Friction-Induced Lines of Attraction and Repulsion for Parts Sliding on an Oscillated Plate

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Abstract—We show that the frictional forces arising from simultaneous small amplitude periodic translation and rotation of a rigid plate cause parts on the plate to converge to or diverge from a line coinciding with the rotation axis. The relative phase between the translation and rotation determines whether the parts are attracted to or repelled from the rotation axis. Assuming that both the translational and rotational accelerations of the plate are “bang-bang” and have identical frequencies, we derive the resultant velocity fields for point parts on the plate. For many choices of phase the speed of the part is approximately proportional to its distance from the rotation axis. The strength of the velocity field can be controlled by modulating the amplitude of the translational acceleration, or modulating the relative phase of the velocity field. The phases that maximize part speed towards and away from the rotation axis. These optimal phases not only maximize part speed but also generate velocity fields that are nearly independent of the coefficient of friction.

Note to Practitioners—Without using sensors, one method of positioning and orienting an isolated part on a plate is to subject the part to a sequence of squeeze fields. Each squeeze field attracts the part to a “squeeze line” until a stable configuration is reached. This paper describes how to vibrate a rigid plate so that frictional forces move an isolated part towards a squeeze line of arbitrary orientation. An optimal plate motion is presented that maximizes the speed of point parts towards the squeeze line. Since the analysis is restricted to point parts, a natural extension entails generalizing the methods presented in this paper to handle parts with planar extent. Such a generalization may indicate that a rigid vibrating plate can serve as a sensorless universal parts feeder.

Index Terms—Friction-induced velocity fields, friction-induced force fields, sensorless part orientation, rigid plate oscillation, rigid plate vibration, squeeze fields, parts feeding, assembly.

I. INTRODUCTION

In [1], we showed that very simple periodic motions of a six degree-of-freedom (DOF) vibratory plate create frictional forces that independently drive multiple parts on the plate. Our Programmable Parts-feeding Oscillatory Device (PPOD) is a six-DOF vibratory plate inspired by the three-DOF Universal Planar Manipulator (UPM) designed by Reznik and Canny [2], [3]. The UPM is a vibratory plate capable of moving only in the horizontal plane of the plate surface (i.e., \(x, y, \theta\)). Asymmetric periodic plate motions cause the plate to slip relative to a part for a longer time in one direction than another, producing a net frictional force on the part. By incorporating the rotational degree of freedom, the UPM is able to create position-dependent frictional forces.

Higashimori et al. built an even more underactuated device that has just one translational and one rotational degree of freedom (the axis of rotation is horizontal, passes through the center of the plate, and is aligned with the direction of translation) [4]. The motions of this device mimic those used by a pizza chef to manipulate a pizza deep in the oven. Higashimori et al. showed that this “pizza manipulator” plate can drive a disk on the plate to any desired position and orientation using feedback from an overhead camera.

Unlike the PPOD, neither the “pizza manipulator” nor the UPM has the motion freedoms necessary to generate divergent fields (i.e., fields exhibiting sink and source behavior). Sink fields are particularly desirable because they can be used to position and orient parts without the use of sensors. For example, B¨ohringer et al. presented an algorithm that positions and orients an arbitrary polygonal part without sensors by applying a sequence of force fields, each of which causes the part to converge towards a nodal line until it reaches a stable configuration [5]. The authors proposed creating these “squeeze fields” with a flexible vibrating plate. When the plate is vibrated and properly clamped, objects on the plate are attracted to a nodal line that forms on the surface. Yet there are drawbacks to this method: to change the orientation of the nodal line for each successive squeeze, the plate must be reclamped; the nodal line is often curved; and the interaction between the flexible plate and the part is complicated and difficult to model.

Another method of generating squeeze fields is with a planar array of actuators or motion pixels. Array-based systems include rolling wheels [6], [7], air jets [8], [9], MEMS devices [10], [11], [12], and individually vibrating plates [13]. The main disadvantage of these systems is increased hardware and software complexity: numerous small actuators must be individually controlled to approximate a continuous force field.

These shortcomings motivated us to consider vibrating a single rigid plate to generate squeeze fields. In this paper we demonstrate that a single rigid plate can be used to generate fields with nonzero divergence if it is allowed to rotate out of the horizontal plane. This idea was first introduced in [1] in which we showed that a squeeze field occurs in theory when the plate is symmetrically oscillated about an axis parallel to and below the plate surface. This particular plate motion creates the equivalent of a (straight) nodal line directly above the rotation axis to which parts are attracted (Figure 1(b)). We
verified this attractive behavior with the 1-DOF device shown in Figure 1(a); however, the mechanical design of this device limits the configuration of the nodal line to a single location and orientation. With the PPDO, the configuration of the nodal line can be easily translated or rotated without any hardware modification. The accompanying videos qualitatively verify that our two devices can generate nodal lines for sinusoidal plate accelerations.

