BROADCASTING IN ONE DIMENSION

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We consider the broadcasting problem for one-dimensional grid graphs with a given neighborhood template. There are two different models that have been considered—shouting (a node informs all of its neighbors in one step) and whispering (a node informs a single neighbor in one step). Let $\sigma(t)$ (respectively $\omega(t)$) denote the maximum number of nodes that can be reached in $t$ steps by shouting (respectively whispering) broadcast from a single source.

We obtain detailed information about the benefits of shouting over whispering. We prove for the one-dimensional case a conjecture by Stout that $\omega(t)$ eventually becomes a polynomial. In particular, we show that there exist constants $i$ and $t_0$ such that $\omega(t) = \sigma(t) - i$ for all $t > t_0$. When the broadcast only goes in one direction (i.e., when all elements of the template are positive), we also determine that $i = d - 1$ and $t_0 \leq 3d$ for a neighborhood template with the furthest neighbor at distance $d$.

1. Introduction

A graph $G = (V, E)$ represents a communications network, so that the set $V$ of vertices corresponds to the members of the network and the set $E$ of edges corresponds to the communication links connecting pairs of members. Various information dissemination processes have been investigated using such a model. One such process is broadcasting. In this process a member of the network (called the originator) has a message to be disseminated to all the other members, as quickly as possible, by a series of calls over the network. This series of calls (called a broadcasting scheme) is constrained by the following rules:

1. Each call requires one unit of time.
2. A member may participate in at most one call per time unit.
3. A member can only call adjacent members.

If each member is allowed to call only one neighbor at a time the process is called whispering and if a member can simultaneously call all of its neighbors the process is called shouting. These two terms are due to Stout [9].

Farley and Hedetniemi were the first to study whispering in finite and infinite
grids [2]. Let \( \omega(n, t) \) be the maximum number of members of an infinite \( n \)-dimensional grid that can be informed (by whispering) of a message from a single vertex in \( t \) time units. Farley and Hedetniemi conjectured that \( \omega(2, t) \) was a certain polynomial in \( t \) of degree 2 and gave a broadcasting scheme achieving this value. Cockayne and Hedetniemi [1] generalized the conjecture to \( \omega(n, t) \) for all \( n \geq 2 \). The conjecture for \( n = 2 \) was proved independently by Ko [6, 7] and Peck [8]. Both Ko and Peck realized that the conjecture was incorrect for \( n \geq 3 \). Ko computed the exact form of \( \omega(3, t) \) which did turn out to be a polynomial in \( t \) of degree 3 for \( t \geq 9 \). Both Ko and Peck gave upper bounds for \( \omega(n, t) \). Peck also gave lower bounds for \( \omega(n, t) \); both upper and lower bounds are polynomials in \( t \) of degree \( n \) with the same coefficients for the two largest powers of \( t \). This gives detailed asymptotic information about \( \omega(n, t) \).

Stout [9] and Klarner [4, 5] considered a more general situation: Given a set of originators in \( \mathbb{Z}^n \) and a set of vectors \( F \), a member \( a \in \mathbb{Z}^n \) is connected to all members \( a + f \) where \( f \in F \). Klarner considered shouting and Stout considered the relationship between shouting and whispering in these generalized grids. In this paper, we consider whispering in these grids, motivated by a conjecture of Stout (cf. below).

A template is a finite set \( F = \{ f_0, f_1, \ldots, f_k \} \subset \mathbb{Z}^n \). The broadcast space of a template is \( S = \langle F \rangle \) where

\[
\langle F \rangle = \left\{ \sum_{i=0}^{k} a_i f_i \mid a_i \in \mathbb{N} \right\}.
\]

Each point \( s \) of \( S \) is a vertex of the graph \( G \) and is connected to vertices \( s + f_0, s + f_1, \ldots, s + f_k \). The message originates at \( s_0 = (0, 0, \ldots, 0) \). The infinite \( n \)-dimensional grid of Cockayne and Hedetniemi [1] uses the template \( F = \{ \pm e_1, \ldots, \pm e_n \} \) where \( e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \). The level of a point \( s \in S \) is

\[
\text{level}(s) = \min \left\{ \sum_{i=0}^{k} a_i \mid \sum_{i=0}^{k} a_i f_i = s \text{ with each } a_i \in \mathbb{N} \right\},
\]

