Packing problems in edge-colored graphs

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Abstract

Let \( F \) be a fixed edge-colored graph. We consider the problem of packing the greatest possible number of vertex disjoint copies of \( F \) into a given complete edge-colored graph. We observe that this problem is NP-hard unless \( F \) consists of isolated vertices and edges or unless there are only two colors and \( F \) is properly 2-edge-colored. Of the remaining problems we focus on the case where \( F \) is a properly 2-edge-colored path of length 2, when we show polynomial solutions based on matching methods. In fact, we prove that a large packing exists if and only if each color class has a matching of a sufficiently large size. We also consider the more general problem where \( \mathcal{F} \) is a fixed family of edge-colored graphs, and an edge-colored complete graph is to be packed with vertex disjoint copies of members of \( \mathcal{F} \). Here we concentrate on two cases in which \( \mathcal{F} \) consists of trees and give sufficient conditions that guarantee the existence of a packing of given size.

1. Introduction

Consider a fixed \( k \)-edge colored graph, i.e., a graph \( F \) with a fixed partition of its edge set \( E(F) \) into \( k \) "color classes" \( E_1, E_2, \ldots, E_k \). Given a \( k \)-edge-colored complete graph we ask whether its vertices can be partitioned into sets \( V_1, V_2, \ldots, V_m \) such that each \( V_i \) contains a spanning copy of \( F \). Such a partition will be called an \( F \)-partition. (Note that considering the partition problem only for complete graphs is not a real restriction, since any \( k \)-edge-colored graph may be viewed as a \((k + 1)\)-edge-colored complete graph where the nonedges are colored by the \((k + 1)\)-th color.) Not surprisingly, most of these \( F \)-partition problems turn out to be NP-complete (see Section 2). The trivial case where \( F \) consists of just one edge (say of color \( c \)) is the familiar
matching problem (regardless of \( k \)), as in this case an \( F \)-partition of a \( k \)-edge-colored complete graph is just a perfect matching in the subgraph formed by the edges colored \( c \). Thus this \( F \)-partition problem has a polynomial-time solution, using e.g. the maximum matching algorithm of [4]. Matching methods also apply when \( F \) consists of an isolated edge and a number of isolated vertices (cf. [6]). Thus we shall assume in the sequel that \( F \) has more than one edge. Then we show that the \( F \)-partition problem is NP-complete unless \( F \) consists of isolated vertices and edges, or unless \( k \) is equal to two and \( F \) is properly two-edge-colored.

In Section 3, we consider the first two interesting cases, namely, where \( F \) is the union of two isolated edges of two distinct colors, and where \( F \) is a properly two-edge-colored path of length two. We show that for the latter the trivial necessary matching conditions are sufficient as well. Namely, when \( F \) is properly two-edge-colored path of length two, a two-colored complete graph of order \( n \) (divisible by 3) admits an \( F \)-partition if and only if each color class contains a matching with at least \( n/3 \) edges. This result leads to a polynomial algorithm for the \( F \)-partition problem. There may be polynomial algorithms for unions of isolated edges of distinct colors, and other properly two-edge-colored graphs. However, it has just been shown by Bruce Bauslaugh [1] that if \( F \) is a properly two-edge-colored four-cycle, then the \( F \)-partition problem is again NP-complete. (On the other hand, the problem of finding long paths or cycles is polynomially solvable [8, 2].)

If an edge-colored complete graph does not admit an \( F \)-partition, it may still be possible to find many disjoint sets \( V_1, V_2, \ldots, V_m \) of vertices such that each \( V_i \) contains a (spanning) copy of \( F \). Such a family of sets is called an \( F \)-packing, and the \( F \)-packing problem consists of finding an \( F \)-packing with the largest possible size. This problem also turns out to be polynomial for the above two special graphs \( F \).

