\|x - \hat{x}_0\| \leq \delta$. Moreover, the inequality

$$\frac{\alpha}{1 + q} \leq \|\hat{x}_0 - \tilde{x}_0\| \leq \frac{\alpha}{1 - q}$$  \hspace{1cm} (2.26)

is valid.

Next we recall the definition of collectively compact approximation and some theorems from Anselone [2] and Ahues et al.[1].

**Definition 2.1.** A sequence $\{\mathcal{K}_n\}$ in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is said to be a collectively compact approximation of $\mathcal{K} \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, i.e., $\mathcal{K}_n \overset{cc}{\rightarrow} \mathcal{K}$ if

(i) $\mathcal{K}_nx \rightarrow \mathcal{K}x$, for each $x \in \mathcal{X}$,

(ii) for some positive integer $n_0$, the set $\bigcup_{n \geq n_0} \{(\mathcal{K}_n - \mathcal{K})x : \|x\| \leq 1, x \in \mathcal{X}\}$ is relatively compact.
Lemma 2.6. Let \( \{K_n\} \) be a uniformly bounded sequence in \( B(\mathbb{X}) \) and \( K_n \xrightarrow{cc} K \) or \( K_n \xrightarrow{n} K \). If \( (I - K) \) is invertible, then \( (I - K_n)^{-1} \) exists and uniformly bounded on \( \mathbb{X} \), for sufficiently large \( n \).

Now we discuss the existence and convergence rates of the approximated solution \( x_n \) to \( x_0 \).

Theorem 2.3. Let \( x_0 \in C^r[-1,1] \) be an isolated solution of the equation (2.2). Assume that 1 is not an eigenvalue of the linear operator \( K'(x_0) \), where \( K'(x_0) \) denotes the Fréchet derivative of \( K(x) \) at \( x_0 \). Let \( P_n : \mathbb{X} \to \mathbb{X}_n \) be either orthogonal or interpolatory projection operator defined by (2.9) or (2.13), respectively. Then the equation (2.18) has a unique solution \( x_n \in B(x_0, \delta) = \{ x : \|x - x_0\|_{L^2} < \delta \} \) for some \( \delta > 0 \) and for sufficiently large \( n \). Moreover, there exists a constant \( 0 < q < 1 \), independent of \( n \) such that

\[
\frac{\alpha_n}{1 + q} \leq \| x_n - x_0 \|_{L^2} \leq \frac{\alpha_n}{1 - q},
\]

where \( \alpha_n = \|(I - T_n'(x_0))^{-1}(T_n(x_0) - T(x_0))\|_{L^2} \).

Proof. Using estimates (2.5) and (2.16), we have

\[
\|(T_n'(x_0) - T'(x_0))x\|_{L^2} = \|(P_n K'(x_0) - K'(x_0))x\|_{L^2} = \|(P_n - I)K'(x_0)x\|_{L^2} \leq cn^{-r}\|K'(x_0)x\|_{L^2} \leq 2cn^{-r}\|K'(x_0)x\|_{L^2}.
\]

This implies \( \|T_n'(x_0) - T'(x_0)\|_{L^2} \to 0 \), as \( n \to \infty \), i.e., \( T_n'(x_0) \) is norm convergent to \( T'(x_0) \). Hence by Lemma 2.6, we have \( (I - T_n'(x_0))^{-1} \) exists and uniformly bounded on \( \mathbb{X} \), for some sufficiently large \( n \), i.e., there exists some \( A_1 > 0 \) such that \( \|(I - T_n'(x_0))^{-1}\|_{L^2} \leq A_1 < \infty \).

Now from the estimate (2.7), we have for any \( x \in B(x_0, \delta) \),

\[
\|(T_n'(x_0) - T_n'(x_0))y\|_{L^2} = \|(P_n K'(x_0) - P_n K'(x))y\|_{L^2} \leq \|P_n\|_{L^2}\|(K'(x_0) - K'(x))y\|_{L^2} \leq \sqrt{2}\|P_n\|_{L^2}\|(K'(x_0) - K'(x))y\|_{L^2} \leq p\sqrt{2}c_2\|x_0 - x\|_{L^2}\|y\|_{L^2} \leq p\sqrt{2}c_2\|y\|_{L^2}.
\]

This implies

\[
\|T_n'(x_0) - T_n'(x)\|_{L^2} \leq p\sqrt{2}c_2\delta.
\]

Hence, we have

\[
\sup_{\|x - x_0\|_{L^2} \leq \delta} \|(I - T_n'(x_0))^{-1}(T_n'(x_0) - T_n'(x))\|_{L^2} \leq A_1p\sqrt{2}c_2\delta \leq q, \text{ (say)}
\]

where \( 0 < q < 1 \), which proves the equation (2.24) of Theorem 2.2.
Taking use of (2.16), we have
\[
\alpha_n = \|(I - T_n')(x_0))^{-1}(T_n(x_0) - T(x_0))\|_{L^2} \\
\leq A_1\|T_n(x_0) - T(x_0)\|_{L^2} \\
= A_1\|P_n(Kx_0 + f) - (Kx_0 + f)\|_{L^2} \\
= A_1\|(P_n - I)(Kx_0 + f)\|_{L^2} \\
= A_1\|(P_n - I)x_0\|_{L^2} \\
\leq A_1cn^{-r}\|x_0^{(r)}\|_{L^2} \to 0, \text{ as } n \to \infty.
\] (2.28)

By choosing \(n\) large enough such that \(\alpha_n \leq \delta(1 - q)\), the equation (2.25) of Theorem 2.2 is satisfied. Hence by applying Theorem 2.2, we obtain
\[
\frac{\alpha_n}{1 + q} \leq \|x_n - x_0\|_{L^2} \leq \frac{\alpha_n}{1 - q},
\]

This completes the proof. \(\square\)