that is, the minimum number of edges (or steps) in a path from \( s_0 \) to \( s \) in \( G \). The \( l \)th layer of \( S \) is \( L_l = \{ s \in S \mid \text{level}(s) = l \} \). We define \( \omega(t) \) to be the maximum number of vertices in \( G \) which can be informed by time \( t \) by any broadcasting scheme using whispering and shouting, respectively. Ko's and Peck's bounds are on \( \omega(t) \) for the Cockayne and Hedetniemi template above. For shouting it is clear that \( \sigma(t) = |\bigcup_{0 \leq i \leq t} L_i| \). Stout [9] proved that \( \omega(t), \sigma(t) \) tends to 1 giving information about the asymptotic benefits of shouting compared to whispering. He also conjectured that, for \( F \subset \mathbb{Z}^n \), \( \omega(t) \) and \( \sigma(t) \) are polynomials of degree \( n \), for \( t \) greater than some \( t_0 \). Klarner [4] proved this conjecture for \( \sigma(t) \) in a more general context. The main result of this paper is that Stout's conjecture holds for \( n = 1 \), i.e., that \( \omega(t) \) is eventually a polynomial when \( F \subset \mathbb{Z} \). It follows from our results that in fact \( \sigma(t) - \omega(t) \) tends to a constant, i.e., that it is a constant for sufficiently large \( t \). We give the exact form of this polynomial in the case \( F \subset \mathbb{N} \) and give examples il-
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Illustrating the difficulty of the general case $F \subseteq \mathbb{N}$. A preliminary version of this work dealing only with the uni-directional case was reported in [3]. We postpone the discussion of higher dimensions to a subsequent paper.

2. Uni-directional broadcasting

We assume here that $F \subseteq \mathbb{N} - \{0\}$ and $S = \langle F \rangle = \mathbb{N}$. In other words, the message originates at 0, the points to be informed are the positive integers, all steps are positive integers and $1 \in F$. (Otherwise, $S = \langle F \rangle \neq \mathbb{N}$.) Hence, let $F$ be fixed, $F = \{f_0, f_1, \ldots, f_k\}$ and $d = f_0 > f_1 > \cdots > f_k = 1$. To avoid trivialities let $k \geq 1$.

A minimal $F$-partition of $s \in S$ is any vector $(a_0, a_1, \ldots, a_k)$ of nonnegative integers such that $\sum_{i=0}^{k} a_i f_i = s$ and $\sum_{i=0}^{k} a_i = \text{level}(s)$.

Lemma 1. In any minimal $F$-partition $(a_0, a_1, \ldots, a_k)$ of $s \in S$, $\sum_{i=1}^{k} a_i < d$.

Proof. Write $s = \sum_{i=0}^{k} a_i f_i$ as $s = a_0 d + s_1 + s_2 + \cdots + s_m$ where, among the $s_i$, there are exactly $a_1$ summands $f_1$, $a_2$ summands $f_2$, etc. so that $\sum_{i=1}^{k} a_i = m$. If $m \geq d$ then, by the pigeon-hole principle, $\sum_{j=1}^{u} s_j \equiv \sum_{j=1}^{u} s_j \pmod{d}$ for some nonnegative integers $0 \leq u < v \leq d$. Thus $\sum_{j=1}^{u+1} s_j$ is a nonzero multiple of $d$ contradicting the fact that $(a_0, a_1, \ldots, a_k)$ is a minimal $F$-partition. \square

Corollary 2. If $\text{level}(s) \geq d$ then $\text{level}(s) = 1 + \text{level}(s - d)$.

Proof. According to Lemma 1, $\text{level}(s) \geq d$ implies that $a_0 > 0$ in any minimal $F$-partition of $s$. Therefore $\text{level}(s - d) \leq \text{level}(s) - 1$. Since the inequality $\text{level}(s) \leq 1 + \text{level}(s - d)$ is trivial, the corollary follows. \square

Theorem 3. For $l \geq 2d$, the $l$th layer $L_l$ consists of exactly $d$ elements, one in each residue class modulo $d$.

Proof. For any fixed $x = 0, 1, \ldots, d - 1$ there is a nonnegative integer $i$ with $\text{level}(id + x) = l$. Indeed, consider

$$m = \min \{ \text{level}(id + x) \mid \text{level}(id + x) \geq l \}$$

and let $m = \text{level}(i_0d + x)$. If $l < m$ then, by Corollary 2,

$$\text{level}((i_0 - 1)d + x) = \text{level}(i_0d + x) - 1 = m - 1 \geq l$$

contradicting the definition of $m$.