In this paper we analyze the generation of nodal lines using bang-bang motions of the rigid plate, instead of sinusoidal motions, for two reasons: the bang-bang motion simplifies the analysis and creates stronger squeeze fields than sinusoidal motions. As our current experimental system is not capable of approximating bang-bang motions, we verify our theoretical results using numerical simulations of the full dynamics of the part-plate interaction. Simulations agree qualitatively with our experimental results over the range of motions we have tested with the experimental systems.

Our simulations assume Coulomb friction (kinetic and static) and incorporate the full dynamics of a point part as long as it remains in contact with the plate surface. From these simulations we observed that for a given periodic plate motion and position \( r \) on the plate, there is a unique average velocity \( \bar{v}(r) \) over one cycle of plate motion such that a point part at \( r \) moving with any other average velocity tends toward \( \bar{v}(r) \). We call \( \bar{v}(r) \) the asymptotic velocity at \( r \). Thus a part’s motion on the plate is given approximately by the position-dependent asymptotic velocity field, where the quality of the approximation depends on the rate of convergence to the field.

In [1] we derived the asymptotic velocity field for parts on a plate that is oscillated with bang-bang angular acceleration about a rotation axis below the plate surface; it has the simple form

\[
v(x) \approx -\frac{3}{8} \mu \alpha Tx,
\]

where \( \mu \) is the coefficient of kinetic friction, \( \alpha \) is the magnitude of the angular acceleration of the plate, \( T \) is the period of oscillation, and \( x \) is the position of the part with respect to the nodal line (i.e., the rotation axis).

Our device in Figure 1(a) is typically operated at \( \alpha \approx 100 \) rad/s\(^2\) to ensure a workspace of roughly 10 cm on each side the rotation axis in which the part remains in continuous contact with the plate. Assuming that \( T \approx 0.03 \) s and \( \mu \approx 0.4 \), the maximum asymptotic speed is only about 4.5 cm/s for a part located at \( x = \pm 10 \) cm. This is not a very strong squeeze field, especially in the context of typical high speed industrial part feeding. Assuming that \( \mu \) is fixed, the strength of the field can be increased only by increasing \( \alpha \) or \( T \). Both of these options have drawbacks. As \( \alpha \) is increased, the region of the plate shrinks for which the part remains in contact at all times. If the part is outside the contact region, (1) is no longer a correct model of part motion due to impacts between the plate and the part. Experimentally, we notice that this situation

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\(^2\)This paper has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the authors. This includes a multimedia .mp4 format movie clip which shows parts converging on nodal lines created by the one-DOF device in Fig. 1(a) and the PPDO. This material is 3 MB in size.
corresponds to chatter and irregular part motion—topics not addressed in this paper. Alternatively, as \( T \) is increased, the displacement of the plate during each cycle becomes large. Large plate motions are undesirable because they are generally more difficult to actuate and control; moreover, they nullify many simplifying assumptions about the theoretical model of the system that help give rise to analytical solutions.

In this paper we introduce a general class of plate motions that generate nodal lines and discuss how to optimize the motion to maximize part speed independent of \( T \) and \( \alpha \). Namely, we examine a class of two-DOF periodic plate motions consisting of simultaneous translation in the horizontal plane and rotation out of the horizontal plane. The translational and rotational components have identical frequencies, small amplitudes, and bang-bang acceleration profiles. Depending on the relative phase between the translational and rotational accelerations, we show that parts move towards the nodal line (LineSink fields), or away from the nodal line (LineSource fields). By controlling the relative phase between translation and rotation, and/or the magnitude of the translational acceleration, fields with much larger part speeds than (1) can be achieved for equivalent values of \( T \) and \( \alpha \). Moreover, many of these fields are less sensitive to variations in \( \mu \) while still retaining the proportionality between the speed and position of the part.

II. RELATED WORK

Part feeding along a single dimension using a vibratory surface has been well studied. All analysis can be performed in the (vertical) \( x-z \) plane in which the plate is modeled as a line with the part sitting on it (Figure 1(c)). When the plate motion is purely translational all parts experience the same linear and rotational accelerations, we show that parts move towards or away from a node in the \( x-z \) plane.

The remainder of this paper is laid out as follows: in Sections III–V we discuss the system dynamics giving rise to nodal lines and the corresponding formation of asymptotic velocity fields; in Sections VI–VIII we characterize the asymptotic velocity cycles for bang-bang plate motions and determine the conditions that give rise to LineSink and LineSource fields; in Sections IX–X we derive optimal asymptotic velocity fields and discuss some of their properties; and in Section XI we give simulation results and discuss the limitations of the asymptotic velocity model.

