The forest consensus theorem

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Abstract

We show that the limiting state vector in the differential model of consensus seeking with an arbitrary communication digraph is obtained by multiplying the eigenprojection of the Laplacian matrix of the model by the vector of initial states. Furthermore, the eigenprojection coincides with the stochastic matrix of maximum out-forests of the weighted communication digraph. These statements make the forest consensus theorem. A similar result for DeGroot’s iterative pooling model requires the Cesàro (time-average) limit in the general case. The forest consensus theorem is useful for the analysis of consensus protocols.

Index Terms

Consensus, Eigenprojection, Matrix exponent, DeGroot’s iterative pooling.

I. INTRODUCTION

The continuous-time model of consensus seeking in a multiagent system has the form [22], [10]

\[
\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, n, \\
u_i(t) = -\sum_{j=1}^{n} a_{ij} (x_i(t) - x_j(t)),
\]

where \(x_i(t)\) is the state of the \(i\)'th agent and \(a_{ij} \geq 0\) is the weight with which agent \(i\) takes into account the discrepancy with agent \(j\). The matrix form of the model (1–2) is:

\[
\dot{x}(t) = -L x(t),
\]

where \(x(t) = (x_1(t), \ldots, x_n(t))^T\), \(L\) is the Laplacian matrix of the model (1–2):

\[
L = \text{diag}(A 1) - A,
\]

\(A = (a_{ij})_{n \times n}\), and \(1 = (1, \ldots, 1)^T\).

The nonsymmetric Laplacian matrices of this kind were studied in [1], [9], [5].

In this paper, we present the forest consensus theorem stating that for an arbitrary non-negative matrix \(A\) and any trajectory \(x(t)\) satisfying (1–2),

\[
\lim_{t \to \infty} x(t) = \tilde{J} x(0)
\]

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holds, where $\tilde{J}$ is the eigenprojection of $L$, which coincides with the matrix $\tilde{J}$ of maximum out-forests of the communication digraph corresponding to $A$.

A similar result, which involves the Cesàro limit, holds for the discretization of the model (1)–(2).

The paper is organized as follows. After introducing the necessary notation and summarizing the preliminary results, in Section III we prove the forest consensus theorem. Section IV is devoted to the properties of the limiting state of the model. Section V contains a numerical example; in Section VI, we show that the classical results on a communication digraph having a spanning diverging tree immediately follow from the forest consensus theorem. Finally, Section VII presents a counterpart of the forest consensus theorem for the discretization of the model (1)–(2).

II. BASIC CONCEPTS AND PRELIMINARY RESULTS

A. Eigenprojections and functions of matrices

Let $A \in \mathbb{C}^{n \times n}$ be an arbitrary square matrix. Let $\nu = \text{ind} A$ be the index of $A$, i.e., the smallest $k \in \{0, 1, \ldots\}$ such that $\text{rank} A^{k+1} = \text{rank} A^k$, where $A^0$ is identified with the identity matrix $I$. $A$ is nonsingular iff $\nu = 0$. The index of a singular matrix is the index of its eigenvalue 0, i.e., the multiplicity of 0 as the root of its minimal polynomial, or, equivalently, the size of the largest Jordan block with zero diagonal in its Jordan form. If $\nu = 1$ then the algebraic and geometric multiplicities of 0 coincide (in this case, the eigenvalue 0 of $A$ is called semisimple).

Let $R(A)$ and $N(A)$ be the range and the null space of $A$, respectively. The eigenprojection of $A$ at eigenvalue 0 is a projection (i.e. an idempotent matrix) $Z$ such that $R(Z) = N(A^\nu)$ and $N(Z) = R(A^\nu)$. In other words, $Z$ is a projection to $N(A^\nu)$ along $R(A^\nu)$. In the case of a singular matrix $A$, following [25], we call $Z$ the eigenprojection of $A$ (without mentioning eigenvalue 0). The eigenprojection is unique because an idempotent matrix is uniquely determined by its range and null space (see, e.g., [7, Sections 2.4 and 2.6]).

Eigenprojections underlie the definition of the components of a matrix which, in turn, are used to define $\varphi(A)$ for differentiable functions $\varphi: \mathbb{C} \to \mathbb{C}$ (see either of [14, Chapter 5], [7, Section 2.5], [17], [16]), in the theory of generalized inverse matrices, as well as in the numerous applications of matrix analysis.