On the other hand, if $s_1 \equiv s_2 \pmod{d}$ both have level $l \geq 2d$ and (say) $s_1 < s_2$, then by Lemma 1, $s_1 > d^2$ and hence all $s = s_1 + id$ have level greater then $l$ by Corollary 2. This contradicts our assumption that $\text{level}(s_1) = \text{level}(s_2)$. \square
This result is useful in constructing a broadcasting scheme for $S$. The following is immediate and elaborates on Klarner's result for the one-dimensional case.

**Corollary 4.** For $t \geq 2d$, $\sigma(t) = dt + \sigma_0$ where $\sigma_0 = \sigma(2d) - 2d^2$.

**Proof.** As noted above $\sigma(t) = |\bigcup_{0 \leq l < t} L_l|$. Hence,

$$\sigma(t) = |\bigcup_{0 \leq l < t} L_l| + |\bigcup_{2d + 1 \leq l \leq t} L_l| = \sigma(2d) + (t - 2d)d = dt + (\sigma(2d) - 2d^2).$$

Our main result is:

**Theorem 5.** There is a (whispering) broadcasting scheme which informs, in any time $t \geq 3d$, all elements of $\bigcup_{0 \leq l < t} L_l$ and one element of $L_t$. Moreover, no scheme can inform more points of $\mathbb{N}$ in $t$ time units, for any fixed $t$.

**Remark.** We give one scheme that works for all $t \geq 3d$. However, the negative result says that it cannot be improved, even by a scheme that works only for some particular $t$, $t \geq 3d$.

**Proof of Theorem 5.** Clearly, no broadcasting scheme can inform in time $t$ a point $s \in L_t$ with $l > t$. It is also easy to see that at most one point $s \in L_t$ can be informed since the information must, in order to reach such an $s$ in time $t$, make a step in each of the time intervals $1, 2, \ldots, t$ and the broadcasting rules imply that there can be at most one such path. Thus, a broadcasting scheme which informs one point of $L_t$ and all points of $\bigcup_{0 \leq l < t} L_l$ in time $t$ is an optimal scheme. We describe such a scheme below for $t \geq 3d$.

It is enough to describe our scheme for $t = 3d$. Indeed, if we have a scheme which informs one point of $L_t$ and all points of $\bigcup_{0 \leq l < t} L_l$ in time $t$, we can inform one point of $L_{t+1}$ and all points of $\bigcup_{0 \leq l \leq t} L_l$ in time $t + 1$ by having every point $s$ of level $t$ and the single informed point of level $t$ make a call $[s, s + d]$ at time $t + 1$. (This follows from Corollary 2.) Hence, let $t = 3d$ for the remainder of this proof.

The single informed point of $L_t$ is chosen to be $dt$ and it is informed by the sequence of calls $[(t - 1)d, id]$ at time $i$ for $i = 1, 2, \ldots, t$. This will be referred to as the main path. For each $s \in L_{t-1}$ we choose a fixed minimal $F$-partition and call it the canonical $F$-partition of $s$. Moreover, we choose the canonical $F$-partitions in such a way that if $(a_0, a_1, \ldots, a_k)$ is the canonical $F$-partition of an $s \in L_{t-1}$ with $a_i > 0$ for some $1 \leq i \leq k$ then $(a_0 + 1, a_1, \ldots, a_{i-1}, \ldots, a_k)$ is the canonical $F$-partition of some $s' \in L_{t-1}$. To see that this is possible, we have to show that $s' = s - f_i + d$ has level $t - 1$. Let $\bar{s} = s - f_i = s' - d$. Clearly, $(a_0, a_1, \ldots, a_{i-1}, \ldots, a_k)$ is a minimal $F$-partition of $\bar{s}$; hence level($\bar{s}$) = $t - 2 \geq 2d$. By Lemma 1, $\bar{s} < d^2$ and so $s' > \bar{s} > d^2$.
level($s') > d$. Now Corollary 2 implies that  

$\text{level}(s') = 1 + \text{level}(\bar{s}) = 3d - 1 = t - 1.$

The \textit{canonical rank} of $s \in L_{t-1}$ is $\sum_{i=1}^{k} a_i$ where $(a_0, a_1, \ldots, a_k)$ is the canonical $F$-partition of $s$. (Note that steps of length $d$ are not counted in the canonical rank; in particular, Lemma 1 implies that the canonical rank is less than $d$.)