**Theorem 2.4.** Let \(x_0 \in C'[−1, 1], r ≥ 1, \) be an isolated solution of the equation (2.2). Assume that 1 is not an eigenvalue of the linear operator \(K'(x_0)\), where \(K'(x_0)\) denotes the Frechet derivative of \(K(x)\) at \(x_0\). Let \(P_n : \mathbb{X} \to \mathbb{X}_n\) be either orthogonal or interpolatory projection operator defined by (2.9) or (2.13), respectively. Then the equation (2.18) has a unique solution \(x_n \in B(x_0, \delta) = \{x : \|x - x_0\|_{\infty} < \delta\}\) for some \(\delta > 0\) and for sufficiently large \(n\). Moreover, there exists a constant \(0 < q < 1\), independent of \(n\) such that
\[
\frac{\alpha_n}{1 + q} \leq \|x_n - x_0\|_{\infty} \leq \frac{\alpha_n}{1 - q},
\]

where \(\alpha_n = \|(I - T_n')(x_0))^{-1}(T_n(x_0) - T(x_0))\|_{\infty}\

**Proof.** Using estimates (2.4) and (2.17), we have
\[
\|(T_n'(x_0) - T'(x_0))x\|_{\infty} = \|(P_nK'(x_0) - K'(x_0))x\|_{\infty} \\
= \|(P_n - I)K'(x_0)x\|_{\infty} \\
\leq cn^{\beta - r}\|K'(x_0)x^{(r)}\|_{\infty} \\
\leq 2cn^{\beta - r}\|k\|_{r, \infty}\|x\|_{\infty}.
\] (2.29)

Since \(0 < \beta < 1\), for \(\beta < r = 1, 2, 3, \ldots, \) it follows that
\[
\|T_n'(x_0) - T'(x_0)\|_{\infty} = O(n^{\beta - r}) \to 0, \text{ as } n \to \infty.
\]

Hence by applying Lemma 2.6, we see that \((I - T_n'(x_0))^{-1}\) exists and uniformly bounded on \(\mathbb{X}\), for some sufficiently large \(n\), i.e., there exists some \(A_2 > 0\) such that \(\|(I - T_n'(x_0))^{-1}\|_{\infty} \leq A_2 < \infty\).

Now, we have for any \(x \in B(x_0, \delta),\)
\[ \| (T'_n(x) - T'_n(x)) y \|_\infty \]
\[ \leq \| (P_n K'(x_0) - P_n K'(x)) y \|_\infty \]
\[ \leq \| (P_n - I) \{ (K'(x_0) - K'(x)) y \} \|_\infty + \| (K'(x_0) - K'(x)) y \|_\infty. \] (2.30)

Using the estimate (2.7), we have
\[ \| (K'(x_0) - K'(x)) y \|_\infty \leq c_2 \| x_0 - x \|_{L^2} \| y \|_{L^2} \]
\[ \leq 2c_2 \| x_0 - x \|_\infty \| y \|_\infty \leq 2c_2 \delta \| y \|_\infty. \] (2.31)

Putting \( r = 1 \) in the estimate (2.17) and using the assumption (iii), we have
\[ \| (P_n - I) \{ (K'(x_0) - K'(x)) y \} \|_\infty \leq cn^{\beta-1} \| [(K'(x_0) - K'(x)) y]^{(1)} \|_\infty \]
\[ \leq cn^{\beta-1} \sup_{t \in [-1,1]} \| [(K'(x_0) - K'(x)) y]^{(1)}(t) \| \]
\[ = cn^{\beta-1} \sup_{t \in [-1,1]} \left| \int_{-1}^{1} \frac{\partial}{\partial t} [k_u(t, s, x_0(s)) - k_u(t, s, x(s))] y(s) \, ds \right| \]
\[ \leq cn^{\beta-1} c_3 \int_{-1}^{1} |(x_0 - x)(s)| |y(s)| \, ds \]
\[ \leq 2cn^{\beta-1} c_3 \| x - x_0 \|_\infty \| y \|_\infty \]
\[ \leq 2cn^{\beta-1} c_3 \delta \| y \|_\infty. \] (2.32)

Hence combining estimates (2.30), (2.31) and (2.32), we have
\[ \| (T'_n(x_0) - T'_n(x)) \|_\infty \leq (2cc_3 n^{\beta-1} + 2c_2) \delta. \]

Since \( \beta < 1 \), we have
\[ \sup_{\| x - x_0 \|_\infty \leq \delta} \| (I - T'_n(x_0))^{-1} (T'_n(x_0) - T'_n(x)) \|_\infty \leq A_2 (2cc_3 n^{\beta-1} + 2c_2) \delta \leq q, \text{ (say)} \]

where \( 0 < q < 1 \), which proves the equation (2.24) of Theorem 2.2.

Since \( \beta < r = 1, 2, ..., \) using the estimate (2.17), we have
\[ \alpha_n = \| (I - T'_n(x_0))^{-1} (T'_n(x_0) - T(x_0)) \|_\infty \]
\[ \leq A_2 \| T_n(x_0) - T(x_0) \|_\infty \]
\[ = A_2 \| P_n (Kx_0 + f) - (Kx_0 + f) \|_\infty \]
\[ = A_2 \| (P_n - I)(Kx_0 + f) \|_\infty \]
\[ = A_2 \| (P_n - I)x_0 \|_\infty \]
\[ \leq cn^{\beta-r} \| x^{(r)}_0 \|_\infty \to 0, \text{ as } n \to \infty. \] (2.33)

By choosing \( n \) large enough such that \( \alpha_n \leq \delta(1 - q) \), the equation (2.25) of Theorem 2.2 is satisfied. Hence by applying Theorem 2.2, we obtain
\[ \frac{\alpha_n}{1 + q} \leq \| x_n - x_0 \|_\infty \leq \frac{\alpha_n}{1 - q}. \]
This completes the proof.

Next we discuss the existence and convergence of the iterated approximate solution $\tilde{x}_n$ to $x_0$.

**Theorem 2.5.** Let $x_0 \in C^r[-1,1]$ be an isolated solution of the equation (2.2). Assume that 1 is not an eigenvalue of $K'(x_0)$. Then for sufficiently large $n$, the operator $I - \tilde{T}_n(x_0)$ is invertible on $C[-1,1]$ and there exist constants $L$, $L_1 > 0$ independent of $n$ such that $\|(I - \tilde{T}_n(x_0))^{-1}\|_\infty \leq L$ and $\|(I - \tilde{T}_n(x_0))^{-1}\|_{L^2} \leq L_1$.