Let $\lambda_1, \ldots, \lambda_s$ be all distinct eigenvalues of $A$; let $\nu_k$ be the index of $\lambda_k$ defined as the index of $A - \lambda_k I$. According to the theory of matrix components [14, Chapter 5], for any function $\varphi: \mathbb{C} \to \mathbb{C}$ having finite derivatives $\varphi^{(j)}(\lambda_k)$ of the first $\nu_k - 1$ orders at $\lambda_1, \ldots, \lambda_s$, $\varphi(A)$ is defined as follows:

$$\varphi(A) := \sum_{k=1}^{s} \sum_{j=0}^{\nu_k - 1} \varphi^{(j)}(\lambda_k) Z_{kj},$$

where the derivative $\varphi^{(0)}$ of order 0 is the value of $\varphi$ and $Z_{kj}$ are the components of $A$ defined by

$$Z_{kj} = (j!)^{-1}(A - \lambda_k I)^j Z_{k0}.$$ 

1Such an eigenprojection is also called the principal idempotent [15].
Here, the component $Z_{k0}$ is the eigenprojection of $A - \lambda_k I$ ($k = 1, \ldots, s$) also called the eigenprojection of $A$ at $\lambda_k$.

For more details on eigenprojections, see, e.g., [3].

B. The stochastic matrix of maximum out-forests

A matrix $A = (a_{ij})$ of the model (1)-(2) determines a weighted communication digraph $\Gamma$ with vertex set $V(\Gamma) = \{1, \ldots, n\}$: $\Gamma$ has the $(j, i)$ arc with weight $w_{ji} = a_{ij}$ whenever $a_{ij} > 0$ (i.e., when agent $j$ influences agent $i$). Thus, arcs of $\Gamma$ are oriented in the direction of influence; the weight of an arc is the degree of influence.

A diverging tree is a weakly connected (i.e., its corresponding undirected graph is connected) digraph in which one vertex, called the root, has indegree zero and the other vertices have indegree one. A diverging tree is said to diverge from its root. Spanning diverging trees are also called out-arborescences or out-branchings [20]. A diverging forest is a digraph all of whose weak components (i.e., maximal weakly connected subdigraphs) are diverging trees. The roots of these trees form the set of roots of the diverging forest.

Definition 1. Any spanning diverging forest of a digraph $\Gamma$ is called an out-forest of $\Gamma$. An out-forest $F$ of $\Gamma$ is a maximum out-forest of $\Gamma$ if $\Gamma$ has no out-forest with a greater number of arcs than in $F$. The out-forest dimension of $\Gamma$ is the number of components in any maximum out-forest of $\Gamma$.

The weight of a weighted digraph is the product of its arc weights. The matrix $\bar{J} = (\bar{J}_{ij})$ of maximum out-forests of a weighted digraph $\Gamma$ is defined as follows:

$$\bar{J}_{ij} = \frac{f_{ij}}{f}, \quad i, j = 1, \ldots, n,$$

(7)

where $f$ is the total weight of all maximum out-forests of $\Gamma$ and $f_{ij}$ is the total weight of those of them that have $i$ belonging to a tree diverging from $j$. In Proposition 1, we list some properties of $L$ and $\bar{J}$ (cf. [10], [8], [4]) which are useful for the analysis of consensus protocols.

Proposition 1. Let $L$ be the Laplacian matrix of the model (1)-(2). Let $\bar{J}$ be the matrix of maximum out-forests of the corresponding communication digraph $\Gamma$ whose out-forest dimension is $d$. Then:

1) $L$ is singular (since $L1 = 0$);
2) If $\lambda \neq 0$ is an eigenvalue of $L$, then $\text{Re}(\lambda) > 0$ [2, Proposition 9];
3) $\text{ind} L = 1$ [9, Proposition 12];
4) rank $L = n - d$; rank $\bar{J} = \text{tr} \bar{J} = d$ [1 Proposition 11];
5) $\bar{J}$ is row stochastic as by definition, $\sum_{j=1}^{n} f_{ij} = f$ for any $i \in \{1, \ldots, n\}$;
6) $\bar{J}$ is the eigenprojection of $L$ [9, Proposition 12], which implies that $\bar{J}^2 = \bar{J}$;
7) $L\bar{J} = \bar{J}L = 0$ [1 Theorem 5]; $\mathcal{N}(\bar{J}) = \mathcal{R}(L)$, $\mathcal{R}(\bar{J}) = \mathcal{N}(L)$ (by items 2 and 11);
8) $\bar{J} = \lim_{\alpha \to \infty} (I + \alpha L)^{-1}$ [1 Theorem 6];
9) \( \bar{J} = C(0)/h(0) \), where \( C(\lambda) \) is the quotient of the matrix polynomial \( \lambda h(\lambda) I \) and the binomial \( \lambda I - L \), \( \lambda h(\lambda) \) being the minimal polynomial of \( L \) (this follows from [14, Eq. (22) in Chapter 5]);