III. SYSTEM DYNAMICS

A. Plate Kinematics

Consider a rigid plate undergoing small amplitude vibration. We define three coordinate systems: a fixed inertial frame \( \mathcal{W} \), a local frame \( \mathcal{S} \) attached to the origin of the plate, and an inertial frame \( \mathcal{S'} \) instantaneously aligned with \( \mathcal{S} \) (Figure 2). The \( z \)-axis of \( \mathcal{W} \) is in the direction opposite the gravity vector. The \( z \)-axis of \( \mathcal{S} \) is always perpendicular to the plate surface.

In general, the configuration of the plate in \( \mathcal{W} \) is given by

\[
\begin{bmatrix}
R(t) & p(t) \\
0 & 1
\end{bmatrix} \in SE(3),
\]

where \( p : \mathbb{R} \to \mathbb{R}^3 \) and \( R : \mathbb{R} \to SO(3) \) are both \( C^1 \) functions of time. The motion of the plate is periodic with period \( T \), so that \( (p(t), R(t)) = (p(t+T), R(t+T)) \).

Without loss of generality, this paper focuses on nodal lines that are aligned with the \( y \)-axis of \( \mathcal{W} \); these are generated when a plate undergoes simultaneous translation along the \( x \)-axis of \( \mathcal{W} \) and rotation about the \( y \)-axis of \( \mathcal{S'} \). This type of plate motion, which we refer to as nodal line motion, has the form

\[
p(t) = \begin{bmatrix} p_x(t) \\ 0 \\ 0 \end{bmatrix},
\]

\[
R(t) = \begin{bmatrix}
\cos \psi(t) & 0 & -\sin \psi(t) \\
0 & 1 & 0 \\
\sin \psi(t) & 0 & \cos \psi(t)
\end{bmatrix},
\]

where \( p_x \) and \( \psi \) are periodic functions with identical frequencies that measure the respective translation and rotation of the plate. We assume that \( p_x \) and \( \psi \) are chosen such that the part does not stick to the plate, ensuring that kinetic friction acts on the part at all times.

We choose to operate the plate in a regime where the linear and angular displacements are very small during each cycle so that we may assume

\[
R \approx I, \quad p = 0,
\]

where \( I \) is the identity matrix. In other words, we assume the \( \mathcal{S} \) and \( \mathcal{W} \) frames are nearly coincident at all times. We discuss the consequences of this assumption in Section XI.

B. Part Kinematics

Let \( q = [x_S, y_S, 0]^T \) be a vector in \( \mathcal{S} \) to a point part with mass \( m \) in contact with the plate, and \( r = [x, y, z]^T \) be the corresponding vector in \( \mathcal{W} \) such that \( r = p + Rq \) (Figure 2).
The acceleration of the part in $\mathcal{W}$ is given by
\[ \ddot{\mathbf{r}} = \ddot{\mathbf{p}} + \omega \times \mathbf{R} \dot{\mathbf{q}} + \alpha \times \mathbf{R} \dot{\mathbf{q}} + 2 \omega \times \mathbf{R} \ddot{\mathbf{q}} + \mathbf{R} \dddot{\mathbf{q}}, \tag{6} \]
where $\omega$ and $\alpha$ are the respective angular velocity and angular acceleration of the plate in $\mathcal{W}$. If the plate undergoes the nodal line motion given by (2) and (3), then (6) reduces to
\[ \ddot{\mathbf{r}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \approx \begin{bmatrix} \ddot{p}_x - \dot{\psi}^2 x + \ddot{x}_S \\ 0 \\ -\ddot{\psi} x - 2 \dot{\psi} \dot{x}_S \end{bmatrix}, \tag{7} \]
under the assumptions of (4) and (5). We further simplify this to
\[ \ddot{\mathbf{r}} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \approx \begin{bmatrix} \ddot{p}_x + \ddot{x}_S \\ 0 \\ -\ddot{\psi} x \end{bmatrix}, \tag{8} \]
by assuming that the Coriolis and centripetal acceleration terms are negligible. These assumptions are valid as long as the part’s position satisfies:
\[ |x(t)| \ll \frac{\ddot{p}_x(t)}{\dot{\psi}(t)} \tag{9} \]
\[ |x(t)| \gg \frac{2\ddot{\psi}(t)\ddot{x}_S(t)}{\dot{\psi}(t)} \]. \tag{10} \]
We note that the periodic plate motion imposes the following two tight bounds: $|\ddot{\psi}(t)| \leq \frac{1}{2} \alpha T$ and $|\ddot{x}_S(t)| \leq \frac{1}{2} a_x T$, where $\alpha$ and $a_x$ are the magnitudes of the maximum angular and linear accelerations during a cycle. Thus, we can write (9) and (10) as
\[ |x(t)| \ll \frac{16|\ddot{p}_x(t)|}{\alpha^2 T^2} \tag{11} \]
\[ |x(t)| \gg \frac{a_x T^2}{4|\dot{\psi}(t)|}. \tag{12} \]
In most cases these assumptions are justified. For example, if $|\ddot{p}_x(t)| = a_x = 10 \text{ m/s}^2$, $|\dot{\psi}(t)| = \alpha = 100 \text{ rad/s}^2$, and $T = 0.03 \text{ s}$, (8) is a good approximation of the part’s acceleration as long as $|x(t)| \ll 17.7 \text{ m}$ and $|x(t)| \gg 0.0022 \text{ m}$; however, we note that as $a_x$ or $T$ grow large, Coriolis accelerations can become significant near the origin and (12) is no longer a valid assumption. We return to this point in Section XI.