In Section 4, we consider a somewhat more general problem. Let \( \mathcal{F} \) be a fixed family of connected \( k \)-edge-colored graphs. Given a \( k \)-edge-colored complete graph we ask whether its vertices can be partitioned into sets \( V_1, V_2, \ldots, V_m \) such that each \( V_i \) contains a spanning copy of some element of \( \mathcal{F} \). Such partition will be called an \( \mathcal{F} \)-partition. An \( \mathcal{F} \)-packing is a family of disjoint sets \( V_1, V_2, \ldots, V_m \) such that each \( V_i \) contains a (spanning) copy of some member of \( \mathcal{F} \). We investigate two cases where \( \mathcal{F} \) consists of trees and develop sufficient conditions that guarantee the existence of \( \mathcal{F} \)-packings covering a prescribed number of vertices.

2. NP-completeness results

In this section we show that the \( F \)-partition problem is NP-complete unless \( F \) is the union of \( k \) disjoint edges or \( k \) is equal to two and \( F \) is properly two-edge-colored.

**Theorem 2.1.** Let \( F \) be a fixed \( k \)-edge-colored graph. If there is a color \( c \) (among the \( k \) allowed colors) such that the graph \( F' \) obtained from \( F \) by deleting all edges of color \( c \) has a component with at least three vertices, the \( F \)-partition is NP-complete.
Proof. Let \( L \) be the largest component of \( F' \) (not containing edges of color \( c \)) viewed as an uncolored graph. Then \( L \) has at least three vertices, and according to [6] it is NP-complete to decide whether the vertices of a given (uncolored) graph \( G \) can be partitioned into subsets \( V_1, V_2, \ldots, V_m \) such that each \( V_i \) contains a spanning copy of \( L \). The proof given in [6] transforms the \(|L|\)-dimensional matching problem to the above problem by the use of modules. A module is constructed from an initial graph isomorphic to \( L \), by attaching a distinct copy of \( L \) to each of its vertices and then in those copies a suitable vertex \( v \) is duplicated. An illustration is given in Fig. 1 below (cf. also [5]). For any instance of the \(|L|\)-dimensional matching problem, we construct a graph \( G \) in which one of these modules is attached to each \(|L|\)-tuple; this graph \( G \) can be partitioned into subsets \( V_1, V_2, \ldots, V_m \) such that each \( V_i \) contains a spanning copy of \( L \) if and only if there was an \(|L|\)-dimensional matching. We now view \( L \) as a \( k \)-edge-colored graph with the color \( c \) not actually appearing, and in the above construction we copy the colors of \( L \) onto the modules in the obvious way (each module being constructed from copies of \( L \)). This is illustrated in Fig. 1. In \( G \) we keep these colors of the modules, and use \( c \) to color all the edges missing from \( G \). In this way we obtain a \( k \)-edge-colored complete graph which admits an \( L \)-partition if and only if there was an \(|L|\)-dimensional matching. Thus the \( L \)-partition problem is NP-complete.

Suppose that \( L \) has \( l \) vertices and \( F \) has \( f \) vertices. We now describe a reduction from the \( L \)-partition problem discussed above to the \( F \)-partition problem. Suppose that \( G \) is an instance of the \( L \)-partition problem, i.e., a \( k \)-edge-colored complete graph with \( ml \) vertices which we want to partition into copies of \( L \). Construct a new \( k \)-edge-colored complete graph \( H \) on \( mf \) vertices by adding to \( G \) \( m \) copies of all components (other than \( L \)) of \( F \) and giving all missing edges color \( c \). Clearly, if \( G \) has an

Fig. 1. (a) \( L \) (\( v \) will be duplicated); (b) The uncolored module; (c) The colored module.
L-partition, then $H$ has an $F$-partition. Conversely, suppose that $H$ has an $F$-partition, and denote by $b$ the number of components of $F'$ isomorphic to $L$. There are $m$ copies of $F$ in an $F$-partition and hence $mb$ copies of $L$. But the only copies of $L$ in $H$ are the $m(b - 1)$ copies specifically added for the other components isomorphic to, but distinct from $L$ plus whatever copies can be found in $G$. Thus $G$ must admit an $L$-partition.

