Krausz dimension and its generalizations in special graph classes

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A krausz \((k, m)\)-partition of a graph \(G\) is the partition of \(G\) into cliques, such that any vertex belongs to at most \(k\) cliques and any two cliques have at most \(m\) vertices in common. The \(m\)-krausz dimension \(kdim_m(G)\) of the graph \(G\) is the minimum number \(k\) such that \(G\) has a krausz \((k, m)\)-partition. 1-krausz dimension is known and studied krausz dimension of graph \(kdim(G)\).

In this paper we prove, that the problem \(^{*}\)\(\text{kdim}(G) \leq 3\)^{*} is polynomially solvable for chordal graphs, thus partially solving the problem of P. Hlineny and J. Kratochvil. We show, that the problem of finding \(m\)-krausz dimension is NP-hard for every \(m \geq 1\), even if restricted to \((1,2)\)-colorable graphs, but the problem \(^{*}\)\(\text{kdim}_m(G) \leq k\)^{*} is polynomially solvable for \((\infty, 1)\)-polar graphs for every fixed \(k, m \geq 1\).

**Keywords:** Krausz dimension, intersection graphs, linear \(k\)-uniform hypergraphs, chordal graphs, polar graphs

1 Introduction

In this paper we consider finite undirected graphs without loops and multiple edges. The vertex and the edge sets of a graph (hypergraph) \(G\) are denoted by \(V(G)\) and \(E(G)\) respectively. \(N(v) = N_G(v)\) is the neighborhood of a vertex \(v\) in \(G\) and \(deg(v)\) is the degree of \(v\). Let \(G(X)\) denote the subgraph of \(G\) induced by a set \(X \subseteq V(G)\) and \(ecc_G(v)\) is the eccentricity of a vertex \(v \in V(G)\).

A krausz partition of a graph \(G\) is the partition of \(G\) into cliques (called clusters of the partition), such that every edge of \(G\) belongs to exactly one cluster. If every vertex of \(G\) belongs to at most \(k\) clusters then the partition is called krausz \(k\)-partition. The krausz dimension \(kdim(G)\) of the graph \(G\) is a minimal \(k\) such that \(G\) has krausz \(k\)-partition.

Krausz \(k\)-partitions are closely connected with the representation of a graph as an intersection graph of a hypergraph. The intersection graph \(L(H)\) of a hypergraph \(H = (V(H), E(H))\) is defined as follows:

1) the vertices of \(L(H)\) are in a bijective correspondence with the edges of \(H\);

2) two vertices are adjacent in \(L(H)\) if and only if the corresponding edges have a nonempty intersection.

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Hypergraph $H$ is called linear, if any two of its edges have at most one common vertex; $k$-uniform, if every edge contains $k$ vertices.

The multiplicity of the pair of vertices $u, v$ of the hypergraph $H$ is the number $m(u, v) = |\{E \in E(H) : u, v \in E\}|$; the multiplicity $m(H)$ of the hypergraph $H$ is the maximum multiplicity of the pairs of its vertices. So, linear hypergraphs are the hypergraphs with the multiplicity 1.

Denote by $H^*$ the dual hypergraph of $H$ and by $H^{[2]}$ the 2-section of $H$ (i.e. the simple graph obtained from $H$ by transformation each edge into a clique). It follows immediately from the definition that

$$L(H) = (H^*)^{[2]}$$ (first this relation was implicitly formulated by C. Berge in [3]). This relation implies that a graph $G$ has krausz $k$-partition if and only if it is intersection graph of linear $k$-uniform hypergraph.

A graph is called $(p, q)$-colorable [4], if its vertex set could be partitioned into $p$ cliques and $q$ stable sets. In this terms $(1, 1)$-colorable graphs are well-known split graphs.

Another generalization of split graphs are polar graphs (see [5], [20]). A graph $G$ is called polar if there exists a partition of its vertex set $V(G) = A \cup B$, $A \cap B = \emptyset$ (2) (bipartition $(A, B)$) such that all connected components of the graphs $\overline{G}(A)$ and $G(B)$ are complete graphs. If, in addition, $\alpha$ and $\beta$ are fixed integers, and the orders of connected components of the graphs $\overline{G}(A)$ and $G(B)$ are at most $\alpha$ and $\beta$ respectively, then the polar graph $G$ with bipartition $(\alpha, \beta)$ is called $(\alpha, \beta)$-polar. Given a polar graph $G$ with bipartition $(\alpha, \beta)$, if its order of connected components of the graph $\overline{G}(A)$ (the graph $G(B)$) is not restricted above, then the parameter $\alpha$ (respectively $\beta$) is supposed to be equal $\infty$. Thus an arbitrary polar graph is $(\infty, \infty)$-polar, and a split graph is $(1, 1)$-polar.