Each point of $L_{t-1}$ has a canonical rank, and there are, by our choice of canonical $F$-partitions, points of $L_{t-1}$ of canonical ranks $0, 1, \ldots, r$ for some $r \in \mathbb{N}_0$. Let $p_1, p_2, \ldots, p_d$ be an enumeration of the points of $L_{t-1}$ in the order of non-decreasing canonical rank. Let $r_j \equiv p_j \pmod{d}$ and $0 \leq r_j < d$. In particular, $p_1$ has rank 0 and hence $p_1 = (t-1)d$. Thus $p_1$ is already informed at time $t-1$ by the path leading to $td$. Furthermore, we note that every element with $s \equiv 0 \pmod{d}$ (residue class $r_1$) with $\text{level}(s) = t$ is informed by this path. We will first describe how to inform the elements $s$ in the range $(t-d) = 2d^2 \leq s \leq td$ with $\text{level}(s) \leq t-1$ and then show how to inform the remaining elements $s < 2d^2$.

We consider the remaining $p_j$ in order of increasing $j$. Let $(a_0, a_1, \ldots, a_k)$ be the canonical $F$-partition of $p_j$. The point $(t-j)d$ receives the information at time $(t-j)$ and calls $(t-j+1)d$ at time $(t-j+1)$. Beginning at time $(t-j+2)$ we branch off from the main path taking $a_1$ steps $f_1$, $a_2$ steps $f_2$, $\ldots$, $a_k$ steps $f_k$, followed by $a_0 - t + j$ steps $d$ in subsequent time units completing the path to $p_j$ at time $t$. Note that $p_j$ and $a_0 - t + j - 1$ other elements in the residue class $r_j$ are informed by this path. In addition, if the canonical rank of $p_j$ is greater than 1, some members of other residue classes are also informed.

In order to inform other points $s$ of residue class $r_j$ in the range $2d^2 \leq s \leq td$ we can, if necessary, replicate the canonical portion of the path to $p_j$, that is, the $a_1$ steps $f_1$, $a_2$ steps $f_2$, $\ldots$, $a_k$ steps $f_k$ beginning at time $t-j+2$ from each main path element $(t-j-1)d$, $(t-j-2)d$, $\ldots$, $(t-m)d$. Note that $m$ is the largest value such that $(t-m)d + \sum_{i=1}^{k} a_i f_i = 2d^2$ so that $m \leq d + r$ where $r$ is the canonical rank. This assures that all vertices of residue $r_j$ in the range are informed by time $t$.

Since each “branching off” begins at time $t-j+2$, from $(t-j)d$ and each of the vertices $(t-j)d$, $(t-j-1)d$, $\ldots$, $(t-m)d$ is informed and available at this time, there is no conflict at the branching off point. However, as we continue this process for larger $j$ we may find that some calls are being done twice. Although the paths for elements of residue classes whose corresponding canonical rank is 1 cannot possibly interfere with each other, already for a residue class of canonical rank 2 we encounter a problem. The path to $p_j$ from $(t-j)d$ which we wish to build consists of a step of size $f_j$ followed by a step of size $f_j$ followed by some steps of size $d$. However, some residue class $r_i$ may require a call to $(t-j)d + f_j$ to occur at time $t-i+2$. Since $i < j$ we can simply replace the call at time $t-i+2$ with a call at time $t-j+2$, which is at least one time unit earlier, and then call $(t-j)d + f_j + f_{j_2}$ from $(t-j)d + f_j$ at time $t-j+3$ without disturbing any subsequent calls in the path to
the element of class \( r_j \). As \( j \) increases we may find that the path for \( r_j \) overlaps several steps of a path to \( r_i \) in this fashion. By Lemma 1, the only interference between two paths leading to elements of different residue classes may occur as overlapping identical prefixes. By replacing the redundant calls at times \( t - i + 2 \) with the same calls at times \( t - j + 2 \), we can ensure that no duplicate calls occur.