\section*{C. Part Dynamics}
Assuming $\mathbf{R} \approx \mathbf{I}$, the motion of the part over the plate surface is governed exclusively by the friction force, which is always directed opposite $\mathbf{q} = [\ddot{x}_S, \ddot{y}_S, 0]$ (i.e., directed opposite the relative velocity vector between the part and the plate). The magnitude of the friction force—assuming a Coulombic friction model—is the product of the coefficient of kinetic friction $\mu$, and the magnitude of the normal force acting on the part. The magnitude of the normal force, in turn, is the product of the part’s mass and effective gravity. If $\mathbf{R} \approx \mathbf{I}$, the effective gravity can be expressed as
\[ g_{\text{eff}} = g + \ddot{z}, \]
where $g$ is the gravitational acceleration. The frictional force acting on the part can therefore be approximated as
\[ \mathbf{f}_a = -\mu m g_{\text{eff}} = -\mu m (g + \ddot{z}) \frac{\dot{\mathbf{q}}}{||\mathbf{q}||}. \tag{13} \]
which is a function of both position and time.
Because friction is the only force acting on the part tangential to the plate surface, Newton’s second law dictates that the part’s acceleration along the plate surface is
\[ \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = -\mu (g + \ddot{z}) \frac{\dot{\mathbf{q}}}{||\mathbf{q}||} \begin{bmatrix} \ddot{x}_S \\ \ddot{y}_S \end{bmatrix}. \tag{14} \]

For the case of nodal line motion given by (2) and (3), (14) becomes
\[ \ddot{x} = -\mu (g - \ddot{\psi} x) \frac{\ddot{x}_S}{||\ddot{x}_S||}, \tag{15} \]
where $\ddot{x}_S = \ddot{x} - \ddot{p}_x$.

We emphasize that the direction of the part’s acceleration along the surface is a function of the plate’s in-plane velocity via $\ddot{p}_x$, whereas the magnitude of the part’s acceleration along the surface is a function of the plate’s out-of-plane acceleration via $\dot{\psi}$. The plate’s in-plane translational motion constrains the direction of the part’s velocity to align with the translation axis; the plate’s out-of-plane rotational motion modulates the effective gravity experienced by the part. Since the effective gravity is always greater than $g$ on one side of the rotation axis and less than $g$ on the other, an asymmetry is introduced that creates distinct behavior on each side of the plate. Depending on the relative phase between the translational and rotational motions, the part is eventually drawn towards or away from the rotation axis.

\section*{IV. Bang-Bang Nodal Line Motion}
To simplify the subsequent analysis, we consider only bang-bang nodal line plate motions in which both the linear acceleration ($\ddot{p}_x$) and the angular acceleration ($\dot{\psi}$) are piecewise constant functions of time with identical frequencies. We define the plate motion as
\[ \ddot{p}_x(t) = \begin{cases} \alpha_x & 0 \leq t < T/2 \\ -\alpha_x & T/2 \leq t < T \end{cases}, \tag{16} \]
\[ \ddot{\psi}(t) = \begin{cases} -\alpha & 0 \leq t < \phi \\ \alpha & \phi \leq t < \phi + T/2 \\ -\alpha & \phi + T/2 \leq t < T \end{cases}. \tag{17} \]
where \( a_x \) and \( \alpha \) are both positive constants. The phase \( \phi \) is restricted to the interval \( 0 \leq \phi < \frac{1}{2}T \) (in Section VI-E we justify the exclusion of the interval \( \frac{1}{2}T \leq \phi < T \) due to symmetry). The phase can be interpreted as the time from the beginning of a cycle until \( \psi_1 \) first transitions from negative to positive. For a part on the right-hand side of the plate (i.e., a part with position \( x > 0 \)), \( \phi \) is the time at which the effective gravity first transitions from a high value to a lower one. For a part on the left-hand side of the plate, \( \phi \) is the time at which the effective gravity first transitions from a low value to a higher one.