Denote by $KDIM(k)$ the problem of determining whether $kdim(G) \leq k$ and by $KDIM$ the problem of finding the krausz dimension.

The class of line graphs (intersection graphs of linear 2-uniform hypergraphs, i.e graphs with krausz dimension at most 2) have been studied for a long time. It is characterized by a finite list of forbidden induced subgraphs [1], efficient algorithms for recognizing it (i.e. solving the problem $KDIM(2)$) and constructing the corresponding krausz 2-partition are also known (see for example [3], [11], [17], [18]).

The situation changes radically if one takes $k = 3$ instead of $k = 2$ : the problem $KDIM(k)$ is NP-complete for every fixed $k \geq 3$ [8]. The case $k = 3$ was studied in the different papers (see [5], [14], [15], [16], [19]), and several graph classes, where the problem $KDIM(3)$ is polynomially solvable or remains NP-complete, were found.

In [8] P. Hlineny and J. Kratochvil studied the computational complexity of the krausz dimension in detail. Besides another results, the following results were obtained in their paper:

1) The problem $KDIM$ is polynomially solvable for graphs with bounded treewidth. In particular, it is polynomially solvable for chordal graphs with bounded clique size.

2) For the whole class of chordal graphs the problem $KDIM(k)$ is NP-complete for every $k \geq 6$.

So, the problem of deciding the complexity of $KDIM(k)$ restricted to chordal graphs for $k = 3, 4, 5$ was posed by P. Hlineny and J. Kratochvil. As a partial answer to it, in the Section 5 we prove that the problem $KDIM(3)$ is polynomially solvable in the class of chordal graphs.
Krausz dimension and its generalizations in special graph classes

In the Section 3 we consider the natural generalization of the krausz dimension. The krausz \((k, m)\)-partition of a graph \(G\) is the partition of \(G\) into cliques (called clusters of the partition), such that any vertex belongs to at most \(k\) clusters of the partition, and any two clusters have at most \(m\) vertices in common. As above, the relation (1) implies the following statement:

**Proposition 1** A graph \(G\) has krausz \((k, m)\)-partition if and only if it is the intersection graph of a \(k\)-uniform hypergraph with the multiplicity at most \(m\).

The \(m\)-krausz dimension \(kdim_m(G)\) of the graph \(G\) is the minimum \(k\) such that \(G\) has a krausz \((k, m)\)-partition. The krausz dimension in this terms is the \(1\)-krausz dimension.

Denote by \(KDIM_m\) the problem of determining the \(m\)-krausz dimension of graph, by \(KDIM_m(k)\) the problem of determining whether \(kdim_m(G) \leq k\) and by \(L^m_k\) the class of graphs with a krausz \((k, m)\)-partition. It was proved in [10] that the class \(L^m_k\) could not be characterized by a finite set of forbidden induced subgraphs for every \(m \geq 2\), but the complexity of the problem \(KDIM_m\) for an arbitrary \(m\) was not established yet. We prove that the problem \(KDIM_m\) is NP-hard for every \(m \geq 1\), even if restricted to the class of \((1, 2)\)-colorable graphs.

The class \(L^m_k\) is hereditary (i.e. closed with respect to deleting the vertices) and therefore can be characterized in terms of forbidden induced subgraphs. We prove that for every fixed integers \(m, k\) such finite characterization of the class exists when restricted to \((\infty, 1)\)-polar graphs. In particular, it follows that the problem \(KDIM_m(k)\) is polynomially solvable for \((\infty, 1)\)-polar graphs for every fixed \(m, k\). In particular, it generalizes the result of [8] and [12], that for every fixed \(k\) the problem \(KDIM(k)\) is polynomially solvable for split graphs.

2 Krausz 3-partitions of chordal graphs

Let \(F\) be a family of cliques of graph \(G\). The cliques from \(F\) are called clusters of \(F\). Denote by \(l_F(v)\) the number of clusters from \(F\) covering the vertex \(v\).

A maximal clique with at least \(k^2 - k + 2\) vertices is called a \(k\)-large clique. For such cliques the following statement holds:

**Lemma 2** [8, 9, 16] Any \(k\)-large clique of a graph \(G\) belongs to every krausz \(k\)-partition of \(G\).

Further in this section 3-large clique will be called simply large clique.

Let \(G\) be a graph with \(kdim(G) \leq 3\) and \(Q\) be some its krausz 3-partition. Any subset \(F \subseteq Q\) is called a fragment of the krausz 3-partition \(Q\) (or simply a fragment).