It now remains to inform the elements \( s \) which lie in the range \( 0 < s < 2d^2 \). Ideally we would like to inform every vertex \( s \) in each "block" of elements \((id, (i + 1)d)\) for \( 0 \leq i < 2d - 1 \) by a sequence of \( d - 1 \) calls of step size 1 beginning from \( id \) at time \( i + 2 \). If no conflict is encountered, this will ensure that all vertices in the block are informed by time \( i + 2 - d - 1 < t \). However, due to the calls required to inform elements outside this range, some vertices \( id \) may not be available at time \( i + 2 \) and other elements in the block may be engaged in other calls at the time such a path would reach them.

Let us adopt the following strategy in an attempt to complete the calling scheme. From every informed vertex \( s \) in the range \( 0 < s < 2d^2 \) begin a sequence of calls of step size 1 at the earliest possible time. This sequence will continue until, when trying to call from \( s \) to \( s + 1 \) at time \( j \), we discover that either \( s + 1 \) has been informed at time \( i \leq j \) or \( s + 1 \) is informed by one of the replicated canonical paths at some time \( i > j \). In the first case, the sequence stops at \( s \). In the second case, we can remove the call at time \( i \) and add the call from \( s \) to \( s + 1 \) at time \( j \) and then continue to extend the sequence by attempting to add a call from \( s + 1 \) to \( s + 2 \) at time \( j + 1 \).

Let us apply this strategy to each block of \( d \) elements beginning at \( 2d^2 - d \) and proceeding in decreasing order. For each element \( s \) of residue \( r_j \) in the range \( id \leq s \leq (i + 1)d \) which is not informed after applying the strategy, we can inform this element by replicating the canonical path used for elements of residue \( r_j \) beginning at the appropriate branching off point at time \( t - j + 2 \). As we consider each block in turn, the relative lateness of the calls in this canonical path works in our favor and more elements can be informed by our strategy in the earlier blocks. Certainly we will not inform fewer vertices by this strategy in earlier blocks.

This replication of canonical paths does not require a canonical path starting at an element <0. By our scheme, all replicated paths begin at times greater than \( t - r \geq 3d - r \). From the definition of canonical F-partition, it is clear that all elements of residue class \( r_1 \) have canonical rank 0 and those classes \( r_2, \ldots, r_{k+1} \) have canonical rank 1. Since there is at least one \( r_i \) with each canonical rank \( 2, \ldots, r \), we know that \( 1 + k + (r - 1) \leq d \) so \( t - r \geq 3d - r \geq 2d + k \). At most \( d - 1 \) replicated paths might branch off in the interval \([0, d - 1]\) and these paths do not intersect except for identical prefixes. However, all vertices in \([0, d - 1]\) are informed by time \( d \) and are available at all times \( \geq d + 1 \). Since the earliest branching would occur at some time not before \( 2d + k \), no further replications will be required.

Two corollaries follow from this result:

**Corollary 6.** For \( t \geq 3d \), \( \omega(t) = \sigma(t) - d + 1 \).
This corollary exactly quantifies the difference between shouting and whispering. Thus, the benefit obtained by allowing shouting is to inform \( d - 1 \) additional members in \( t \) units of time for large enough \( t \).

**Corollary 7.** For \( t \geq 3d \), \( \omega(t) = dt + \omega_0 \) where \( \omega_0 = \omega(3d) - 3d^2 \).

**Proof.**

\[
\omega(t) = \left| \bigcup_{0 \leq l \leq t-1} L_l \right| + 1 = \left| \bigcup_{0 \leq l \leq 3d-1} L_l \right| + \left| \bigcup_{3d \leq l \leq t-1} L_l \right| + 1
\]
\[
= \omega(3d) - 1 + (t - 3d)d + 1 = \omega(3d) + dt - 3d^2. \quad \Box
\]

Hence, for \( t \geq 3d \), \( \omega(t) \) is a polynomial of degree one as conjectured by Stout [9].

### 3. One-dimensional broadcasting

In this section, we extend the previous results to the general case of one dimension. Hence, let \( F \subset \mathbb{Z} - \{0\} \); specifically, let \( F = \{f_0, f_1, \ldots, f_p, f_{p+1}, \ldots, f_k\} \) where \( d = f_0 > f_1 > \cdots > f_p > 0 > f_{p+1} > \cdots > f_k = -e \) with \( k, d, e \) positive integers.