In the case of bang-bang nodal line motion the acceleration of a part in the \( x \)-direction can be modeled as piecewise constant if we assume that its position does not change significantly during the cycle. For a given position on the plate, we can capture all the information about the part’s dynamics over the course of a cycle with a plot of velocity in the \( x \)-direction (of \( \mathcal{W} \)) vs. time (Figure 3). The plate’s velocity in the \( x \)-direction is the same for parts on the right and left-hand sides of the plate; however, the rotational motion of the plate about the origin causes the effective gravity experienced by parts on opposite sides of the plate to be out of phase—i.e., when the effective gravity is greater than \( g \) on the right-hand side of the plate it is less than \( g \) by the same amount on the left-hand side of the plate, and vice-versa. As can be seen from (15), the effective gravity is proportional to the acceleration of the part in the \( x \)-direction of \( \mathcal{W} \). Therefore, the effective gravity is also proportional to the slope of the curve representing part’s velocity in the \( x \)-direction over time. Thus, two parts beginning at rest, but located on opposite sides of the rotation axis, have different velocity profiles in the \( x \)-direction during the cycle. For the phase depicted in Figure 3, the part on the right-hand side of the plate (\( x > 0 \)) has a negative average velocity over the cycle. Conversely, the part on the left-hand side of the plate (\( x < 0 \)) has a positive average velocity over the cycle. It follows that the two parts are squeezed towards the rotation axis during this cycle.

V. ASYMPTOTIC VELOCITY

Numerical simulations suggest that all periodic plate motions drive point parts to a state in which their velocities at the beginning and end of each cycle are nearly identical. We refer to the state in which they are truly identical as the asymptotic velocity state. Parts generally reach a nearly asymptotic velocity state within a small number of cycles regardless of their initial position and velocity, provided that their initial speed is not substantially larger than the plate’s maximum speed.

At each position on the plate, we define the asymptotic velocity, \( \mathbf{v} \), as the average velocity of a point part during one cycle in an asymptotic velocity state:

\[
\mathbf{v}(r) = \frac{1}{T} \int_{t}^{t+T} \mathbf{v}(t) \, dt \quad \text{such that} \quad \dot{\mathbf{v}}(t) = \dot{\mathbf{v}}(t+T).
\]

This notion is an approximate one, as we assume that the part’s position does not change during the cycle.

The asymptotic velocities at each position form an asymptotic velocity field that provides a means of characterizing part motion.

VI. NODEL LINE ASYMPTOTIC VELOCITY CYCLES

A. The Four Asymptotic Velocity Cycles

Consider a part on the right-hand side of the plate. Recall that at \( t = \phi \) the effective gravity experienced by this part first becomes less than \( g \). The velocity of the plate at this time is \(-a_x(\frac{1}{4}T - \phi)\). At \( t = \phi + T/2 \) the effective gravity experienced by this part transitions back to a value larger than \( g \). The plate’s velocity at this later time is \( a_x(\frac{1}{4}T - \phi) \). We refer to these two time-velocity points as \( p_1 \) and \( p_2 \), respectively. In addition to marking the times at which the effective gravity changes, \( p_1 \) and \( p_2 \) also provide a convenient way of delineating the four types of asymptotic velocity cycles: (Figure 4).

AA Cycles pass above \( p_1 \) and above \( p_2 \).
AB Cycles pass above \( p_1 \) and below \( p_2 \).
BA Cycles pass below \( p_1 \) and above \( p_2 \).
BB Cycles pass below \( p_1 \) and below \( p_2 \).

Each of the four asymptotic velocity cycles can be represented by a set of eleven equations based on its general depiction in Figure 4. For example, the equations for the AA cycle are:

\[
\begin{align*}
\mathbf{v}_B &= \mathbf{v}_A - \mu(g + \alpha x)t_1 \\
\mathbf{v}_D &= \mathbf{v}_C + \mu(g - \alpha x)t_3 \\
\mathbf{v}_F &= \mathbf{v}_E - \mu(g + \alpha x)t_5 \\
\mathbf{v}_C &= -a_x \left( \frac{1}{4}T - \phi \right) + a_xt_2 \\
\mathbf{v}_D &= a_x \left( \frac{1}{4}T - \phi \right) + a_xt_4 \\
\frac{1}{2}T &= t_1 + t_5 \\
\phi &= t_1.
\end{align*}
\]
The system of equations given by (18) is valid regardless of whether the part is on the right or left-hand side of the plate.