**Proof.** Consider

$$|\tilde{T}_n'(x_0)x(t)| = |K'(P_n x_0)P_n x(t)| \leq |(K'(P_n x_0) - K'(x_0))P_n x(t)| + |K'(x_0)P_n x(t)| \quad (2.34)$$

Now using estimates (2.7), (2.16) and the fact that $\|P_n x\|_{L^2} \leq p_1 \|x\|_\infty$, we have

$$\|\{K'(P_n x_0) - K'(x_0)\}P_n x\|_\infty \leq c_2 \|P_n x_0 - x_0\|_{L^2} \|P_n x\|_{L^2} \leq c_2 p_1 n^{-r} \|x_0^{(r)}\|_{L^2} \|x\|_\infty \to 0,$$  \quad as $n \to \infty. \quad (2.35)$$

Again using the Cauchy-Schwarz inequality, we have

$$\|K'(x_0)P_n x\|_\infty = \max_{t \in [-1,1]} |K'(x_0)P_n x(t)| \leq \max_{t \in [-1,1]} \left| \int_{-1}^{1} k_n(t,s,x_0(s))P_n x(s) ds \right| \leq \sqrt{2} M \|P_n x\|_{L^2} \leq \sqrt{2}Mp_1 \|x\|_\infty. \quad (2.36)$$

Now combining the estimates (2.34), (2.35) and (2.36) we obtain

$$\|\tilde{T}_n'(x_0)\|_\infty \leq (c_2 p_1 n^{-r} \|x_0^{(r)}\|_{L^2} + \sqrt{2}Mp_1) < \infty.$$ 

This shows that $\|\tilde{T}_n'(x_0)\|_\infty$ is uniformly bounded.

Now

$$|\tilde{T}_n'(x_0)x(t) - \tilde{T}_n'(x_0)x(t')| = |K'(P_n x_0)P_n x(t) - K'(P_n x_0)P_n x(t')| \leq T_1 + T_2 + T_3. \quad (2.37)$$

where

$$T_1 = |K'(P_n x_0)P_n x(t) - K'(x_0)P_n x(t)|, \quad (2.38)$$

$$T_2 = |K'(x_0)P_n x(t) - K'(x_0)P_n x(t')|, \quad (2.39)$$

$$T_3 = |K'(x_0)P_n x(t') - K'(P_n x_0)P_n x(t')|. \quad (2.40)$$
This implies

\[ T_1 = \| (\mathcal{K}'(\mathcal{P}x_0) - \mathcal{K}'(x_0))\mathcal{P}x(t) \| \]
\[ \leq \| (\mathcal{K}'(\mathcal{P}x_0) - \mathcal{K}'(x_0))\mathcal{P}x\|_{\infty} \]
\[ \leq c_{\mathcal{P}} p_1 n^{-r} \| x_0^{(r)} \|_{L^2} \| x \|_{\infty} \rightarrow 0 , \text{ as } n \rightarrow \infty \]

(2.41)

and

\[ T_3 = \| (\mathcal{K}'(\mathcal{P}x_0) - \mathcal{K}'(x_0))\mathcal{P}x(t') \| \]
\[ \leq \| (\mathcal{K}'(\mathcal{P}x_0) - \mathcal{K}'(x_0))\mathcal{P}x\|_{\infty} \]
\[ \leq c_{\mathcal{P}} p_1 n^{-r} \| x_0^{(r)} \|_{L^2} \| x \|_{\infty} \rightarrow 0 , \text{ as } n \rightarrow \infty . \]

(2.42)

Since \( k_u(t, s, u) \in \mathcal{C}([-1, 1] \times [-1, 1] \times \mathbb{R}) \), \( k_u(t, s, u) \) is uniformly continuous in first variable \( t \).

Hence for any \( \epsilon > 0 \), however small, there exists some number \( \delta > 0 \) such that

\[ |k_u(t, s, u) - k(t', s, u)| < \epsilon , \text{ whenever } |t - t'| < \delta . \]

Hence

\[ T_2 = \left| \int_{-1}^1 [k_u(t, s, x_0(s)) - k_u(t', s, x_0(s))]\mathcal{P}x(s)ds \right| \]
\[ \leq \sup_{-1 \leq s \leq 1} |k_u(t, s, x_0(s)) - k_u(t', s, x_0(s))| \int_{-1}^1 |\mathcal{P}x(s)|ds \]
\[ \leq \epsilon \sqrt{2} \| \mathcal{P}x \|_{L^2} , \text{ as } t \rightarrow t' \]
\[ \leq \epsilon \sqrt{2} p_1 \| x \|_{\infty} \rightarrow 0 , \text{ as } t \rightarrow t' . \]

(2.43)

Hence combining estimates (2.37), (2.41), (2.42) and (2.43) we have

\[ |\mathcal{T}_n^t(x_0)x(t) - \mathcal{T}_n^t(x_0)x(t')| \rightarrow 0 , \text{ as } t \rightarrow t' \text{ and } n \rightarrow \infty . \]

(2.44)

This implies \( \{ \mathcal{T}_n^t(x_0) \}_{n=1}^{\infty} \) is collectively compact.

Hence using Lemma 2.6, we can conclude that \( (I - \mathcal{T}_n^t(x_0))^{-1} \) is invertible on \( \mathcal{C}[-1, 1] \) and there exist constants \( L > 0 \) independent of \( n \) such that \( \| (I - \mathcal{T}_n^t(x_0))^{-1} \|_{\infty} \leq L \).

On similar lines it can be shown that the result is true for \( L^2 \)-norm, i.e., there exists a constants \( L_1 > 0 \) independent of \( n \) such that \( \| (I - \mathcal{T}_n^t(x_0))^{-1} \|_{L^2} \leq L_1 \).