**Lemma 8.** Let \((a_0, a_1, \ldots, a_k)\) be a minimal \( F \)-partition of \( s \in S \).

(a) If \( a_0 \geq e \), then \( \sum_{i=1}^{p} a_i < d \), \( \sum_{i=p+1}^{k} a_i < d \), and \( s > 0 \).

(b) If \( a_k \geq d \), then \( \sum_{i=0}^{p} a_i < e \), \( \sum_{i=p+1}^{k} a_i < e \), and \( s < 0 \).

**Proof.** (a) Lemma 1 implies that \( \sum_{i=0}^{p} a_i < d \). Suppose that \( a_0 \geq e \) and \( \sum_{i=p+1}^{k} a_i \geq d \), and write \( \sum_{i=0}^{p} a_i f_i \) as \( s = \sum_{i=0}^{p} a_i f_i + s_1 + s_2 + \cdots + s_m \), where \( a_{p+1} s_i \) equal \( f_{p+1} \), \( a_{p+2} s_i \) equal \( f_{p+2} \), ..., and \( a_k s_i \) equal \( f_k \). As in Lemma 1, the pigeon-hole principle implies that

\[
\sum_{j=p+1}^{u} s_j \equiv \sum_{j=p+1}^{v} s_j \pmod{d} \quad \text{for some } 0 \leq u < v \leq d.
\]

(Recall that \( m = \sum_{i=p+1}^{k} a_i > d \).) Thus, some \( \sum_{i=p+1}^{u} s_j \) is a nonzero multiple of \( d \); as \(-e \leq s_j < 0 \) for all \( j = 1, \ldots, m \), we may write \( \sum_{j=u+1}^{v} s_j = -\alpha d \) with \( 0 < \alpha \leq e \).

Denote by \( \tilde{a} \) the value of \( a_i \) minus the number of \( s_j \) with \( u+1 \leq j \leq v \) and \( s_j = f_i \).

Since \( a_0 \geq e \geq \alpha \), the vector \((a_0 - \alpha, a_1, \ldots, a_p, \tilde{a}_{p+1}, \ldots, \tilde{a}_k)\) is also an \( F \)-partition of \( s \), contradicting the minimality of \((a_0, a_1, \ldots, a_k)\). Now,

\[
s = \sum_{i=0}^{p} a_i f_i + \sum_{i=p+1}^{k} a_i f_i \geq a_0 d + \sum_{i=p+1}^{k} a_i (-e) > ed - de = 0.
\]

The proof of (b) is similar. \( \Box \)
Corollary 9. In any minimal $F$-partition $(a_0, a_1, \ldots, a_k)$ of $s \in S$, $a_0 < e$ or $a_k < d$.

(There is an easier direct proof of this, by observing that $a_0 \geq e$ and $a_k \geq d$ implies that $(a_0 - e, a_1, \ldots, a_k - d)$ is also an $F$-partition of $s$.)

Corollary 10. Let $(a_0, a_1, \ldots, a_k)$ be a minimal $F$-partition of $s \in S$ with $\text{level}(s) > 2(d + e)$. Then either

\[ s > 0 \quad \text{and} \quad \sum_{i=1}^{k} a_i < 2d, \]

or

\[ s < 0 \quad \text{and} \quad \sum_{i=0}^{k-1} a_i < 2e. \]

Proof. If $\text{level}(s) \geq 2(d + e)$ then $a_0 < e$ and $a_k < d$ is impossible, because $\sum_{i=1}^{p} a_i < d$ and $\sum_{i=p+1}^{k} a_i < e$ follow by Lemma 1, and $\text{level}(s) = \sum_{i=0}^{k} a_i$. Thus either (a) or (b) of Lemma 8 must occur. \(\square\)

Corollary 11. If $\text{level}(s) \geq 2(d + e)$ then either

\[ s > 0 \quad \text{and} \quad \text{level}(s) = 1 + \text{level}(s - d) \]

or

\[ s < 0 \quad \text{and} \quad \text{level}(s) = 1 + \text{level}(s + e). \]

Proof. Similar to the proof of Corollary 2. \(\square\)

Theorem 12. For $t \geq 4(d + e)$, the $l$th layer $L_l$ consists of exactly $d + e$ elements, of which $d$ are positive (one in each residue class modulo $d$) and $e$ negative (one in each residue class modulo $e$).

Proof. Similar to the proof of Theorem 3. \(\square\)

Corollary 13. For $t > 4(d + e)$, $\sigma(t) = (d + e)t + \sigma_0$ where $\sigma_0 = \sigma(4d + 4e) - 4(d + e)^2$.