B. Loss of Contact and Sticking

It is implicit in (18) that the part maintains permanent contact with the plate, or equivalently, $g_{\text{eff}} > 0$ for all time. Thus, in the model, the part’s position is limited to a maximum distance from the origin, $x_{\text{max}}$. Since the minimum value of $g_{\text{eff}}$ is $g - \alpha x_{\text{max}}$, the valid domain of the part’s position has the following bound:

$$|x| < x_{\text{max}} = \frac{g}{\alpha}. \quad (19)$$

It is also implicit in (18) that the part does not stick to the plate. Sticking does not occur if the plate always has a greater absolute acceleration than the part in the $x$-direction:

$$a_x > \mu(g + \alpha x). \quad (20)$$

Clearly the part is most likely to stick when its distance from the origin is $x_{\text{max}}$. Thus, (20) reduces to

$$a_x > 2\mu g. \quad (21)$$

C. Dimensionless Quantities

The asymptotic velocity analysis that follows is greatly simplified by introducing the following set of dimensionless quantities:

$$\tilde{x} = x \left(\frac{\alpha}{g}\right), \quad \tilde{a}_x = \frac{a_x}{\mu g}, \quad \tilde{\phi} = \frac{\phi}{T}$$

For example, the equations characterizing the AA cycle in (18) transform to

$$\tilde{v}_B = \tilde{v}_A - (1 + \tilde{x})\tilde{t}_1 \quad \tilde{v}_C = \tilde{v}_B - (1 - \tilde{x})\tilde{t}_2$$
$$\tilde{v}_D = \tilde{v}_C + (1 - \tilde{x})\tilde{t}_3 \quad \tilde{v}_E = \tilde{v}_D - (1 - \tilde{x})\tilde{t}_4$$
$$\tilde{v}_F = \tilde{v}_E - (1 + \tilde{x})\tilde{t}_5 \quad \tilde{v}_A = \tilde{v}_F$$
$$\tilde{v}_C = -\tilde{a}_x \left(\frac{1}{4} - \tilde{\phi}\right) + \tilde{a}_x \tilde{t}_2 \quad \tilde{v}_D = \tilde{a}_x \left(\frac{1}{4} - \tilde{\phi}\right) + \tilde{a}_x \tilde{t}_4$$
$$\frac{1}{2} = \tilde{t}_1 + \tilde{t}_5 \quad \frac{1}{2} = \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4$$
$$\tilde{\phi} = \tilde{t}_1$$

When expressed in dimensionless quantities.

The dimensionless asymptotic velocity, $\tilde{v}$, for any of the four cycles can be expressed as

$$\tilde{v} = \frac{1}{2} \left[ (\tilde{v}_A + \tilde{v}_B)\tilde{t}_1 + (\tilde{v}_B + \tilde{v}_C)\tilde{t}_2 + (\tilde{v}_C + \tilde{v}_D)\tilde{t}_3 + (\tilde{v}_D + \tilde{v}_E)\tilde{t}_4 + (\tilde{v}_E + \tilde{v}_F)\tilde{t}_5 \right]. \quad (23)$$
Δv is a minimum between \( \tau \) and is given by (Figure 5). In this case, the ratio of \( \Delta v \) always occurs before \( P \). Between \( \tau_1 \) and \( \tau_2 \), the difference between the velocities of the parts decreases from \( \Delta v_0 \) to \( \Delta v_2 \). This decrease is smallest when \( P_1 \) and \( P_2 \) both pass above or below \( v_0 \) and the effective gravity is a minimum between \( \tau_1 \) and \( \tau_2 \) (as shown in Figure 5). In this case, the ratio of \( \Delta v_2 \) to \( \Delta v_0 \) is maximized and is given by

\[
\frac{\Delta v_2}{\Delta v_0} = \frac{\tilde{a}_x - (1 - |\tilde{x}|)}{\tilde{a}_x + (1 - |\tilde{x}|)}.
\]

Between \( \tau_3 \) and \( \tau_4 \), the difference between the velocities further decreases from \( \Delta v_2 \) to \( \Delta v_4 \). This decrease is smallest when \( P_1 \) and \( P_2 \) both pass above or below \( v_2 \) and the effective gravity is a minimum between \( \tau_3 \) and \( \tau_4 \) (unlike what is shown in Figure 5). In this case, the ratio of \( \Delta v_3 \) to \( \Delta v_0 \) is maximized and is identical to (24). Thus, during any cycle,

\[
\frac{\Delta v_4}{\Delta v_0} = \left( \frac{\tilde{a}_x - (1 - |\tilde{x}|)}{\tilde{a}_x + (1 - |\tilde{x}|)} \right)^2 = \gamma.
\]

Because we have already assumed that \( |\tilde{x}| < 1 \) in order to maintain part-plate contact, (25) is a contractive mapping (i.e., \( \gamma < 1 \)) of the velocity difference between \( P_1 \) and \( P_2 \) during each cycle. Therefore, a unique asymptotic velocity exists at each location on the surface of a plate undergoing bang-bang nodal line motion.