Let \(F\) be some fragment of krausz 3-partition \(Q\) and \(H\) be the subgraph of \(G\) obtained by deleting edges covered by \(F\) (\(F\) could be empty). Fix some vertex \(a \in V(H)\) and positive integer \(k\). Denote by \(B_k[a]\) the \(k\)th neighborhood of \(a\) in \(H\), i.e. the set of vertices at distance at most \(k\) from \(a\). A family of cliques \(F_k(a)\) in \(H\) is called \((a, k)\)-local fragment (or simply a local fragment), if

1. every edge with at least one end in \(B_k[a]\) is covered by some cluster of \(F_k(a)\);
2. every vertex \(v \in B_k[a]\) belongs to at most \(3 - l_F(v)\) clusters of \(F_k(a)\).
3. every two clusters of \(F_k(a)\) have at most one common vertex.

A clique \(C\) is called special, if \(C\) is a cluster of every \((a, k)\)-local fragment for some \(a\) and \(k\). In particular, by Lemma 2 large cliques are special.

The following statements are evident.
Lemma 3  
1) If $\deg(v) \geq 19$ for some vertex $v \in V(G)$, then $v$ is contained in some large clique.

2) If $l_F(v) = 2$, then $C = N_H(v) \cup \{v\}$ is a special clique.

3) If $v \in B_k[a]$ is adjacent to at least $4 - l_F(v)$ vertices of the cluster $C$ of some local fragment $F_k(a)$, then $v \in C$.

4) for every $a \in V(H)$ and every $k$ there exists at least one $(a, k)$-local fragment;

5) If the clique $C$ is special, then $F \cup \{C\}$ is a fragment.

Proof: Let’s illustrate, for example, 3) and 5). If $v \in B_k[a]$ is adjacent to vertices $v_1, \ldots, v_{4-l_F(v)} \in C \in F_k(a)$, but $v \not\in C$, then the edges $vv_1, \ldots, vv_{4-l_F(v)}$ should be covered by different clusters of $F_k(a)$. It contradicts (2).

The family of cliques $X = \{C \in Q \setminus F : C \cap B_k[a] \not= \emptyset\}$ is a local fragment. Since $C$ is special, $C \in X$ and therefore $C \in Q \setminus F$.

Denote by $lc(H)$ the length of a longest induced cycle of the graph $H$.

Lemma 4 Let $G$ be a chordal graph with $kdim(G) \leq 3$. Let further there are no special cliques in $H$. Then $lc(H) \leq 6$.

Proof: Suppose contrary, i.e. let $a_1, \ldots, a_k$ form the induced cycle $S \cong C_k$ in $H$, $k \geq 7$, $a_i a_{i+1} \in E(H)$, indices are taken modulo $k$.

Since for every $a_i$ there are two nonadjacent neighbors in $H$, then in every local fragment with center in $a_i$ it is covered by at least 2 clusters. It implies $l_F(a_i) \leq 1$ for every $i = 1, \ldots, k$. As $G$ is a chordal graph, there exist chords of the cycle $S$ covered by the fragment $F$. It is easy to see, that for every two consecutive vertices $a_i, a_{i+1}$ of $S$ at least one of them belongs to some chord of $S$ (indices are taken modulo $k$). Indeed, let without lost of generality $a_i = a_k, a_{i+1} = a_1$. If our statement is not true, then one can choose the chord $a_p a_q, 1 < p < q < k$ such, that $(p - 1) + (k - q)$ is minimal. But then $G(a_1, \ldots, a_p, a_q, \ldots, a_k)$ is a chordless cycle.

Assume without lost of generality, that one of chords of $S$ contains $a_1$. As $l_F(a_i) \leq 1$, for every vertex $a_i$ chords incident to this vertex are covered by exactly one cluster of $F$. It implies that there are no pairs of chords of the form $\{a_i a_j, a_j a_{j+1}\}$, since in this case the vertices $a_i, a_j, a_{j+1}$ are covered by one cluster of $F$ and thus the edge $a_j a_{j+1}$ should be covered by $F$.

Let us show, that all chords of $S$ are covered by the cluster $C_{chord} \supseteq \{a_1, a_3, \ldots, a_{k-1}\}$ (and thus $k$ is even). Indeed, suppose that some chords of $S$ are covered by the cluster $C \supseteq \{a_{i_1}, \ldots, a_{i_r}\}, i_1 < i_2 < \ldots < i_r, i_1 = 1, C \not= C_{chord}$. Then there exist $1 \leq p < q \leq r$ such, that $q - p \geq 3$. So, $G(a_{i_p}, a_{i_p+1}, \ldots, a_{i_q-1}, a_{i_q})$ is a cycle of length at least 4, where without lost of generality $a_{i_p}$ belongs to some chord. That chord should be covered by a cluster $C' \in F, C' \not= C$. So, we have $l_F(a_{i_p}) \geq 2$, the contradiction.