Proof. Similar to the proof of Corollary 4. \(\square\)

As was the case in unidirectional broadcasting, the information on $\sigma(t)$ we obtained above will help us prove that $\omega(t)$ is eventually a polynomial. (At the same time, Theorem 12 and Corollary 13 further the knowledge of $\sigma(t)$, [4,5].)

Theorem 14. There exist constants $t_0$ and $i$, $d + e - 1 \leq i \leq 2(d + e - 1)$ such that:

1. There is a (whispering) broadcasting scheme which informs in time $t$ (for all $t \geq t_0$) all but $i$ points of $\bigcup_{0 \leq s \leq t} L_s$.

2. Every broadcasting scheme for any fixed time $t \geq t_0$ misses at least $i$ points of $\bigcup_{0 \leq s \leq t} L_s$. 

Proof. We shall show below that there exists a $\tau_0$ and a broadcasting scheme that informs in any time $t \geq \tau_0$ all points of $\bigcup_{0 \leq i \leq t-2} L_i$, one point on $L_t$ and at least one point on $L_{t-1}$, thus all but $i \leq 2(d+e-1)$ points of $\bigcup_{0 \leq i \leq t} L_i$. Thus, there exist constant $\tau_0$ and $i$ satisfying (1) (namely $\tau_0$ and $\bar{i}$). Notice that (1) implies that $i \geq d + e - 1$, because no broadcasting scheme can inform more than one point of $L_t$. (Recall that at most one such point can be informed, since in order to reach such a point the information must make a step in each of the time intervals 1, 2, ..., $t$. The broadcasting rules imply that there can be at most one such path.) Let $\tau_0$ and $i$ satisfy (1) with $i$ as small as possible. By the above, $d + e - 1 \leq i \leq 2(d+e-1)$, moreover, (2) is satisfied by the choice of $i$. We observe that any given broadcasting scheme for large enough time $t$ which misses $i$ points of $\bigcup_{0 \leq i \leq t} L_i$ can be extended, by making calls $[s, s + d]$ or $[s, s - e]$, to a scheme for time $t + i$, which misses at most $i$ points of $\bigcup_{0 \leq i \leq t+1} L_i$.

It remains to show that some scheme informs all of $\bigcup_{0 \leq i \leq t-2} L_i$ and two other points, for some sufficiently large $t$. We make the calls $[(i-1)d, id]$ at times $i = 1, 2, ..., t$, and the calls $[-(i-2)e, -(i-1)e]$ at times $i = 2, 3, ..., t$—obtaining a primary positive main path and a secondary negative main path. Note that $dt \in L_t$ and $-e(t-1) \in L_{t-1}$ are informed by these paths. Let $q = 2(d+e)^2 - (d+e) - 1$. All positive elements of $\bigcup_{t-q \leq i \leq t-1} L_i$ are informed as in the proof of Theorem 5. (Note that this part of the proof of Theorem 5 does not require $1 \in F$.) That is, calls are added to the scheme to inform the elements of $L_{t-1}$ in order of nondecreasing canonical rank and the canonical portion of each path is replicated to inform other points of the same residue class on the lower levels. This technique is used to inform all positive elements of levels $L_t$, $t-q \leq i \leq t-1$. All negative elements of levels $L_t$, $t-q \leq i \leq t-2$, are informed similarly. (There is a delay of one time unit, as the secondary main path begins at $i = 2$.)

Before describing how to complete calling paths to the remaining members of levels $L_t$, $1 \leq i < t-q$, we observe that there is a calling scheme to inform all elements of $\bigcup_{0 \leq i < p} L_i$ by time $p + k - 1$. We can begin such a scheme with the calls $[(i-1)d, id]$ at time $i = 1, 2, ..., p - 1$. Other calls can be added as necessary to complete paths to the other elements. It is easy to complete such a scheme which requires at most $p + k - 1$ time units by considering the other elements in decreasing lexicographic order of their canonical F-partitions. We will call such a scheme a filling scheme. (This is more complicated than the corresponding part of the proof of Theorem 5 since that proof relied on the fact that $1 \in F$ which is not assumed here.)