If the part’s position never changed from its initial location, its velocity profile would always converge to the asymptotic velocity state at that location. Further, (25) suggests that the convergence occurs more rapidly for smaller values of \( \tilde{a}_x \). In most cases, it is indeed reasonable to assume that the part converges to a unique asymptotic velocity at each location on the plate (i.e., for a given initial position, \( \tilde{a}_x \) is small enough to allow the part to reach the asymptotic velocity state before its position has been significantly altered). However, plate motions with very large values of \( \tilde{a}_x \) may induce such a slow convergence to the asymptotic velocity state at the initial location of the part that we can no longer assume the part remains stationary. In this case, the part may not reach the asymptotic velocity state at its initial location (or any subsequent location other than the origin). We return to this point in Section XI.

### E. Periodicity of Asymptotic Velocity

Parts on the right and left-hand sides of the plate are subject to the same plate velocity in the \( x \)-direction, but experience effective gravities out of phase with each other. It follows that the asymptotic velocity cycle of a part located at position \( \tilde{x} \) on a plate with phase \( \tilde{\phi} \) is equivalent to the asymptotic velocity cycle of a part located at position \( -\tilde{x} \) on a plate with phase \( \tilde{\phi} + \frac{1}{2} \). This relationship naturally extends to asymptotic velocities and ensures that all possible asymptotic cycle behavior manifests itself on the interval \( 0 \leq \tilde{\phi} < \frac{1}{2} \) as previously asserted. Mathematically,

\[
\tilde{v}(\tilde{x}; \tilde{a}_x, \tilde{\phi}) = \tilde{v}(-\tilde{x}; \tilde{a}_x, \tilde{\phi} + \frac{1}{2} \).
\]

Another interpretation of (26) is that for a fixed value of \( \tilde{a}_x \), increasing \( \tilde{\phi} \) by \( \frac{1}{2} \) changes a LineSink field into a LineSource field of equal strength.

### F. Domain of Asymptotic Velocities

From (19), (21), and (26), we conclude that all unique asymptotic velocity behavior for which the part does not stick to or lose contact with the plate exists in the domain

\[
-1 < \tilde{x} < 1 \quad 2 < \tilde{a}_x < \infty \quad 0 \leq \tilde{\phi} < \frac{1}{2},
\]

which we refer to as \( D \). Every point in \( D \) can be mapped to a single asymptotic velocity cycle and its associated asymptotic velocity (Figure 7). We refer to asymptotic velocities directed towards (away from) the origin as attractive (repulsive). If for fixed values of \( \tilde{a}_x \) and \( \tilde{\phi} \) the asymptotic velocity is attractive (repulsive) at all positions, then the resulting field is a LineSink (LineSource).

### G. The AA Cycle

By solving the system of equations given by (22), the non-dimensional times of AA cycles are:

\[
\tilde{t}_1 = \tilde{\phi}
\]

\[
\tilde{t}_2 = \frac{2\tilde{a}_x - 4\tilde{a}_x\tilde{\phi} - 1}{4\tilde{a}_x} + \frac{1}{4(\tilde{x} - 1)}
\]

\[
\tilde{t}_3 = \frac{1}{2(1 - \tilde{x})}
\]
minimized when \( \tilde{x} \) is greater than \( \frac{1}{4} \) to make physical sense. From (29), \( \tilde{t}_3 \) is greater than \( \frac{1}{2} \) unless \( \tilde{x} < 0 \). Thus, AA cycles can only exist on the left-hand side of the plate. From (23), the asymptotic velocity of AA cycles is

\[
\tilde{v}_{AA}(\tilde{x}) = L_{AA} \tilde{x} + N_{AA} \frac{\tilde{x}}{\tilde{x} - 1},
\]

where

\[
L_{AA} = \frac{4\tilde{a}_x \tilde{\phi} - \tilde{a}_x + 1}{4\tilde{a}_x},
\]

\[
N_{AA} = \frac{1}{2} - \tilde{\phi}.
\]