**Theorem 2.6.** Let \( x_0 \in \mathcal{C}'[-1, 1] \) be an isolated solution of the equation (2.2). Let \( \mathcal{P}_n : \mathbb{K} \rightarrow \mathbb{K}_n \) be either orthogonal or interpolatory projection operator defined by (2.9) or (2.13), respectively. Assume that 1 is not an eigenvalue of \( \mathcal{K}'(x_0) \), then for sufficiently large \( n \), the iterated solution \( \mathcal{x}_n \) defined by (2.23) is the unique solution in the sphere \( B(x_0, \delta) = \{ x : \| x - x_0 \|_{\infty} < \delta \} \). Moreover, there exists a constant \( 0 < q < 1 \), independent of \( n \) such that

\[ \frac{\beta_n}{1 + q} \leq \| \mathcal{x}_n - x_0 \|_{\infty} \leq \frac{\beta_n}{1 - q} , \]

where

\[ \beta_n = \| (I - \mathcal{T}_n^t(x_0))^{-1}(\mathcal{T}_n^t(x_0) - \mathcal{T}(x_0)) \|_{\infty} . \]
Proof. From Theorem 2.5, there exists a constant $L > 0$ such that $\|(I - \tilde{T}_n'(x_0))^{-1}\|_\infty \leq L$, for sufficiently large value of $n$.

Using the estimate (2.7) of Lemma 2.1 and the fact that $\|P_n x\|_{L^2} \leq p_1 \|x\|_\infty$, for any $x \in B(x_0, \delta)$, we have

$$\|\{\tilde{T}_n'(x) - \tilde{T}_n'(x_0)\} y\|_\infty = \|\{\mathcal{K}'(P_n x) P_n - \mathcal{K}'(P_n x_0) P_n\} y\|_\infty$$

$$= \|\{\mathcal{K}'(P_n x) - \mathcal{K}'(P_n x_0)\} P_n y\|_\infty$$

$$\leq c_2 \|P_n(x - x_0)\|_{L^2} \|P_n y\|_{L^2}$$

$$\leq c_2 p_2 \|x - x_0\|_\infty \|y\|_\infty \leq c_2 p_2^2 \delta \|y\|_\infty. \quad (2.45)$$

This implies

$$\sup_{\|x-x_0\|_\infty \leq \delta} \|(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n'(x) - \tilde{T}_n'(x_0))\|_\infty \leq L c_2 p_2^2 \delta \leq q, \text{ (say)}$$

where $0 < q < 1$, which proves the equation (2.24) of Theorem 2.2.

Now using the estimate (2.6), we have

$$\|\{\tilde{T}_n(x_0) - T(x_0)\} y\|_\infty \leq \|\{\mathcal{K}(P_n x_0) - \mathcal{K}(x_0)\} y\|_\infty$$

$$\leq c_1 \|(I - P_n) x_0\|_{L^2} \|y\|_{L^2}$$

$$\leq \sqrt{2} c_1 \|(I - P_n) x_0\|_{L^2} \|y\|_\infty$$

Hence using estimate (2.16), we have

$$\|\tilde{T}_n(x_0) - T(x_0)\|_\infty \leq \sqrt{2} c_1 n^{-r} \|x_0^{(r)}\|_{L^2} \to 0, \text{ as } n \to \infty. \quad (2.46)$$

Hence

$$\beta_n = \|(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - T(x_0))\|_\infty$$

$$\leq L \sqrt{2} c_1 n^{-r} \|x_0^{(r)}\|_{L^2} \to 0, \text{ as } n \to \infty.$$

Choose $n$ large enough such that $\beta_n \leq \delta(1 - q)$. Then the equation (2.25) of Theorem 2.2 is satisfied. Thus by applying Theorem 2.2, we obtain

$$\frac{\beta_n}{1 + q} \leq \|\tilde{x}_n - x_0\|_\infty \leq \frac{\beta_n}{1 - q}$$

where

$$\beta_n = \|(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - T(x_0))\|_\infty.$$

This completes the proof. \qed

Theorem 2.7. Let $x_0 \in C^r[-1, 1]$ be an isolated solution of the equation (2.2). Let $P_n : \mathbb{X} \to \mathbb{X}_n$ be either orthogonal or interpolatory projection operator defined by (2.9) and (2.13) respectively. Assume that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$, then for sufficiently large $n$, the iterated solution $\tilde{x}_n$ defined by (2.23) is the unique solution in the sphere $B(x_0, \delta) = \{x : \|x - x_0\|_{L^2} < \delta\}$. Moreover, there exists a constant $0 < q < 1$, independent of $n$ such that

$$\frac{\beta_n}{1 + q} \leq \|\tilde{x}_n - x_0\|_{L^2} \leq \frac{\beta_n}{1 - q},$$

where

$$\beta_n = \|(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - T(x_0))\|_{L^2}.$$
Proof. From Theorem 2.5, we have \((I - \tilde{T}_n'(x_0))^{-1}\) exists and it is uniformly bounded in \(L^2\)-norm on \(C[-1,1]\), i.e., there exists a constant \(L_1 > 0\) such that \(||(I - \tilde{T}_n'(x_0))^{-1}\||_{L^2} \leq L_1\).

Now using the estimate (2.7) and the fact that \(||P_n||_{L^2} \leq p\), we have for any \(x \in B(x_0, \delta)\),

\[
||\tilde{T}_n'(x) - \tilde{T}_n'(x_0)||_{L^2} \leq \sqrt{2}||\{\tilde{T}_n'(x) - \tilde{T}_n'(x_0)\}y||_{\infty} = \sqrt{2}||\{K'(P_nx)P_n - K'(P_nx_0)P_n\}y||_{\infty} = \sqrt{2}||\{K'(P_nx) - K'(P_nx_0)\}P_ny||_{\infty} \leq \sqrt{2c_2}\|P_n(x - x_0)||_{L^2}\|P_ny||_{L^2} \leq \sqrt{2c_2}p^2\|x - x_0||_{L^2}\|y||_{L^2} \leq \sqrt{2c_2}p^2\|y||_{L^2}.
\]

Thus we obtain

\[
\sup_{||x - x_0||_{L^2} \leq \delta} ||(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n'(x) - \tilde{T}_n'(x_0))||_{L^2} \leq L_1\sqrt{2c_2}p^2 \delta \leq q, \text{(say)}
\]

where \(0 < q < 1\), which proves the equation (2.24) of Theorem 2.2.