**Corollary 2.2.** The $F$-partition problem is NP-complete unless $F$ consists of isolated vertices and edges, or unless $k = 2$ and $F$ is properly two-edge-colored.

**Proof.** If $k \geq 3$ and some component of $F$ has at least two edges then there is a color $c$ not used on either of these edges. Therefore deleting all edges of color $c$ leaves a component with at least three vertices. If $k = 2$ and $F$ is not properly 2-edge-colored then one of the colors contains two incident edges and hence removing all edges of the other color will result in a component with at least three vertices.

3. **Properly two-colored graphs**

   The first interesting example is the case where $F$ is a path of length two, with two edges of different colors. We give a polynomially verifiable condition for the existence of an $F$-partition in this case.

**Theorem 3.1.** Let $F$ be the path of length two with its two edges having different colors. A two-edge-colored complete graph of order at least $3k$ admits a packing of $k$ copies of $F$ if and only if each color class admits a matching of size $k$.

**Proof.** Any packing with $k$ copies of $F$ yields two matchings with $k$ edges each. Hence, it remains to prove that the existence of two matchings with $k$ edges each (in the respective colors) implies the existence of an $F$-packing with $k$ copies of $F$. Otherwise we take a smallest counterexample, i.e., a smallest two-edge-colored complete graph on at least $3k$ vertices (for some $k$) in which there is a matching $M_1$ with $k$ edges of color 1 and a matching $M_2$ with $k$ edges of color 2, but which admits no $F$-packing with $k$ copies of $F$. We may assume that the number of vertices, $n$, is at least six.

   In order to obtain a contradiction, it will suffice to show that for a suitable choice of the matchings $M_i$, the graph $M = M_1 \cup M_2$ contains a component isomorphic to $F$. Indeed, after deleting this component, we would obtain a graph with $n - 3$ vertices in which there is both a matching of color one and a matching of color two, having $k - 1$ edges. By minimality we would obtain $k - 1$ disjoint copies of $F$ which together with the deleted component of $M$ would produce an $F$-packing with $k$ copies of $F$, contrary to the assumption.

   Clearly, the components of $M$ are alternating paths and alternating even cycles. Since each $M_i$ has $2k$ vertices of degree 1 and $n - 2k$ vertices of degree 0, the number of
vertices of degree 1 in \( M \) incident to edges of \( M_1 \) is the same as that incident to edges of \( M_2 \). In particular, if \( M \) has an isolated edge of color 1 then it has another component which is a path, possibly of length one, in which the first and the last edge both have color 2. We also observe that if \( M \) has no isolated edge and no component isomorphic to \( F \) then it must have an isolated vertex, i.e., there must be a vertex \( w \) which is not incident to any edge of \( M \). To see this, note that \( k \leq n/3 \) and hence the average degree in \( M \) is \( 4k/n \leq 4/3 \); if all components of \( M \) were cycles or paths of length at least 3 then the average degree would be at least 3/2.

To obtain a component isomorphic to \( F \) we shall consider the following modifications of \( M_1 \) and \( M_2 \):

1. If \( M \) has two isolated edges \( xx', yy' \) of opposite color, then depending on the color of the edge \( xy \) either \( M_1 \) or \( M_2 \) can be modified by replacing either \( xx' \) or \( yy' \) with \( xy \), creating a component of \( M \) (either \( xyy' \) or \( yxx' \)) isomorphic to \( F \).

2. If \( M \) has an isolated edge \( xx' \) of color 1 then it has a component which is a path \( yy'y''...zz' \) with both \( yy' \) and \( zz' \) of color 2. If \( xy \) has color 2, then we modify \( M_2 \) by replacing \( yy' \) with \( xy \), yielding a component \( xyy' \) of \( M \) isomorphic to \( F \) and a shorter path component \( y'y''...zz' \). A similar operation applies if \( x'z' \) has color 2. Otherwise we modify \( M_1 \) by replacing \( xx' \) and \( y'y'' \) with \( xy \) and \( x'z' \), yielding a component \( xyy' \) of \( M \) isomorphic to \( F \) and a shorter path component \( y''...zz'x' \).