In particular, this proposition implies that for any odd $i$ and even $j$ such that $a_i a_j$ is not the edge of $S$, the vertices $a_i$ and $a_j$ are nonadjacent in $G$ (otherwise $l_F(a_i) \geq 2$).

Let us denote by $C(v_1, \ldots, v_r)$ the clique containing vertices $v_1, \ldots, v_r$.

Let $e = \min\{ecc_H(a_1), 5\}$. Since there is no special cliques in $H$ there exists two different local fragments $F_e(a_1) \supseteq \{C(a_1, a_2), C(a_1, a_k)\}$ and $F_e(a_1) \supseteq \{C'(a_1, a_2), C'(a_1, a_k)\}$, such that without lost of generality $C(a_1, a_2) \setminus C'(a_1, a_2) \not= \emptyset$.

Let $v \in C(a_1, a_2) \setminus C'(a_1, a_2)$. Then $v \in C'(a_1, a_k)$ and therefore $a_1 v, a_2 v, a_k v \in E(H)$.
The vertices $a_2, v, a_k, a_k - 1, a_3$ form a cycle in $G$. It should have at least 2 chords. Since $a_2a_k - 1, a_3a_k \notin E(G)$, there are edges $va_3, va_k - 1 \in E(G)$. The edges $va_3, va_k - 1$ are not covered by $F$ (otherwise $v \in C_{chord}$ and thus $\{a_1, v\} \in C_{chord} \cap C(a_1, a_2)$ and hence $va_3, va_k - 1 \in E(H)$. It implies, that $v \in C(a_3, a_4) \in F_e(a_1)$. So, $va_4 \in E(H)$ (see Figure 1). Note, that since $k \geq 7$ we have $a_4a_k - 1 \notin E(H)$.

Let us remind, that in the local fragment $F'_e(a_1)$ the vertex $v$ is covered by the cluster $C'(v, a_1, a_k)$. So, all other edges of $H$ incident to $v$, should be covered by at most two clusters of $F_e(a_1)$. But it is impossible, since the vertices $a_2, a_4, a_k - 1$ are pairwise nonadjacent. This contradiction proves Lemma 4.

The considerations above suggest the following algorithm which reduces the problem of recognition chordal graphs with Krausz dimension at most 3 to the same problem for graphs with bounded maximum degree and maximum induced cycle length.

**Algorithm 1**

**Input:** chordal graph $G$.

**Output:** One of the following:
1) graph $H$ with $\Delta(H) \leq 18$ and $lc(H) \leq 6$ such that $kdim(G) \leq 3$ if and only if $kdim(H) \leq 3$;
2) the answer "$kdim(G) > 3$".

**begin**

$F := \emptyset$; $H := G$; $isContinue := \text{true}$;

**while** ($isContinue = \text{true}$)

**if** there exists a vertex $v \in V(H)$ such that $l_F(v) = 2$

$C := N(v) \cup \{v\}$;

**if** $C$ is a clique

$F := F \cup \{C\}$; **continue** to the next iteration of the cycle;

**else** the answer is "$kdim(G) > 3$"; **stop**;

**if** there exists a vertex $v \in V(H)$ with $\deg(v) \geq 19$

**if** $v$ is contained in a clique $C$ with $|C| \geq 8$

extend $C$ to a maximal clique; $F := F \cup \{C\}$;

**continue** to the next iteration of the cycle;

**else** the answer is "$kdim(G) > 3$"; **stop**;

**For** every non-isolated vertex $v \in V(H)$ **generate** all $(v, e)$-local fragments, $e = \min\{ecc_H(v), 5\}$;
if there exists a vertex \( v \in V(H) \) such that there is no \((v, e)\)-local fragments
the answer is ”\( kdim(G) > 3 \)”; stop;
if there exists a special clique \( C \)
\[
F := F \cup \{C\}; \quad \text{continue to the next iteration of the cycle;}
\]
isContinue := false
endwhile;
add a pendant edge \( v p_v \) to every vertex \( v \in V(H) \) with \( l_F(v) = 1 \);
end.

**Theorem 5** [3] Let \( lc(H) \leq s + 2, \Delta(H) \leq \Delta \). Then \( \text{treewidth}(H) \leq \Delta(\Delta - 1)^{s-1} \).