10) \( \bar{J} = \bar{J}_{n-d} \), where \( \bar{J}_{n-d} \) is defined recursively: \( \bar{J}_k = I - k \frac{L J_{k-1}}{\text{tr}(L J_{k-1})} \), \( k = 1, \ldots, n-d \), \( \bar{J}_0 = I \), and \( L \bar{J}_{n-d} = 0 \) [2, Section 4] or [9, Section 5].

An elementwise characterization of \( \bar{J} \) is given in [1, Theorem 2'].

### III. The Forest Consensus Theorem

**Theorem 1.** Let \( x(t) \) be a solution of (3). Then

\[
\lim_{t \to \infty} x(t) = \tilde{J} x(0),
\]

where \( \tilde{J} \) is the eigenprojection of \( L \). Moreover, \( \tilde{J} \) coincides with the matrix \( \bar{J} \) of maximum out-forests of the communication digraph corresponding to \( L \).

**Proof.** All solutions of (3) satisfy the identity [14, Eq. (46) in Chapter 5]

\[
x(t) = e^{-Lt} x(0).
\]

According to (5) \( e^{-Lt} \) is representable in the form (cf. Eq. (12) of Chapter 4 in [13])

\[
e^{-Lt} = \sum_{k=1}^{s} \sum_{j=0}^{\nu_k-1} Z_{kj} t^j e^{-\lambda_k t},
\]

where \( \lambda_1, \ldots, \lambda_s \) are all distinct eigenvalues of \( L \).

Since \( L \) is singular, we can set \( \lambda_1 = 0 \). Then \( Z_{1j} \) are the components of \( L \) corresponding to the characteristic root 0. By item 5 of Proposition 1 \( \nu_1 = \text{ind } L = 1 \) and by item 6 of Proposition 1 \( Z_{10} \), the eigenprojection of \( L \) denoted by \( \bar{J} \), coincides with the matrix \( \bar{J} \) of maximum out-forests of the communication digraph \( \Gamma \).

Since the components \( Z_{kj} \) of \( L \) are independent of \( t \) while, by item 2 of Proposition 1 \( \Re(\lambda_k) > 0 \) \( (k \geq 2) \), we have

\[
\lim_{t \to \infty} \sum_{k=2}^{s} \sum_{j=0}^{\nu_k-1} Z_{kj} t^j e^{-\lambda_k t} = 0.
\]

Finally, (9)–(11) and \( \nu_1 = 1 \) yield

\[
\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{-Lt} x(0) = Z_{10} x(0) = \tilde{J} x(0).
\]

\[\square\]

\[\text{In some cases, the expression } 3 \text{ Theorem 1] can be more convenient for calculations.}\]
IV. THE PROPERTIES OF THE ASYMPTOTIC STATE

Now we need some additional notation. A basic bicomponent of a digraph $\Gamma$ is any maximal (by inclusion) strongly connected weighted subdigraph of $\Gamma$ such that there are no arcs coming into it from outside. By [1, Proposition 6], the number of basic bicomponents in $\Gamma$ is equal to the out-forest dimension $d$ of $\Gamma$.

Let $x(\infty)$ be the limiting state vector of the model (1)–(2): $x(\infty) = \lim_{t \to \infty} x(t)$.

**Corollary 1** (of Theorem 1). Let $K$ be a basic bicomponents of $\Gamma$; let $j$ be a vertex of $K$. It holds that:

1) If $i$ is a vertex of $K$ or $i$ is reachable (by a directed path) from $K$ and not reachable from the other basic bicomponents of $\Gamma$, then $x_i(\infty) = x_j(\infty)$ and $x_i(\infty)$ is equal to the value of consensus for the communication digraph $K$ alone;

2) If vertex $i$ is reachable from several basic bicomponents of $\Gamma$, then $x_i(\infty)$ is between the minimum and maximum elements of $x(\infty)$ that correspond to these basic bicomponents (and is strictly between them if the minimum and maximum differ);

3) If vertex $i$ is not in a basic bicomponent of $\Gamma$, then $x(\infty)$ is independent of $x_i(0)$.