Now using the estimate (2.46), we have

\[
\beta_n = ||(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - \mathcal{T}(x_0))||_{L^2} \leq L_1\sqrt{2}||\tilde{T}_n(x_0) - \mathcal{T}(x_0)||_{\infty} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

Choose \(n\) large enough such that \(\beta_n \leq \delta(1 - q)\). Hence the equation (2.25) of Theorem 2.2 is satisfied. Then applying Theorem 2.2, we get

\[
\frac{\beta_n}{1 + q} \leq \|	ilde{x}_n - x_0\|_{L^2} \leq \frac{\beta_n}{1 - q},
\]

where

\[
\beta_n = ||(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - \mathcal{T}(x_0))||_{L^2}.
\]

This completes the proof. \(\square\)

**Theorem 2.8.** Let \(x_0 \in C[-1,1]\) be an isolated solution of the equation (2.2). Let \(\tilde{x}_n\) defined by the iterated scheme (2.23). Then the following hold

\[
\|	ilde{x}_n - x_0\|_{\infty} \leq c\{\|x_0 - P_nx_0\|_{L^2}^2 + |< g_t, (I - P_n)x_0 |}\}, \tag{2.47}
\]

and

\[
\|	ilde{x}_n - x_0\|_{L^2} \leq c\{\|x_0 - P_nx_0\|_{L^2}^2 + |< g_t, (I - P_n)x_0 |}\}, \tag{2.48}
\]

where \(g_t(s) = k_u(t, s, x_0(s))\) and \(c\) is a constant independent of \(n\).

**Proof.** From Theorem 2.6, we have

\[
\frac{\beta_n}{1 + q} \leq \|	ilde{x}_n - x_0\|_{\infty} \leq \frac{\beta_n}{1 - q},
\]

where

\[
\beta_n = ||(I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - \mathcal{T}(x_0))||_{\infty}.
\]
Hence using Theorem 2.5, we get
\[
\|\bar{x}_n - x_0\|_\infty \leq \beta_n = \| (I - \tilde{T}_n'(x_0))^{-1}(\tilde{T}_n(x_0) - T(x_0))\|_\infty \\
\leq \| (I - \tilde{T}_n'(x_0))^{-1}\|_\infty \| \tilde{T}_n(x_0) - T(x_0)\|_\infty \\
\leq L\|\mathcal{K}(P_n x_0) - \mathcal{K}(x_0)\|_\infty. \tag{2.49}
\]

We denote \(g(t, s, x_0, x, \theta) = k_u(t, s, x_0 + \theta(x - x_0))\) and \(g_t(s) = k_u(t, s, x_0(s))\).

Now
\[
|\mathcal{K}(P_n x_0)(t) - \mathcal{K}(x_0)(t)| = \left| \int_{-1}^{1} [k(t, s, P_n x_0(s) - k(t, s, x_0(s))]ds \right|
\]
\[
= \left| \int_{-1}^{1} [k_u(t, s, x_0(s) + \theta(P_n x_0(s) - x_0(s)))](x_0 - P_n x_0)(s)ds \right|
\]
\[
= \left| \int_{-1}^{1} g(t, s, x, x_0, \theta)(x_0 - P_n x_0)(s)ds \right|
\]
\[
\leq \left| \int_{-1}^{1} g(t, s, x, x_0, \theta) - g(t)(x_0 - P_n x_0)(s)ds \right|
\]
\[
+ \left| \int_{-1}^{1} g_t(s)(x_0 - P_n x_0)(s)ds \right| \tag{2.50}
\]

For the first term of the above estimate (2.50), we have
\[
\left| \int_{-1}^{1} g(t, s, x, x_0, \theta) - g_t(s)(x_0 - P_n x_0)(s)ds \right|
\]
\[
= \left| \int_{-1}^{1} [k_u(t, s, x_0(s) + \theta(P_n x_0(s) - x_0(s)))] - k_u(t, s, x_0(s)) ](x_0 - P_n x_0)(s)ds \right|
\]
\[
\leq c_2 \int_{-1}^{1} |x_0 - P_n x_0)(s)||x_0 - P_n x_0)(s)|ds
\]
\[
= c_2 \int_{-1}^{1} |x_0 - P_n x_0)(s)|^2ds \leq \sqrt{2} c_2 \|x_0 - P_n x_0\|^2_{L^2}. \tag{2.51}
\]

Now for the second term of (2.50), we have
\[
\left| \int_{-1}^{1} g_t(s)(x_0 - P_n x_0)(s)ds \right| = |< g_t(\cdot), (I - P_n)(x_0) >|. \tag{2.52}
\]

Hence combining estimates (2.50), (2.51) and (2.52), we have
\[
\|\mathcal{K}(P_n x_0) - \mathcal{K}(x_0)\|_\infty \leq \sqrt{2} c_2 \|x_0 - P_n x_0\|^2_{L^2} + |< g_t(\cdot), (I - P_n)(x_0) >|. \tag{2.53}
\]

Therefore from estimates (2.49) and (2.53), we have
\[
\|\bar{x}_n - x_0\|_\infty \leq L\|\mathcal{K}(P_n x_0) - \mathcal{K}(x_0)\|_\infty \\
\leq c\{\|x_0 - P_n x_0\|^2_{L^2} + |< g_t(\cdot), (I - P_n)(x_0) > |\}. \tag{2.54}
\]
where \( c \) is a constant independent of \( n \), this proves the estimate (2.47).

Similarly for \( L^2 \)-norm, we can show that
\[
\| \tilde{x}_n - x_0 \|_{L^2} \leq \sqrt{2} \| \tilde{x}_n - x_0 \|_{\infty}
\]
\[
\leq c \{ \| x_0 - P_n x_0 \|_{L^2}^2 + | < g_t(\cdot), (I - P_n)(x_0) > | \},
\]
this proves the estimate (2.48). This completes the proof. \( \square \)

Now we discuss the convergence rates for the approximate and iterated approximate solutions. To distinguish between the Legendre Galerkin solutions and Legendre collocation solutions, we set the following notations. In case of Legendre Galerkin solutions, we denote the approximate solution and the iterated approximate solution by \( x_n = x_n^G \) and \( \tilde{x}_n = \tilde{x}_n^G \), respectively. For Legendre collocation method, we write the approximate solution and iterated approximate solution as \( x_n = x_n^C \) and \( \tilde{x}_n = \tilde{x}_n^C \), respectively.