**Theorem 6** The problem \( KDIM(3) \) is polynomially solvable for chordal graph.

**Proof:** The correctness of algorithm 1 follows from the considerations above. Let us show, that the Algorithm 1 is polynomial. Indeed, the procedure of finding large clique which contains the fixed vertex \( v \in V(H) \) has the complexity \( O(m) \). We start to generate all possible \((v, e)\)-local fragments for a vertex \( v \in V(H) \) only then \( \text{deg}(v) \leq 18 \). It implies \( |B_e[v]| \leq \text{const} \) and thus the complexity of this procedure is constant. The outer loop of the algorithm 1 is performed at most \( m \) times.

After performing the Algorithm 1 we obtained the graph \( H \) with bounded maximum degree and the length of a longest induced cycle. By Theorem 5 \( H \) has bounded treewidth. For such a graph the problem of determining its krausz dimension is polynomially solvable [3]. \( \square \)

### 3 m-krausz dimension of graphs

We will start with proving the NP-hardness of the problem \( KDIM_m \). In order to make the proof more clear, we firstly will prove, that \( KDIM_m \) is NP-hard for general graphs, and then we will use the developed construction to prove, that \( KDIM_m \) is NP-hard for \((1, 2)\)-colorable graphs.

**Theorem 7** The problem \( KDIM_m \) is NP-hard for every fixed \( m \geq 1 \).

**Proof:** Let us reduce to the problem \( KDIM_m \) the following special case of the 3-dimensional matching problem (which we will call the problem A):

Given: Non-intersecting sets \( X, Y, Z \), such that \( |X| = |Y| = |Z| = q \); \( M \subseteq X \times Y \times Z \), such that the following condition holds:

(*) if \((a, b, w), (a, x, c), (y, b, c) \in M\), then \((a, b, c) \in M\).

The question: Does \( M \) contain a subset \( M' \subseteq M \) (3-dimensional matching) such, that \( |M'| = q \) and every two elements of \( M' \) have not common coordinates?

It is known, that the problem A is NP-complete [7]. Let \( X, Y, Z, M, |X| = |Y| = |Z| = q \), be the input of the problem A. Let us reduce the problem A to the problem of determining, if \( kdim_m(G) \leq 2q \). Construct the graph \( G \) as follows:

\[
V(G) = X \cup Y \cup Z \cup \{v, v_1, \ldots, v_q\};
\]

\[
E(G) = \bigcup_{(a, b, c) \in M} \{ab, bc, ac\} \cup \{v_i v : i = 1, \ldots, q\} \cup \{vd : d \in X \cup Y \cup Z\}
\]
we have $p \cup X$ 

Let us show, that the problem A could be reduced to the problem $KDIM_m$. Then $Q = Q_1 \cup Q_2 \cup Q_3$ isundra (2q, m)-partition of $G$, since $\text{deg}(u) \leq 2q$ for every vertex $u \in V(G) \setminus \{v\}$ and the vertex $v$ is covered by exactly $2q$ clusters of $Q$.

Let now $Q$ be kundra (2q, m)-partition of $G$. Denote by $Q(v)$ the set of clusters of $Q$, which contain the vertex $v$. Since the vertices $v_i$, $i = 1, \ldots, q$, have degree 1, there exist q clusters from $Q(v)$ of the form $vv_i$, $i = 1, \ldots, q$. Let $C_1, \ldots, C_p$ be the remaining clusters from $Q(v)$, $p \leq q$. Then $(C_1 \cup \ldots \cup C_p) \setminus \{v\} = X \cup Y \cup Z$. Since $X, Y, Z$ are stable sets of $G$, we have $|C_i| \leq 4$, $i = 1, \ldots, p$. As $|X \cup Y \cup Z| = 3q$, we have $p = q$, $|C_i| = 4$, $C_i \cap C_j = \{v\}$, $i, j = 1, \ldots, p, i \neq j$.

Let $C_i = \{a_i, b_i, c_i, v : a_i \in X, b_i \in Y, c_i \in Z\}$, $i = 1, \ldots, q$. The property (*) implies, that $M' = \{(a_i, b_i, c_i) : i = 1, \ldots, q\} \subseteq M$ and, by the consideration above, $M'$ is the 3-dimensional matching.

**Corollary 8** The problem $KDIM_m$ is NP-hard in the class of (1, 2)-colorable graphs for every fixed $m \geq 1$.

**Proof:** Let us show, that the problem A could be reduced to the problem $KDIM_m$ in the class of (1, 2)-colorable graphs.

Let $G$ be the graph constructed in the proof of Theorem 8. Let us construct the graph $G'$ as follows:

$V(G') = V(G) \cup V'_1 \cup V'_2$, where

$$V'_1 = \{w, w_1, \ldots, w_{2q}\};$$

$$V'_2 = \{f_u : u \in V(G) \setminus (X \cup \{v_1, \ldots, v_q\}\};$$

$E(G') = E(G) \cup E'_1 \cup E'_2 \cup E'_3 \cup E'_4$, where

$$E'_1 = \{ww_i : i = 1, \ldots, 2q\};$$

(see Figure 2). Let us show that $M$ contains the 3-dimensional matching $M'$ if and only if there exists a krausz (2q, m)-partition of $G$.