3. If \( M \) has an isolated vertex \( w \) and a path component \( xx'yy' \), then consider the edge \( xw \). Depending on the color of this edge we can modify \( M_1 \) or \( M_2 \) to obtain a component of \( M \) isomorphic to \( F \), and an isolated edge in \( M \).

4. If \( M \) has an isolated vertex \( w \) and a component \( xx'yy'z... \) which is a path of length \( l \geq 4 \), then using the edge \( wy \) we can modify \( M_1 \) or \( M_2 \) to obtain a new \( M \), with either an isolated edge and a path component of length \( l - 1 \) or a path component of length three and a path component of length \( l - 3 \geq 1 \), and by the observation above it must contain an isolated vertex.

5. If \( M \) has an isolated vertex \( w \) and a cycle component then we can modify \( M_1 \) or \( M_2 \) by replacing an edge of such a cycle by an edge incident to \( w \), obtaining a new \( M \) with a new path component (and one fewer cycle components).

By the minimality of our counterexample we see that none of the operations (1–3) can be applicable because they produce a component of \( M \) isomorphic to \( F \). Similarly, operation (4) cannot be applicable because it results in a situation in which one of modifications (1)–(3) applies. Finally, operation (5) results in a situation in which modification (4) is applicable. Since modifications (1)–(5) describe all possible situations, we have obtained a final contradiction.

It is easy to see that the above proof can be transformed to a polynomial-time algorithm to modify any two matchings \( M_1 \) and \( M_2 \) to produce an \( M \) that solves the \( F \)-packing problem. Therefore, using any polynomial-time maximum matching algorithm, e.g. [4], we have a polynomial algorithm to actually find a maximum packing of the properly colored path of length 2.

The corollary below is an immediate consequence of Theorem 3.1.
Corollary 3.2. Let $F$ be a path of length two, with its two edges having different colors. A two-edge-colored complete graph of order $n \equiv 0 \mod 3$ admits an $F$-partition whenever each color class admits a matching with $n/3$ edges.

Another simple example not excluded by Corollary 2.2 is the union of two isolated edges of different colors. We find a solution similar to the above.

Proposition 3.3. Let $F$ be the union of two isolated edges of different colors. A two-edge-colored complete graph on more than $4k$ vertices admits an $F$-packing with $k$ copies of $F$ if and only if each color class admits a matching of size $k$.

Proof. Any packing with $k$ copies of $F$ yields two matchings with $k$ edges each. To see that two such matchings (in the two colors) also yields the existence of an $F$-packing on $4k$ vertices, let $M_1$ and $M_2$ be matchings with $k$ edges of color 1 and 2, respectively, and assume that $M_1 \cup M_2$ contains as many vertices as possible. If $M_1 \cup M_2$ were not a matching, then it would contain a connected component $C$ containing at least one edge of each color; and there were at least two vertices, say $x$ and $y$, not covered by $M_1 \cup M_2$. Replacing by $xy$ an edge of $C$ whose color is the same as that of $xy$, we would obtain a larger union of two monochromatic matchings, a contradiction to the choice of $M_1$ and $M_2$. \qed

We note that the above proposition is not valid anymore if the graph is assumed to have just $4k$ vertices, as shown by the two-edge-colorings in which color 1 forms a complete bipartite graph $K_{2k,2k}$ ($k$ odd) or $K_{2k-1,2k+1}$ ($k$ even).

The proof of Proposition 3.3 can also be transformed to a polynomial-time algorithm that solves the $F$-packing problem. We do not know, however, whether or not all properly two-colored paths $F$ have polynomially solvable $F$-partition (or packing) problems. In the case where $F$ is a path of length three, with the central edge of color 2 and the end edges of color 1, one may be tempted to guess that a two-colored complete graph admits an $F$-partition if and only if color 1 admits a perfect matching and color 2 admits a matching that covers half of the vertices. However, this is false. Indeed, take the two-edge-colored complete graph on $n$ vertices consisting of a complete bipartite graph $K_{2k,n-2k}$ in color 2 and its complement in color 1. For $n$ even and $k < n/4$, there is no $F$-partition, although for $n$ even and $k \geq n/8$ the matching conditions are satisfied. The following result shows that this example is in a sense best possible.