Corollary 1 is easily proved using the row stochasticity of $\bar{J}$ and two simple facts which follow from [1, Theorem 2']. The facts are: (1) $\bar{J}_{ij} \neq 0$ if and only if $j$ belongs to a basic bicomponent of $\Gamma$ and $i$ is reachable from $j$; (2) If $i$ and $j$ belong to the same basic bicomponents in $\Gamma$, then the $i$-row and $j$-row of $\bar{J}$ are equal, while the $i$-column and $j$-column are proportional.

Using time shift and item 7 of Proposition 1 we have

**Corollary 2** (of Theorem 1). Let $x(t)$ be a solution of (3). Then for any $t \in \mathbb{R}$, $\bar{J} x(t) = x(\infty)$. Consequently, for any $t_1, t_2 \in \mathbb{R}$, $\bar{J} (x(t_1) - x(t_2)) = 0$, i.e., $(x(t_1) - x(t_2)) \in \mathcal{N}(\bar{J}) = \mathcal{R}(L)$.

V. EXAMPLE

Consider the weighted communication digraph $\Gamma$ shown in Fig. 1. It has two basic bicomponents whose vertex sets are $\{1, 2\}$ and $\{3, 4, 5\}$.

Figure 1. A communication digraph $\Gamma$.

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Cf. [6], equation between (16) and (17).
The Laplacian matrix (4) of the model (1)–(2) corresponding to $\Gamma$ is:

$$L = \begin{pmatrix}
2 & -2 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -2 & 4 & -2 & 0 & 0 \\
0 & 0 & 0 & -2 & 2 & 0 & 0 \\
0 & -3 & -2 & 0 & 0 & 5 & 0 \\
0 & 0 & -4 & 0 & 0 & -1 & 5 \\
\end{pmatrix}. $$

The spectrum of $L$ is real: $(0, 0, 2, 3, 5, 5, 5)$, which is not generally the case for a Laplacian matrix of a weighted digraph. On the other hand, $L$ is not diagonalizable as the geometric multiplicity of the triple eigenvalue $5$ is $1$. The minimal polynomial of $L$ is

$$\lambda h(\lambda) = \lambda(\lambda - 2)(\lambda - 3)(\lambda - 5)^3. $$

To find $Z_{10} = \tilde{J}$, one can use item 9 of Proposition 1:

$$Z_{10} = \frac{C(0)}{h(0)}, $$

where $h(0) = (-2)(-3)(-5)^3 = -750$, while $C(0)$ can be determined using (55) and (56) in [14, Chapter 4]:

$$C(0) = \Psi(0 \cdot I, L) = (L - 2I)(L - 3I)(L - 5I)^3. $$

Substituting $C(0)$ and $h(0)$ into (12) yields

$$Z_{10} = \tilde{J} = \frac{1}{750} \begin{pmatrix}
250 & 500 & 0 & 0 & 0 & 0 \\
250 & 500 & 0 & 0 & 0 & 0 \\
0 & 0 & 300 & 150 & 300 & 0 \\
0 & 0 & 300 & 150 & 300 & 0 \\
0 & 0 & 300 & 150 & 300 & 0 \\
150 & 300 & 120 & 60 & 120 & 0 \\
30 & 60 & 264 & 132 & 264 & 0 \\
\end{pmatrix}. $$

The matrix $\tilde{J}$ can also be found using item 10 of Proposition 1 as well as (13), this involves four matrix multiplications, but it does not require the knowledge of the nonzero eigenvalues or the minimal polynomial of $L$. Direct enumeration of forests has no practical value for the computation of $\tilde{J}$. However, to give the reader a little taste of this “forestry”, in Fig. 2 we present all 32 maximum out-forests of $\Gamma$ and their weights. The total weight of them is $f = 750$; by the definition (7), it is the common denominator of the entries of $\tilde{J} = (\tilde{J}_{ij})$. The numerator of $\tilde{J}_{ij}$ is the total weight $f_{ij}$ of the maximum out-forests in which $i$ belongs to a tree diverging from $j$. For example, for $\tilde{J}_{65}$, these are the forests #10, 12, 14, 16, 26, 28, 30, and 32 whose weights are $16, 4, 16, 4, 32, 8, 32$, and $8$, respectively, so that $f_{65} = 120$ and $\tilde{J}_{65} = \frac{120}{750}$, in concordance with (14).