![Figure 2](image-url)
\[ E'_2 = \{wx : x \in X\}; \quad (8) \]
\[ E'_3 = \{x_1x_2 : x_1, x_2 \in X, x_1 \neq x_2\}; \quad (9) \]
\[ E'_4 = \{uf_u : u \in V(G) \setminus (X \cup \{v_1, \ldots, v_q\})\} \quad (10) \]

(see Figure 3). The set \( X \cup \{v\} \) is a clique, and the sets \( Y \cup \{f_z : z \in Z\} \cup \{v_1, \ldots, v_q, f_v, w\} \) and \( Z \cup \{f_y : y \in Y\} \cup \{w_1, \ldots, w_{2q}\} \) are stable sets of \( G' \). So, \( G' \) is \((1,2)\)-colorable graph.

![Figure 3:](image)

It is evident, that \( Q \) is the krausz \((2q, m)\)-partition of \( G \) if and only if

\[ Q \cup \{X \cup \{w\}\} \cup \{\{ww_i\} : i = 1, \ldots, 2q\} \cup \{\{u, f_u\} : u \in V(G) \setminus (X \cup \{v_1, \ldots, v_q\})\} \quad (11) \]

is the krausz \((2q + 1, m)\)-partition of \( G' \). \hfill \Box

Now we turn to the complexity of the recognition problem \( KDIM_m(k) \) in the class of \((\infty,1)\)-polar graphs.

A maximal clique with at least \( f(k, m) = m(k^2 - k + 1) + 1 \) vertices is called a \((k,m)\)-large clique.

In [13] the following two statements were proved. Since they were published only in Russian in a journal, which is difficult of access for a general reader, we repeat their proofs here.

**Theorem 9** Any \((k,m)\)-large clique \( C \) of a graph \( G \) belongs to every krausz \((k, m)\)-partition of \( G \).

**Proof:** Let \( A \) be a krausz \((k,m)\)-partition of graph \( G \), \( A_1, A_2, \ldots, A_t \) be those clusters of \( A \) which have common vertices with \( C \). Assume that \( C \not\in A \). Then the family \( B = (B_1, B_2, \ldots, B_t) \), where
\( B_i = A_i \cap C \) is a krausz \((k, m)\)-partition of the graph \( G(C) \), and (by maximality of \( C \)) \( B_i \neq C \) for every \( i = 1, 2, \ldots, t \).

Let us show, that \( |B_i| \leq mk \) for any \( i = 1, 2, \ldots, t \). Consider a cluster of \( B \), say \( B_1 \), and a vertex \( u \in C \setminus B_1 \). No edge of the form \( ux \), where \( x \in B_1 \), is contained in \( B_1 \). Moreover, each cluster of \( B \) different from \( B_1 \) contains at most \( m \) of such edges (by the definition of krausz \((k, m)\)-partition). Taking into account that the vertex \( u \) belongs to at most \( k \) clusters of \( B \), we obtain the inequality \( |B_1| \leq mk \).

Now we will prove that if \( B_1 \setminus B_j \neq \emptyset \) for some clusters \( B_j \in B \), then \( |B_j \setminus B_i| \leq m(k-1) \). Consider a vertex \( u \in B_1 \setminus B_j \). Any edge of the form \( ux \), where \( x \in B_j \setminus B_i \) (if such one exists) is contained neither in \( B_1 \), nor in \( B_j \). Besides, no cluster of \( B \) contains more than \( m \) of such edges by definition of krausz \((k, m)\)-partition. Taking into account that \( u \) belongs to at most \( k-1 \) clusters of \( B \) different from \( B_i \), we obtain the inequality \( |B_j \setminus B_i| \leq m(k-1) \).

Consider an arbitrary vertex \( v \) of the clique \( C \). Let, without loss of generality, it belongs to the clusters \( B_1, B_2, \ldots, B_s \) of \( B \), \( s \leq t \). We show that \( |B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1) \). The following equality is obvious

\[
|B_1 \cup B_2 \cup \ldots \cup B_s| = |B_1| + |B_2 \setminus B_1| + |B_3 \setminus (B_1 \cup B_2)| + \ldots + |B_s \setminus (B_1 \cup B_2 \ldots \cup B_{s-1})|.
\] (12)