Theorem 3.4. Let $F$ be a path of length three with the central edge of color 2 and the end edges of color 1. If a two-edge-colored complete graph on at least $4k$ vertices has a perfect matching in color 1 and a matching of size $2k$ in color 2, then it also contains a packing of $k$ copies of $F$. 
Proof. Consider the union \( M \) of a perfect matching in color 1 and a matching of size \( 2k \) in color 2. The connected components of \( M \) are alternating even cycles and/or alternating paths starting and ending with edges of color 1. If there is a nontrivial path component \( P \) (i.e., a path \( P \) containing at least one edge of color 2) then we can delete its first four vertices – they induce \( F \) in \( P \) – and the assertion follows by induction since the remaining graph still has a perfect matching in color 1 and a matching of size at least \( 2k - 2 \) in color 2. Similarly, if a cycle component \( C \) has length divisible by 4, then \( C \) can be partitioned into copies of \( F \), without violating the condition in the remaining graph.

Suppose that no such reduction can be applied. Then each cycle component of \( M \) has a length \( \equiv 2 \pmod{4} \). Moreover, since the number \( 2k \) of color-2 edges is even, \( M \) consists of an even number of cycle components (and isolated edges in color 1). Take two alternating cycles \( C = x_1x_2 \ldots x_{2s} \) and \( C' = y_1y_2 \ldots y_{2t} \) in \( M \). Assume that the edges \( x_1x_2 \) and \( y_1y_2 \) have color 1. If \( x_{2s}y_{2t} \) had color 2, then \( C \cup C' \) would have an alternating Hamiltonian path \( x_1x_2 \ldots x_{2s}y_{2t}y_{2t-1} \ldots y_1 \) of length divisible by 4 and starting with color 1, and we could easily find an \( F \)-partition of \( C \cup C' \). Hence, by symmetry reasons, we may assume that all edges joining \( C \) with \( C' \) have color 1. In this case, however, \( x_1y_1y_{2t}y_{2t-1} \ldots y_2x_2x_3 \ldots x_{2s} \) is an alternating Hamiltonian path on \( C \cup C' \). \( \square \)

4. Families of graphs

We now assume that \( \mathcal{F} \) is a fixed family of \( t \)-edge-colored graphs. We explore sufficient conditions that guarantee \( \mathcal{F} \)-packings of large size. Let \( \mathcal{F} = \mathcal{F}_t \) be the family of all trees with \( t \) edges, under all possible \( t \)-edge-colorings in which all edges have pairwise distinct colors. When \( t = 2 \), \( \mathcal{F}_t \) consists just of the properly two-colored path of length two. According to our result from Section 3, if a two-edge-colored complete graph admits large matchings in both colors then there is a large \( \mathcal{F}_2 \)-packing. Now we prove an analog of this theorem for three-edge-colorings.

**Theorem 4.1.** If \( n \) is sufficiently large with respect to \( k \), then every three-edge-colored complete graph on \( n \) vertices in which each color class admits a matching of size \( 2k - 1 \) has a \( \mathcal{F}_3 \) packing of \( k \) trees. On the other hand, having matchings of size \( 2k - 2 \) in each color is not sufficient for the same conclusion.