4On this correspondence, see Section II-B.
Figure 2. The maximum out-forests of the communication digraph in the Example.
Using [1, Proposition 9] the set of all maximum out-forests of \( \Gamma \) can be described as follows.  
1. Choose an arbitrary spanning diverging tree in each basic bicomponent of \( \Gamma \).  
2. Choose any maximum out-forest in the digraph obtained from \( \Gamma \) by removing all arcs belonging to the basic bicomponents. Combining the chosen trees and forest gives a maximum out-forest of \( \Gamma \); every desirable forest can be obtained in this way. A more detailed algorithm for constructing maximum out-forests can be found in [1, Section 5].

Let \( x(0) = (1, 10, 5, 7, 9, *, *)^T \) (the last two components are “free”: they correspond to “nonbasic” vertices which, by Corollary [1] do not affect the limiting state vector). By Theorem 1,  
\[
\lim_{t \to \infty} x(t) = \tilde{J} x(0) = (7, 7, 7, 7, 7, 7)^T,
\]
i.e., asymptotic consensus is achieved. On the other hand, if \( x(0) = (0, 6, 3, 9, 10, *, *)^T \), then  
\[
\lim_{t \to \infty} x(t) = \tilde{J} x(0) = (4, 4, 7, 7, 5.2, 6.64)^T,
\]
and asymptotic consensus is achieved only within the basic bicomponents having vertex sets \( \{1, 2\} \) and \( \{3, 4, 5\} \), but not for the whole set of agents.

Thus, a system satisfying (1)–(2) has its domain of convergence to consensus, that is, the set of initial states \( x(0) \) such that the product \( \tilde{J} x(0) \) gives a vector with all equal components. In [4], this domain (obviously, it is a subspace of \( \mathbb{R}^n \)) is characterized and it is shown that when \( x(0) \) does not belong to the domain, then there is still some reasonable “quasi-consensus”. It can by obtained by first, projecting \( x(0) \) onto the domain of convergence and second, applying the coordination protocol (1)–(2). Say, for the initial states of the form \( x(0) = (0, 6, 3, 9, 10, *, *)^T \) which were considered above, the value of such a “quasi-consensus” is 5.82.

VI. On communication digraphs of out-forest dimension 1

Suppose that the communication digraph \( \Gamma \) has a spanning diverging tree or, equivalently, the out-forest dimension of \( \Gamma \) is one \((d = 1)\). In this (and only this) case, by item[4] of Proposition[1] rank \( \bar{J} = 1 \) holds, so by item[5]  
\[
\bar{J} = v^T_1 1,
\]
where \( v_1^T \) is any row of \( \bar{J} \) and \( v_1^T 1 = 1 \). By items[7] and[4] \( v_1^T \) and \( 1 \) span the left and right null spaces of \( L \), respectively. Thus, Theorem[1] yields the following familiar necessary and sufficient condition of achieving consensus.

**Corollary 3** (of Theorem[1]). If the communication digraph \( \Gamma \) of the model (1)–(2) has a spanning diverging tree, then for any initial state \( x(0) \), \( x(t) \) converges to the consensus  
\[
\lim_{t \to \infty} x(t) = (v_1^T x(0)) 1,
\]
where \( v_1 \) is the unique left eigenvector of \( L \) associated with 0 and satisfying \( v_1^T 1 = 1 \). Conversely, if for each initial state \( x(0) \), \( x(t) \) tends to a consensus, then \( \Gamma \) has a spanning diverging tree.

For the more restricted case of a strongly connected digraph \( \Gamma \), a representation similar to (16) was obtained in [22, Theorem 3]. In this case, it was shown that \( \lim_{t \to \infty} e^{-Lt} = v_r v_r^T \), where
\(v_l\) and \(v_r\) are, respectively, the left and right eigenvectors of \(L\) associated with 0 and satisfying \(v_l^T v_r = 1\). Before Theorem 1, the authors of [22] mention that 1 can be substituted for \(v_r\).