If \( B_1 \setminus B_2 \neq \emptyset \), \( B_2 \setminus B_3 \neq \emptyset \), \ldots, \( B_s \setminus B_{s-1} \neq \emptyset \), then by proved above each term in the right part of the equality (12), starting from the second, does not exceed \( m(k-1) \). Hence we have \( |B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1) \). Let, on the contrary, \( i \in \{2, \ldots, s\} \) is the maximal number such, that \( (B_1 \cup \ldots \cup B_{i-1}) \setminus B_i = \emptyset \). Then \( B_1 \subseteq B_i \), \( B_2 \subseteq B_i \), \ldots, \( B_{i-1} \subseteq B_i \), and the sum of the \( i \) terms in the right part of (12) is equal to \( |B_1 \cup B_2 \cup \ldots \cup B_s| = |B_i| \leq mk \). Each of the other terms does not exceed \( mk \) by the maximality of \( i \). Hence

\[
|B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-i)m(k-1) < mk + (s-1)m(k-1).
\]

So, in any case we obtain that the inequality \( |B_1 \cup B_2 \cup \ldots \cup B_s| \leq mk + (s-1)m(k-1) \) holds. Taking into account that \( C = B_1 \cup B_2 \cup \ldots \cup B_s \) and \( s \leq k \), we have

\[
|C| \leq mk + (k-1)m(k-1) = mk^2 - k + 1 < f(k, m).
\]

The obtained contradiction proves the lemma. □

**Theorem 10** There exists a finite set \( \mathcal{F}_0 \) of forbidden induced subgraphs such that a split graph \( G \) belongs to the class \( L^m_k \) if and only if no induced subgraph of \( G \) is isomorphic to an element of \( \mathcal{F}_0 \).

**Proof:** Denote by \( R_p \) the graph obtained from the complete graph \( H \cong K_{f(k, m)} \) by adding a new vertex and connecting it with exactly \( p \) vertices of \( H \). Put \( \mathcal{F}_0 = \{ R_p : km + 1 \leq p \leq f(k, m) - 1 \} \cup \{ K_1, k+1 \} \).

Using Theorem 8 one can immediately verify that no graph from \( \mathcal{F}_0 \) belongs to \( L^m_k \).

Let, without loss of generality, \( G \) be connected graph, and \( V(G) = C \cup S \) be a bipartition of \( V(G) \) into clique \( C \) and stable set \( S \) such, that \( C \) is a maximal clique. Let also no induced subgraph of \( G \) be isomorphic to an element of \( \mathcal{F}_0 \). Put \( S = \{ v_1, \ldots, v_s \} \). Consider two cases:

1) \( |C| > (km - 1)k + 1 \).

In this case we have

\[
|C| \geq (km - 1)k + 2 = mk^2 - (k - 1) + 1 \geq mk^2 - m(k - 1) + 1 = f(k, m).
\]
Then, since no induced subgraph of $G$ is isomorphic to a graph $R_p$, $km + 1 \leq p < f(k, m) - 1$, we have $\deg(v_i) \leq km$ for any $i = 1, 2, \ldots, s$. Since $G$ contains no induced $K_{1, k+1}$, we have $|N(u) \cap S| \leq k$ for any vertex $u$ from $C$. Moreover, we prove that for any vertex $u$ from $C$ the inequality $|N(u) \cap S| \leq k - 1$ holds. Assume this is not true. Let, without loss of generality, some vertex $u$ from $C$ be adjacent to the vertices $v_1, \ldots, v_k$ from $S$, $k \leq s$. Since $\deg(v_i) \leq km$, $i = 1, 2, \ldots, k$, and $u \in \bigcap_{i=1}^{k} N(v_i)$,

then $|\bigcup_{i=1}^{k} N(v_i)| \leq \sum_{i=1}^{k} (\deg(v_i) - 1) + 1 \leq (km - 1)k + 1 < \varphi(G)$. Hence, there exists a vertex $u'$ from $C$, which is not adjacent to any vertex from $v_1, \ldots, v_k$. But then $G(u, u', v_1, \ldots, v_k) \cong K_{1, k+1}$, a contradiction.

Now we can construct a krawusz $(k, m)$-partition of $G$. Since $\deg(v_i) \leq km$ for any $i = 1, 2, \ldots, s$, then there exists a partition $N(v_i) = C_{i_1} \cup \ldots \cup C_{i_s}$, where $C_{i_j} \cap C_{i_l} = \emptyset$, $j, l \in \{1, \ldots, s\}$, $j \neq l$, $|C_{i_j}| \leq m$, $s_i \leq k$. Obviously, the list of cliques $\{C_{i_j} \cup \{v_i\} : i = 1, s, j = 1, s_i\}$ together with the clique $C$ is a krawusz $(k, m)$-partition of graph $G$.

2) $|C| \leq (km - 1)k + 1$.