**Theorem 2.9.** Let \( x_0 \in C^r[-1,1] \) be a isolated solution of the equation (2.1) and \( x_n = x_n^G \) be the Legendre Galerkin solution or \( x_n = x_n^C \) be the Legendre collocation approximation of \( x_0 \). Then we get the following convergence rates.
\[
\| x_0 - x_n^G \|_{L^2}, \| x_0 - x_n^C \|_{L^2} = O(n^{-r})
\]
and
\[
\| x_0 - x_n^G \|_{\infty}, \| x_0 - x_n^C \|_{\infty} = O(n^{\frac{1}{2} - r}).
\]

**Proof.** From Theorem 2.3, we have
\[
\frac{\alpha_n}{1 + q} \leq \| x_n - x_0 \|_{L^2} \leq \frac{\alpha_n}{1 - q},
\]
where \( \alpha_n = \| (I - T_n'(x_0))^{-1}(T_n(x_0) - T(x_0)) \|_{L^2} \).

Hence we have from the estimate (2.28)
\[
\| x_n - x_0 \|_{L^2} \leq \alpha_n \leq A_1 \| (P_n - I)x_0 \|_{L^2}
\]
\[
\leq A_1 cn^{-r} \| x_0^{(r)} \|_{L^2} = O(n^{-r}).
\]
where \( c \) is a constant independent of \( n \).

Now for the error in infinity norm, using the estimate (2.33), we have
\[
\| x_n - x_0 \|_{\infty} \leq \alpha_n \leq A_2 \| (P_n - I)x_0 \|_{\infty}.
\]

Hence for Legendre Galerkin solution \( x_n = x_n^G \), using estimate (2.12) of Lemma 2.3, we have
\[
\| x_n^G - x_0 \|_{\infty} \leq A_2 \| (P_n^G - I)x_0 \|_{\infty} \leq A_2 cn^{\frac{1}{2} - r}V(x_0^{(r)}) = O(n^{\frac{1}{2} - r}),
\]
and for Legendre collocation solution \( x_n^C \), using estimate (2.15), we have
\[
\| x_n^C - x_0 \|_{\infty} \leq A_2 \| (P_n^C - I)x_0 \|_{\infty} \leq A_2 cn^{\frac{1}{2} - r} \| x_0^{(r)} \|_{\infty} = O(n^{\frac{1}{2} - r}).
\]

Hence the proof follows. \( \square \)

Next we will discuss the error bounds for the iterated Legendre Galerkin and iterated Legendre collocation solutions separately.
Theorem 2.10. Let \( x_0 \in C^r[-1,1] \) be a isolated solution of the equation (2.1) and \( \tilde{x}_n^G \) be the iterated Legendre Galerkin approximations of \( x_0 \). Then we get the following superconvergence rates.

\[
\| x_0 - \tilde{x}_n^G \|_{L^2} = O(n^{-2r}),
\]
\[
\| x_0 - \tilde{x}_n^G \|_{\infty} = O(n^{-2r}).
\]

Proof. From Theorem 2.8, we have

\[
\| x_0 - \tilde{x}_n^G \|_{\infty} \leq c\{\| x_0 - P_n^G x_0 \|_{L^2}^2 + | < g_t(\cdot), (I - P_n^G)(x_0) > |\},
\]

(2.58)

where \( c \) is a constant independent of \( n \).

Using the orthogonality of the projection operators \( P_n^G \) and Cauchy-Schwarz inequality, we obtain

\[
| < g_t(\cdot), (I - P_n^G)(x_0) > | = | < (I - P_n^G)g_t(\cdot), (I - P_n^G)(x_0) > | \\
\leq \| (I - P_n^G)g_t(\cdot) \|_{L^2} \| x_0 - P_n^G x_0 \|_{L^2}
\]

(2.59)

Hence using estimates (2.58), (2.59) and (2.10), we have

\[
\| \tilde{x}_n^G - x_0 \|_{\infty} \leq c\{\| x_0 - P_n^G x_0 \|_{L^2}^2 + \| (I - P_n^G)g_t(\cdot) \|_{L^2} \| x_0 - P_n^G x_0 \|_{L^2}\}
\]
\[
\leq c n^{-2r} \| x_0^{(r)} \|_{L^2}^2 + c n^{-2r} \| x_0^{(r)} \|_{L^2} \| (g_t(\cdot))^{(r)} \|_{L^2}
\]
\[
= O(n^{-2r}).
\]

(2.60)

And also

\[
\| \tilde{x}_n^G - x_0 \|_{L^2} \leq \sqrt{2} \| \tilde{x}_n^G - x_0 \|_{\infty} = O(n^{-2r}).
\]

(2.61)

Hence the proof follows.

\[\square\]

Theorem 2.11. Let \( x_0 \in C^r[-1,1] \) be a isolated solution of the equation (2.1) and \( \tilde{x}_n^C \) be the iterated Legendre collocation approximations of \( x_0 \). Then we have the following convergence rates.

\[
\| x_0 - \tilde{x}_n^C \|_{L^2} = O(n^{-r}),
\]
\[
\| x_0 - \tilde{x}_n^C \|_{\infty} = O(n^{-r}).
\]

Proof. Using Theorem 2.8, Lemma 2.5, we have for the interpolatory projection operator \( P_n^C \)

\[
\| \tilde{x}_n^C - x_0 \|_{\infty} \leq c\{\| x_0 - P_n^C x_0 \|_{L^2}^2 + | < g_t(\cdot), (I - P_n^C)(x_0) > |\}
\]
\[
\leq c\{\| x_0 - P_n^C x_0 \|_{L^2}^2 + \| g_t \|_{L^2} \| x_0 - P_n^C x_0 \|_{L^2}\}
\]
\[
\leq c \{n^{-2r} \| x_0^{(r)} \|_{L^2}^2 + n^{-r} \| g_t \|_{L^2} \| x_0^{(r)} \|_{L^2}\}
\]
\[
= O(n^{-r}),
\]

(2.62)

and

\[
\| \tilde{x}_n^C - x_0 \|_{L^2} \leq \sqrt{2} \| \tilde{x}_n^C - x_0 \|_{\infty} = O(n^{-r}).
\]

(2.63)
Hence the proof follows.