**Proof.** We start with three matchings \( M_i \) of size \( 2k - 1 \) in color \( i (i = 1, 2, 3) \), such that \( M = M_1 \cup M_2 \cup M_3 \) has a maximum number of isolated edges (i.e., edges disjoint from every other edge of \( M \)). Since \( n \) is large, some of the \( M_i \), say \( M_1 \), is an isolated matching, i.e., formed by isolated edges. Indeed, \( M \) does not cover all vertices, so we can take an edge \( xy \) disjoint from \( M \). If \( xy \) has color 1, and \( M_1 \) is not an isolated matching, then replacing a color-1 edge of a tree component of \( M \) by \( xy \), the number of isolated edges increases, contradicting the choice of the \( M_i \). If \( M_2 \cup M_3 \) has
a vertex $z$ of degree 2 in $M$, then take an $xy$ in $M_1$ and consider the color of the edge $xz$. In any case, we find a 3-colored tree containing $z$. Since the two neighbors of $z$ have degree no more than 2 in $M$, deleting the tree we obtain a three-edge-colored complete graph on $n - 4$ vertices, in which each color contains a matching of size at least $2k - 3$, and the statement follows by induction on $k$. Otherwise each isolated $M_i$ is a matching. Let $xx' \in M_1, yy' \in M_2$ and $zz' \in M_3$. If $yz$ has color 1 then $y'yzz'$ is a 3-colored path of length four and the induction applies. If $yz$ has color 2, then we let $M_2' = (M_2 - \{yy'\}) \cup \{yz\}$ and obtain a path component of length two in $M_2' \cup M_3$ for which we can apply the previous deletion with respect to the color of the edge $xz$. For a $yz$ of color 3 the transformation is similar, implying that large matchings indeed guarantee large $\mathcal{F}_3$-packings.

To see that $2k - 1$ cannot be replaced by $2k - 2$, let the color classes $C_2$ and $C_3$ be matchings of size $2k - 2$ and let $C_2 \cup C_3$ consist of $k - 1$ vertex-disjoint 4-cycles. Then in the union of any collection of vertex-disjoint 3-colored trees we have at most 2 edges in each of those 4-cycles, and consequently we cannot pack $k - 3$-colored trees into this particular coloring of the complete graph. \(\blacksquare\)

In its present form, the previous theorem cannot be extended for any larger number $t \geq 4$ of colors, even if we assume that as many as $n/(2t - 2)$ (or even more) pairwise disjoint edges exist in each color class. Motivated by this disappointing fact, we now turn to forests and denote by $\mathcal{F}_t$ the family of all forests with $t$ edges in which each edge has a distinct color. We ask for the minimum integer $m = m(t, k, n)$ such that any $t$-edge-colored complete graph on $n$ vertices which admits a matching of size $m$ in each color class also admits an $\mathcal{F}_t$-packing of size $k$.

For $t > 3$, we do not know the exact value of $m(t, k, n)$. (The cases $t = 2, 3$ are solved by the previous results.) However, it is easy to verify an upper bound independent of $n$.

**Proposition 4.2.** For all $t, k$ and $n$, $m(t, k, n) \leq 2tk$.

**Proof.** Fix a matching $M_i$ of size $2t - 1$ in each color $i$, $1 \leq i \leq t$. Let $F_1$ be a matching of size $t$ in $M = M_1 \cup M_2 \cup \ldots \cup M_t$, such that $F_1$ contains precisely one edge of each color (the edge $F_1 \cap M_i$ can be chosen successively for $i = 1, 2, \ldots, t$, as the $M_i$ are supposed to be large enough). Since $F_1$ has $2t$ vertices, it meets at most $2t$ edges of each color. Thus deleting $F_1$, we obtain a $t$-colored complete graph in which each color class has a matching of size at least $2(tk - 1)$. Hence, by induction on $k$, there are $k - 1$ vertex-disjoint $t$-colored forests $F_2, \ldots, F_k$ which, together with $F_1$, form a packing with the required properties. \(\blacksquare\)

Certainly, the upper bound of $2tk$ can be improved by a more careful counting.