Corollary 3 coincides with [20, Theorem 3.12] (see also [20, Proposition 3.11] and Lemma 1.3 in [23]). The case of a communication digraph \(\Gamma\) having a spanning diverging tree was recently considered in [6] where Lemma 3 presents an analog of (16). However, the multiplier \(1/\sqrt{n}\) in [6, Eq. (18)] is not correct due to an invalid step in the proof.

Finally, observe that Theorem III.8 in [26] can also be derived from Theorem 1.

VII. A DISCRETE COUNTERPART OF THE FOREST CONSENSUS THEOREM

Consider the discretization of the model (3):

\[
\frac{x(t+\tau) - x(t)}{\tau} = -Lx(t)
\]

with a small fixed \(\tau \in \mathbb{R}\). Let \(y(k) := x(k\tau), k = 0, 1, \ldots,\) be the state vector with the discrete-time dynamics determined by (17). Then

\[
y(k) = (I - \tau L) y(k-1), \quad k = 1, 2, \ldots
\]

Setting

\[
P := I - \tau L
\]

and observing [1, Section 8] that \(P\) is row stochastic whenever

\[
0 < \tau \leq \left(\max_i \sum_{j \neq i} a_{ij}\right)^{-1}
\]

we obtain DeGroot’s iterative pooling model [12]:

\[
y(k) = P y(k-1), \quad k = 1, 2, \ldots
\]

Matrix (19) is sometimes called the Perron matrix with parameter \(\tau\) of the weighted digraph \(\Gamma\). Obviously, \(P\) is the linear part of the series expansion of \(e^{-\tau L}\).

Let us compare the asymptotic properties of the model (3) and its discrete analog (21). From (21) one has

\[
y(k) = P^k y(0), \quad k = 0, 1, \ldots
\]

A necessary and sufficient condition of the convergence of \(\{P^k\}\) under (20) is the aperiodicity of \(P\). On the other hand, the Cesàro (time-average) limit

\[
P^\infty := \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} P^i
\]

exists for any stochastic \(P\) and coincides with \(\lim_{k \to \infty} P^k\) whenever the latter exists. Otherwise, if \(P\) is periodic with period \(s\), then \(P^\infty = s^{-1} (P^{(1)} + \ldots + P^{(s)})\), where \(P^{(1)}, \ldots, P^{(s)}\) are the limits of the converging subsequences of \(\{P^k\}\): \(P^{(i)} = \lim_{j \to \infty} P^{js+i}, i = 1, \ldots, s\).

The discrete-time counterpart of Theorem [1] is an immediate consequence of well-known results. Yet, for ease of comparison with Theorem [1] we represent it in the form of a theorem.
Theorem 2. Let sequence \( y(k) \) satisfy (21), where \( P \) is defined by (19)–(20). Then

\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} y(i) = \bar{J} y(0),
\]

where \( \bar{J} \) is the eigenprojection of \( L \), which coincides with the matrix \( \bar{J} \) of maximum out-forests of the communication digraph \( \Gamma \) corresponding to \( L \). Moreover, if (20) is satisfied strictly, then

\[
\lim_{k \to \infty} y(k) = \bar{J} y(0).
\]

Proof. Meyer [21] and Rothblum [24] have shown that \( P^\infty \) is the eigenprojection of \( I - P \). Hence, by (19) and the definition of eigenprojection, \( P^\infty = \bar{J} \). Now applying the Cesàro limit to (22) and using (23) and item 6 of Proposition 1 one obtains the first assertion of Theorem 2.

Alternatively, the identity \( P^\infty = \bar{J} \) coincides with the Markov chain tree theorem first proved by Wentzell and Freidlin [27] and rediscovered in [19], [18]. This identity provides the first assertion of Theorem 2 along the same lines.

Finally, if (20) is satisfied strictly, then \( P \) has a strictly positive diagonal. In this case, by Geršgorin’s theorem, \( P \) has no eigenvalues of modulus 1 except for 1. Hence, \( P \) is not periodic and \( \{P^k\} \) converges, which yields (24).

Obviously, the only essential difference between Theorem 2 and Theorem 1 is the use of the Cesàro limit in the case of a periodic matrix \( P \). With a similar “Cesàro” addendum one can easily formulate a discrete-time counterpart of the Corollary 3 of Section VI.

To compute the matrix \( \bar{J} = \bar{J} \), one can use items 8–10 of Proposition 1, constructive characterizations (h), (j) or (l) in [3, Section 2], or [11, Proposition 2].

REFERENCES