**Remark:** From Theorems 2.9, 2.10, 2.11 we observe that the Legendre Galerkin and Legendre collocation solutions of Urysohn integral equation have same order of convergence, $O(n^{-r})$ in $L^{2}$-norm and $O(n^{\frac{1}{2}-r})$ in infinity norm. The iterated Legendre Galerkin solution converges with the order $O(n^{-2r})$ in both $L^{2}$-norm and infinity norm, whereas the iterated Legendre collocation solution converges with the order $O(n^{-r})$ in both $L^{2}$-norm and in infinity norm. This shows that iterated Legendre Galerkin method improves over the iterated Legendre collocation method.

### 3. Numerical Example

In this section we present the numerical results. To apply Legendre Galerkin and Legendre collocation methods, we choose the approximating subspaces $X_n$ to be the Legendre polynomial subspaces of degree $\leq n$. Legendre polynomials can be generated by the following three-term recurrence relation

\[
\phi_0(s) = 1, \phi_1(s) = s, \quad s \in [-1,1].
\]

and

\[
(i + 1)\phi_{i+1}(x) = (2i + 1)x\phi_i(x) - i\phi_{i-1}(x), \quad x \in [-1,1], \quad i = 1, 2, ..., n - 1. \tag{3.1}
\]

We denote, the Galerkin and iterated Galerkin solutions by $x_n^G$ and $\tilde{x}_n^G$, respectively and also the collocation and iterated collocation solutions by $x_n^C$ and $\tilde{x}_n^C$, respectively, in the following tables. We present the errors of the approximation solutions and the iterated approximation solutions in both $L^{2}$-norm and infinity norm. In Tables 1 and 2, $n$ represents the highest degree of the Legendre polynomials employed in the computation. The numerical algorithms are compiled by using Matlab.

**Example 3.1.** We consider the following integral equation

\[
x(t) - \int_{-1}^{1} k(t, s, x(s))ds = f(t), \quad -1 \leq t \leq 1 \tag{3.2}
\]

with the kernel function $k(t, s, x(s)) = \left(3\sqrt{2\pi}\right)\cos\left(\frac{\pi |s-t|}{4}\right)[x(s)]^2$ and the function $f(t) = \left(\frac{-1}{4}\right)\cos\left(\frac{\pi t}{4}\right)$ where the exact solution is given by $x(t) = \cos\left(\frac{\pi t}{4}\right)$. 

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Table 1: Legendre Galerkin method

| n  | $||x - x_n^G||_{L^2}$ | $||x - x_n^G||_\infty$ | $||x - \tilde{x}_n^G||_{L^2}$ | $||x - \tilde{x}_n^G||_\infty$ |
|----|------------------------|------------------------|------------------------|------------------------|
| 2  | 0.16612096278e-02      | 0.3471163964e-02       | 0.27140926077e-04       | 0.21215376302e-04      |
| 4  | 0.00867599248e-03      | 0.0000023431e-04       | 0.00000026700e-08       | 0.00000006217e-07      |
| 5  | 0.00867599248e-03      | 0.0000023431e-04       | 0.00000026700e-08       | 0.00000006217e-07      |
| 7  | 0.0024104785e-04       | 0.00000131351e-04      | 0.00000026700e-08       | 0.00000006217e-07      |
| 8  | 0.000000041535e-04     | 0.00000026700e-08      | 0.00000006217e-07       | 0.00000006217e-07      |

Table 2: Legendre Collocation method

| n  | $||x - x_n^C||_{L^2}$ | $||x - x_n^C||_\infty$ | $||x - \tilde{x}_n^C||_{L^2}$ | $||x - \tilde{x}_n^C||_\infty$ |
|----|------------------------|------------------------|------------------------|------------------------|
| 2  | 0.23728226126e-02      | 0.63959128045e-02      | 0.50686226008e-03       | 0.39620142716e-03      |
| 4  | 0.01227979480e-03      | 0.00400877378e-02      | 0.01552173562e-05       | 0.00012132949e-03      |
| 5  | 0.00867602639e-03      | 0.00218077988e-02      | 0.00036041468e-05       | 0.00028172709e-05      |
| 7  | 0.00024104404e-04      | 0.00068986980e-04      | 0.00001183440e-08       | 0.00000096811e-07      |
| 8  | 0.00000058748e-04      | 0.00002505430e-05      | 0.00000026700e-08       | 0.00000006217e-07      |

From Tables 1 and 2, we see that the numerical results agree with the theoretical results.

Example 3.2. We consider the following integral equation

$$
\int_0^1 k(t, s, x(s)) ds = f(t), \quad -1 \leq t \leq 1
$$

with the kernel function $k(t, s, x(s)) = \frac{1}{5} \cos(\pi t) \sin(\pi s)[x(s)]^3$ and the function $f(t) = \sin(\pi t)$ where the exact solution is given by $x(t) = \sin(\pi t) + \frac{1}{3}(20 - \sqrt{391}) \cos(\pi t)$.

For this example, we have compared our results with the piecewise polynomial-based Galerkin and collocation methods proposed in [3] and [4]. To do this, we consider a uniform partition of $[0, 1]$:

$$
0 = t_0 < t_1 < t_2 < ... < t_{n+1} = 1
$$

where $t_i = \frac{i-1}{n}$, $i = 1, 2, ..., n + 1$.

We choose the approximating subspaces as the space of piecewise constant functions, which has dimension $n$. The collocation points are taken to be the roots of Legendre polynomial of degree 1 in $[0, 1]$, shifted to $(t_{i-1}, t_i)$, which are actually the mid points of each sub-intervals, i.e, we choose collocation points as:

$$
s_i = \frac{2i - 1}{2n}, \quad i = 1, 2, ..., n.
$$

In Tables 3 and 4, we present the errors in Legendre Galerkin and Legendre collocation methods and in Tables 5 and 6, we give the errors for Galerkin and collocation methods with approximating subspace as the space of piecewise constant functions. In Tables 3 and 4, $n$
denote the highest degree of Legendre polynomial employed in the computation and in Tables 5 and 6, \( n \) denote the the dimension of the approximating subspace.