Since $G$ contains no induced $K_{1, k+1}$, we have $|N(u) \cap S| \leq k$ for any vertex $u$ from $C$. Therefore, as $G$ is connected,

$$|G| = |C| + |S| \leq |C| + \sum_{u \in C} |N(u) \cap S| \leq ((km - 1)k + 1) + ((km - 1)k + 1)k = ((km - 1)k + 1)(k + 1),$$

i.e. the order of graph $G$ is bounded above by a value, depending on $k$ and $m$. Add to the list $\mathcal{F}_0$ all such split graphs $H$, that $H \not\in L^m_k$ and $|H| \leq ((km - 1)k + 1)(k + 1)$.

Obviously, the constructed in the cases 1) and 2) finite list $\mathcal{F}_0$ is a required list of forbidden induced subgraphs.

Since $K_{1, k+1} \not\in L^m_k$, the heredity of $L^m_k$, immediately implies

**Lemma 11** A bipartite graph $G$ belongs to the class $L^m_k$ if and only if no induced subgraph of $G$ is isomorphic to $K_{1, k+1}$.

**Theorem 12** There exists a finite set $\mathcal{F}$ of forbidden induced subgraphs such that an $(\infty, 1)$-polar graph $G$ belongs to the class $L^m_k$ if and only if no induced subgraph of $G$ is isomorphic to an element of $\mathcal{F}$.

**Proof:** Without loss of generality we can suppose that $(\infty, 1)$-polar graph $G$ is connected. Let $G$ have bipartition $(A, B)$; $A_i, i = 1, 2, \ldots, t$, be the vertex sets of connected components of $\overline{G}(A)$; $\mathcal{F}_0$ be the set of split graphs from Theorem [8]. Denote by $\mathcal{F}_1$ the set of $(\infty, 1)$-polar graphs which do not belong to the class $L^m_k$ and have order at most $(k + 1)k(f(k, m) - 1)$.

Put $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \{K_{1, k+1}, K_f(k, m)+1 - e\}$, where $K_f(k, m)+1 - e$ is the graph obtained from the complete graph $K_f(k, m)+1$ after deleting an edge. The set $\mathcal{F}$ is finite, since $\mathcal{F}_0$ and $\mathcal{F}_1$ are finite. According to Theorem [8] there is no krawusz $(k, m)$-partition for $K_f(k, m)+1 - e$. Therefore $K_f(k, m)+1 - e \not\in L^m_k$. Thus, $\mathcal{F} \cap L^m_k = \emptyset$. The necessity of the statement follows from the heredity of the class $L^m_k$.

Now let $G$ contain no induced subgraph isomorphic to an element from $\mathcal{F}$. If $G(A)$ is complete, then $G$ is split graph and by Theorem [10] $G \in L^m_k$. If $G(A)$ is empty, then $G$ is bipartite graph and by Lemma [11] $G \in L^m_k$.

Now suppose that $G(A)$ is neither complete nor bipartite graph. Then $2 \leq t \leq |A| - 1$. Since $K_{1, k+1} \in \mathcal{F}$, then $|A_i| \leq k$ for any $i = 1, 2, \ldots, t$. Now we will prove that since $K_f(k, m)+1 - e \in \mathcal{F}$, then
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$t \leq f(k, m) - 1$. Let, to the contrary, $t \geq f(k, m)$. As $G(A)$ is not complete graph, there exists an index $i_0 \in \{1, 2, \ldots, t\}$ such that $|A_{i_0}| \geq 2$. Consider the set $S = \{a_1, a_2, \ldots, a_{i_0-1}, a'_{i_0}, a''_{i_0}, a_{i_0+1}, \ldots, a_t\}$, where $a_i \in A_i$ for any $i \in \{1, 2, \ldots, t\} \setminus \{i_0\}$ and $a'_{i_0}, a''_{i_0} \in A_{i_0}$. Then $G(S)$ contains $K_{f(k, m)+1 - e}$ as induced subgraph, a contradiction. Therefore

$$|A| \leq \sum_{i=1}^{t} |A_i| \leq k(f(k, m) - 1).$$

Since $|N(a) \cap B| \leq k$ for any vertex $a \in A$ and $G$ is connected, we have

$$|G| \leq |A| + |B| \leq |A| + \sum_{a \in A} |N(a) \cap B| \leq k(f(k, m) - 1) + k^2(f(k, m) - 1) = (k+1)k(f(k, m) - 1).$$

It follows from the inclusion $F_{1} \subseteq F$ that $G \in L_m^{\infty}$. \hfill \Box

**Corollary 13** The problem $K DIM_m(k)$ is polynomially solvable in the class of $(\infty, 1)$-polar graphs for every fixed $k, m \geq 1$.

**References**