**Corollary 4.3.** For $n \geq n_0 = n_0(t, k)$, $m(t, k, n)$ is constant.
Proof. There are less than $2tk$ values of $n$ for which $m(t,k,n+1) < m(t,k,n)$. Of course, $\leq$ holds for all values of $n$. □

Although Proposition 4.2. is easy, it is best possible apart from a multiplicative constant:

**Proposition 4.4.** $m(t,k,n) \geq (tk/2) - 1$ for $k > t$ and $n$ large.

Proof. We define a $t$-coloring of the complete graph on $n$ vertices, in which the edges of colors $1,2,\ldots,t-1$ are concentrated to less than $tk$ vertices when $k > t$. Such a coloring can be obtained easily since the complete graph on either $tk - 1$ or $tk - 2$ vertices has an edge partition into perfect matchings, according to the parity of $tk$. If each color class contains at least one perfect matching, then it also has a matching of size $\geq (tk - 2)/2$. However, any forest of $t-1$ edges has at least $t$ vertices, i.e., in a packing of $k$ forests the edges of colors $1,\ldots,t-1$ should contain at least $tk$ vertices, which is impossible in our coloring. □

It remains an open problem to determine the asymptotic behavior of $m(t,k,n)$ as $t$ and $k$ get large. We expect that the ratio tends to a limit as $k \to \infty$, for every fixed $t$ (and perhaps this limit is the same for every $t$).

We next examine a different family $\mathcal{F}$. Consider two colors, and let $\mathcal{S}_k$ denote the family consisting of the two monochromatic stars, a color-1 $S_k$ and a color-2 $S_k$ (the star with $k$ vertices). The smallest $n$ such that any two-edge-colored complete graph on $n$ vertices admits an $\mathcal{S}_k$-packing of size $p$ will be denoted by $f(k,p)$. It is obvious that $f(k,p)$ is finite for every $k$ and $p$, for instance $kp \leq f(k,p) \leq k(p + 1)$ is easy to show. Moreover, for small cases, $k = 2,3$, one can see that $f(k,p) = kp$. Below we prove this for $k = 4$. In the other cases we do not know the exact value of $f(k,p)$.

**Lemma 4.5.** The value of $f(4,2)$ equals eight, i.e., any two-edge-coloring of the complete graph with 8 vertices contains two vertex-disjoint monochromatic stars $S_4$.

Proof (sketch). The argument is rather technical and needs a case-by-case analysis, therefore we do not include it here in full detail. A relatively simple way to do it is that we start with a vertex $x$ of largest monochromatic degree, say $d(x)$ in color 1. If $d(x) = 7$, then two disjoint stars obviously exist. If $d(x) = 6$, then $x$ and the vertex $x'$, adjacent to $x$ in color 2, can be chosen as the centers of two disjoint monochromatic stars. If $d(x) = 5$, then the way two monochromatic stars can be found depends on the color of the edge joining the color-2 neighbors of $x$. Finally, if each degree in each color is 3 or 4, then it is convenient to distinguish between four cases, according to the color distribution on the triangle induced by the three vertices joined to $x$ by color 2. Three of those four cases are relatively simple, while the fourth one (where precisely one edge of the triangle has color 1) can be settled by analyzing the subgraph.
$F$ induced by the color-1 neighbors of $x$. The degree assumption (i.e., that each monochromatic degree is 3 or 4) implies that $F$ has minimum degree $\geq 1$ and maximum degree $\leq 2$ in each color; hence, it induces either a path or a cycle with four vertices, or two disjoint edges of color 2. Those cases can be handled by the degree assumption. □

**Proposition 4.6.** Any two-edge-colored complete graph on $n = 4p$ vertices, $p \geq 2$, admits an $\mathcal{A}_4$-packing of size $p$.

**Proof.** Looking at the red and blue degrees of any vertex, we can find a monochromatic star $S_4$ whenever $p \geq 3$. Deleting this $S_4$ and applying the same argument on the vertices not in $S_4$ until the remaining graph has 8 vertices, the proof can be completed by using Lemma 4.5. □

Proposition 4.6 shows that $f(4, p) = 4p$. However, it is not always the case that $f(k, p) = kp$.

**Theorem 4.7.** There exists a two-edge-colored complete graph with at least $8k/3 - o(k)$ vertices in which there is no $\mathcal{A}_4$-packing with two stars; thus, $f(k, 2) > 2.66k$ for sufficiently large $k$.