Table 3: Legendre Galerkin method

<table>
<thead>
<tr>
<th>( n )</th>
<th>( | x - x_n^G |_{L^2} )</th>
<th>( | x - x_n^G |_{\infty} )</th>
<th>( | x - \tilde{x}<em>n^G |</em>{L^2} )</th>
<th>( | x - \tilde{x}<em>n^G |</em>{\infty} )</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>0.18447865e-01</td>
<td>0.64441433e-01</td>
<td>0.10769066e-02</td>
<td>0.15229748e-02</td>
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<tr>
<td>3</td>
<td>0.17294403e-01</td>
<td>0.52172170e-01</td>
<td>0.10103846e-02</td>
<td>0.14288986e-02</td>
</tr>
<tr>
<td>4</td>
<td>0.42427202e-03</td>
<td>0.19748104e-02</td>
<td>0.63371262e-05</td>
<td>0.89620430e-05</td>
</tr>
<tr>
<td>5</td>
<td>0.36907755e-03</td>
<td>0.12964794e-02</td>
<td>0.53242459e-05</td>
<td>0.75296151e-05</td>
</tr>
<tr>
<td>6</td>
<td>0.52214001e-05</td>
<td>0.28375175e-04</td>
<td>0.96363220e-08</td>
<td>0.13627807e-07</td>
</tr>
<tr>
<td>7</td>
<td>0.41553615e-05</td>
<td>0.16651422e-04</td>
<td>0.70737260e-08</td>
<td>0.10003751e-07</td>
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<tr>
<td>8</td>
<td>0.39931165e-07</td>
<td>0.24426326e-06</td>
<td>0.05210533e-11</td>
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<tr>
<td>9</td>
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<td>0.12912277e-06</td>
<td>0.32443617e-11</td>
<td>0.45881077e-11</td>
</tr>
</tbody>
</table>

Table 4: Legendre Collocation method

<table>
<thead>
<tr>
<th>( n )</th>
<th>( | x - x_n^C |_{L^2} )</th>
<th>( | x - x_n^C |_{\infty} )</th>
<th>( | x - \tilde{x}<em>n^C |</em>{L^2} )</th>
<th>( | x - \tilde{x}<em>n^C |</em>{\infty} )</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>0.25334382e-01</td>
<td>0.1066739308</td>
<td>0.14589879e-02</td>
<td>0.20633190e-02</td>
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<td>3</td>
<td>0.17300974e-01</td>
<td>0.51950086e-01</td>
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<td>0.56284026e-03</td>
<td>0.30247078e-02</td>
<td>0.16697387e-04</td>
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<tr>
<td>5</td>
<td>0.36911801e-03</td>
<td>0.12940851e-02</td>
<td>0.55660729e-05</td>
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<tr>
<td>6</td>
<td>0.66747628e-05</td>
<td>0.43031650e-04</td>
<td>0.52125995e-07</td>
<td>0.73717234e-07</td>
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<td>7</td>
<td>0.4155544e-05</td>
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<td>0.74765899e-08</td>
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<td>0.35599727e-06</td>
<td>0.45669483e-10</td>
<td>0.64586392e-10</td>
</tr>
<tr>
<td>9</td>
<td>0.28887128e-07</td>
<td>0.12690817e-06</td>
<td>0.36173459e-11</td>
<td>0.51156718e-11</td>
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</table>
Table 5: Piecewise polynomial based Galerkin method

<table>
<thead>
<tr>
<th>n</th>
<th>$|x - x_n^G|_{L^2}$</th>
<th>$|x - x_n^G|_{\infty}$</th>
<th>$|x - \tilde{x}<em>n^G|</em>{L^2}$</th>
<th>$|x - \tilde{x}<em>n^G|</em>{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.30981458760</td>
<td>0.68983512701</td>
<td>0.30029759e-01</td>
<td>0.42468461e-01</td>
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<tr>
<td>4</td>
<td>0.15921432377</td>
<td>0.38594513556</td>
<td>0.52877651e-02</td>
<td>0.74780235e-02</td>
</tr>
<tr>
<td>8</td>
<td>0.0819028e-01</td>
<td>0.19641976481</td>
<td>0.13699122e-02</td>
<td>0.19373469e-02</td>
</tr>
<tr>
<td>16</td>
<td>0.40166351e-01</td>
<td>0.96868820e-01</td>
<td>0.34553767e-03</td>
<td>0.48866369e-03</td>
</tr>
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<td>32</td>
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<td>0.48426304e-01</td>
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<tr>
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<td>0.53890086e-05</td>
<td>0.76212033e-05</td>
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<td>0.60119542e-02</td>
<td>0.13135942e-05</td>
<td>0.18577013e-05</td>
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</tbody>
</table>

Table 6: Piecewise polynomial based collocation method

<table>
<thead>
<tr>
<th>n</th>
<th>$|x - x_n^C|_{L^2}$</th>
<th>$|x - x_n^C|_{\infty}$</th>
<th>$|x - \tilde{x}<em>n^C|</em>{L^2}$</th>
<th>$|x - \tilde{x}<em>n^C|</em>{\infty}$</th>
</tr>
</thead>
<tbody>
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<td>0.31698305338</td>
<td>0.74927708817</td>
<td>0.21307836e-01</td>
<td>0.30133808e-01</td>
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<td>4</td>
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<td>0.38899230331</td>
<td>0.13752503e-02</td>
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<td>0.47868563e-06</td>
</tr>
</tbody>
</table>

From Tables 3, 4, 5 and 6, we see that the Legendre Galerkin and Legendre collocation methods are more rapid than the piecewise polynomial based Galerkin and collocation methods. It is clear from the numerical results that, in case of Legendre Galerkin and Legendre collocation methods, we obtain better errors by solving much smaller nonlinear system of equations. For example, we see that, in iterated Legendre Galerkin and iterated Legendre collocation method, to obtain the error of order $10^{-5}$, a system of size $5 \times 5$ is needed to be solved, whereas in piecewise polynomial based iterated Galerkin and iterated collocation method, we need to solve a system of size $128 \times 128$.

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