To complete the calling paths to the remaining members of levels $L_t$, $1 \leq i < t-q$, note that each such element $x$ is reachable from some member of either the positive or negative main path by fewer than $2(d+e)$ calls. This follows from Corollary 10 if $\text{level}(x) \geq 2(d+e)$ and from the definition of level otherwise. Consider the elements which can be so reached from a given member of the main path. We will attempt to reach these elements using a filling scheme as described above beginning as early as possible from the appropriate element of the main path. In particular, to inform the remaining elements of levels $L_t$, $1 \leq i < t-q$, we begin filling schemes
with \( p = 2(d+e) \) from each vertex \( id, 0 \leq i < t - q - 1 \), at time \( i+1 \) and from each vertex \( -ie, 1 \leq i < t - q - 1 \) at time \( i+2 \). Thus, all of these filling schemes begin by time \( t - q \). Conflicts between these calls and existing calls can be resolved by postponing the new calls as necessary. Since each element on each path of the filling scheme needs exactly one incoming call and at most \( k+1 \) outgoing calls, each path in the filling scheme can be delayed no more than \( k \) time units at each vertex on the path. Recalling that \( k + 1 \leq d+e \), we see that each filling scheme will be completed no more than

\[
(p+k-1) + (p-1)k \\
\leq [2(d+e) - 1] + [(d+e) - 1] - 1 + [2(d+e) - 2][(d+e) - 1] \\
= 2(d+e)^2 - (d+e) - 1
\]
time units after it begins. Thus, all of the filling schemes will be completed by time \( t - q + 2(d+e)^2 - (d+e) - 1 = t \) and the result follows. \( \square \)

**Corollary 15.** There exist constants \( t_0 \) and \( i \), \( d+e - 1 \leq i \leq 2(d+e - 1) \), such that \( \omega(t) = \sigma(t) - i \) for all \( t \geq t_0 \). Hence, \( \omega(t) \) is a polynomial for \( t \geq t_0 \).

It should be suspected by the reader that the value of \( i \) in Theorem 14 can be much smaller than \( 2(d+e - 1) \), in that the above construction which does the positive and the negative sides separately can be improved by combining them. Indeed, \( i = d+e - 1 \) when \( |F| = 2 \): Given \( F = \{d,-e\} \) with \( d,e > 0 \), there is a (whispering) broadcasting scheme that informs for any time \( t \) all elements of \( \bigcup_{0 \leq i < t} L_i \) and one element of \( L_i \). Moreover, no scheme can inform more points in time \( t \) for any fixed \( t \).

**Proof.** The single informed point of \( L_i \) is chosen to be \( d \) and it is informed by the sequence of calls \( [(i-1)d, id] \) at time \( i \) for \( i = 1, 2, \ldots, t \). At times \( i = 2, 3, \ldots, t \), make the calls \( [(i-2)d, (i-2)d - e] \). For each point \( s \) which receives a call from \( s+e \) at time \( i < t \), make the call \( [s, s - e] \) at time \( i + 1 \) unless \( s - e \) has been informed previously. It is easy to verify that every point of \( \bigcup_{0 \leq i < t} L_i \) and one point of \( L_i \) is informed by this scheme. \( \square \)

On the other hand, it is easy to construct examples for which \( i > d+e - 1 \):

**Example 16.** Given \( F = \{3, 1, -3\} \), no (whispering) broadcasting scheme can inform all points of \( \bigcup_{0 \leq i < t} L_i \) and one point of \( L_i \) in time \( t \) for any fixed \( t \geq 3 \).

We consider three cases depending on which of the three calls \( ([0, 3]), [0, 1], [0, -3] \) is made at time \( 1 \).

Suppose that the call \( [0, 3] \) is made at time \( 1 \). Consider the points \( -3(t-1) \) and \( -3(t-2)+1 \). Note that it is not possible to inform either of these points in \( t-1 \) steps if the calling path includes the call \( [0, 3] \). Therefore, the calling path reaching each of these points must consist of calls made at times \( 2, 3, \ldots, t \). Thus, at most one of these points can be informed by time \( t \) if the call \( [0, 3] \) is made at time \( 1 \).
Similarly, if the call [0, -3] is made at time 1, it is possible to inform at most one of the points 3(t − 1) and 3(t − 2) + 1 by time t. If the call [0, 1] is made at time 1, it is possible to reach at most one of the points 3(t − 1) and −3(t − 1) by time t. In this case, the two informing paths conflict only at point 0.

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References