**Proof.** We prove that almost all 2-colorings of the complete graph with $n$ vertices contain no pair of monochromatic stars when $n$ is less than $8k/3 - f(k)$, for some function $f(k) = o(k)$. This fact will imply the lower bound.

Take a random 2-coloring $C$. For each pair $x, y$ of vertices we consider the four sets $V(i, j), i, j = 1, 2$ defined by the rule that $v$ belongs to $V(i, j)$ if $xv$ has color $i$ and $yu$ has color $j$ in $C$. Assuming that the colors 1 and 2 are taken with probability 1/2 on each edge independently, the random variables $\langle v \text{ is in } V(i, j) \rangle$ are independent (for $x, y$ fixed) and have binomial distribution with probability 1/4. Thus, applying a well-known estimate, we obtain that the size of each $V(i, j)$ is at least $n/4 - cn^{1/2} \log n$ and at most $n/4 + cn^{1/2} \log n$ for some (not very large) $c$, with probability greater than $1 - o(1/n(n - 1))$. Then it has a probability tending to 1 that the sets $V(i, j)$ are that large for all possible choices of the pair $x, y$ (because the “bad” probability is small for each pair, and we have just $n(n - 1)/2$ pairs of vertices).

Suppose now that the coloring contains two monochromatic stars $S_1$ and $S_2$, each of size $k$. Denote by $x, y$ the centers of them, and by $i, j$ their colors. Observe that the union of those two stars has no vertex in $V(3 - i, 3 - j)$. Consequently, they are packed into a set of size $3n/4 + 3cn^{1/2} \log n$. This number should be at least $2k$, a contradiction when $n$ is smaller than $8/3k - f(k)$, for some suitably chosen $f(k) = o(k)$. □
Nevertheless, for every $k$, the value of $f(k,p)$ eventually becomes equal to $kp$ when $p$ is large enough, for instance if $p \geq 4k - 4$. What is more, a similar result can be verified for any number of colors and any size of the stars.

**Theorem 4.8.** For every $k$ and $t$ there is an $n_0 = n_0(k,t)$ such that if $n \geq n_0$ and $n$ is a multiple of $k$, then the vertex set of every $t$-edge-colored complete graph of order $n$ can be partitioned into $n/k$ monochromatic stars $S_k$.

**Proof.** We apply an idea similar to the methods of [7] and [3]. Put $s = \lfloor (k - 2)t/(k - 1) \rfloor + 1$. For large $n$ say $n \geq ((k - 1)(s + t((k - 1)))^{s-1}$, we can (sequentially) choose $s - 1$ vertices $v_1, \ldots, v_{s-1}$ and a set $Y = (k - 1)(s + t(k - 2))$, such that the edges from $v_i$ to $Y$ have the same color, for each $i = 1, \ldots, s - 1$. Let us set $Z = V(G) - (Y \cup \{v_1, \ldots, v_{s-1}\})$. First, we cover all but $t(k - 2) + 1$ vertices of $Z$ by stars $S_k$. The remaining vertices of $Z$ are taken one by one; for each such $z \in Z$ we can find a monochromatic star with center $z$ and the other $k - 1$ vertices in $Y$. Deleting this star from the graph, we repeat the procedure until $Z$ becomes empty. (Even the last step can be executed, since the remaining part of $Y$ has cardinality $s(k - 1) > t(k - 2)$.) Finally, each vertex $v_i$ forms a monochromatic star with any $k - 1$ vertices of $Y$. □

We note that with some much larger bounds on the number of vertices, Theorem 4.8 can be proved in the following stronger form which also generalizes the results of [7]:

For any integers $k$ and $t$, and every sufficiently large multiple $n$ of $k$, every $t$-edge-colored complete graph of order $n$ has a vertex decomposition into monochromatic stars on $k$ vertices such that the endpoints of each star induce a monochromatic complete graph of order $k - 1$.

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**References**