Let us now examine under what conditions \( \tilde{v}_{AA} \) is positive. Given that \( \tilde{x} < 0 \), \( \tilde{v}_{AA}(\tilde{x}) > 0 \) as long as

\[
\tilde{\phi} < \frac{1}{4} - \frac{1}{4\tilde{a}_x} - \frac{\tilde{a}_x}{4(\tilde{x} - 1)}.
\]

Of all \( \tilde{x} \) belonging to \( \mathcal{D} \), the right-hand side of (35) is minimized when \( \tilde{a}_x = 2 \) and \( \tilde{x} = -1 \), conservatively implying that \( \tilde{v}_{AA} \) is positive if

\[
\tilde{\phi} < -\frac{3}{8}.
\]

In Section VIII-A we show that (36) is satisfied for all points in \( \mathcal{D} \) corresponding to AA cycles. Thus, all AA cycles have positive asymptotic velocities and exist at negative positions. In other words, AA cycles are attractive, driving parts on the left-hand side of the plate towards the center.

### H. The AB, BA, and BB Cycles

The asymptotic velocities for the other three types of cycle can be computed from (23) based on systems of equations analogous to (22). They have the following forms:

\[
\tilde{v}_{AB}(\tilde{x}) = L_{AB} \tilde{x} + C_{AB} \tilde{x}^3
\]

\[
\tilde{v}_{BA}(\tilde{x}) = L_{BA} \tilde{x} + C_{BA} \tilde{x}^3
\]

\[
\tilde{v}_{BB}(\tilde{x}) = L_{BB} \tilde{x} - N_{BB} \frac{\tilde{x}}{\tilde{x} + 1},
\]

where

\[
L_{AB} = \frac{8\tilde{a}_x^2 \tilde{\phi} + 16\tilde{a}_x^2 \tilde{\phi}^2 - 3\tilde{a}_x^2 + 8\tilde{a}_x \tilde{\phi} + 1}{8\tilde{a}_x^2}.
\]

\[
C_{AB} = \frac{-16\tilde{a}_x^2 \tilde{\phi}^2 - 8\tilde{a}_x \tilde{\phi} - 1}{8\tilde{a}_x^2}
\]

\[
L_{BA} = \frac{8\tilde{a}_x^2 \tilde{\phi} - 4\tilde{a}_x^2 - 16\tilde{a}_x \tilde{\phi}^2 + 16\tilde{a}_x \tilde{\phi}}{8\tilde{a}_x^2} + \frac{-\tilde{a}_x^2 - 8\tilde{a}_x \tilde{\phi} + 4\tilde{a}_x - 1}{8\tilde{a}_x^2}
\]

\[
C_{BA} = \frac{16\tilde{a}_x^2 \tilde{\phi}^2 - 16\tilde{a}_x^2 \tilde{\phi} + 4\tilde{a}_x^2 + 8\tilde{a}_x \tilde{\phi} - 4\tilde{a}_x + 1}{8\tilde{a}_x^2}
\]

\[
L_{BB} = L_{AA} = \frac{4\tilde{a}_x \tilde{\phi} - \tilde{a}_x + 1}{4\tilde{a}_x}
\]

\[
N_{BB} = N_{AA} = \frac{\tilde{a}_x}{4}.
\]

As a point of comparison, we note that the dimensionless form of (1) is

\[
\tilde{v}(\tilde{x}) \approx -\frac{3}{8} \tilde{x},
\]

which is a special case of (37) when \( \tilde{\phi} = 0 \) and \( \tilde{a}_x \gg 1 \).

### I. Oddness of Asymptotic Velocities

The AB and BA asymptotic velocities are odd functions of \( \tilde{x} \):

\[
\tilde{v}_{AB}(-\tilde{x}) = -\tilde{v}_{AB}(\tilde{x})
\]

\[
\tilde{v}_{BA}(-\tilde{x}) = -\tilde{v}_{BA}(\tilde{x}).
\]

Similarly, the AA and BB asymptotic velocities are odd with respect to each other:

\[
\tilde{v}_{AA}(-\tilde{x}) = -\tilde{v}_{BB}(\tilde{x}).
\]

Since AA cycles only exist for \( \tilde{x} < 0 \) and always have positive asymptotic velocity, (49) implies that BB cycles only exist for \( \tilde{x} > 0 \) and always have negative asymptotic velocity. Thus, all points in \( \mathcal{D} \) corresponding to BB cycles are also attractive, driving parts on the right-hand side of plate towards the center.

### VII. Transition Surfaces

For fixed values of \( \tilde{a}_x \) and \( \tilde{\phi} \), a part may transition from one type of asymptotic velocity cycle to another as it changes its position on the plate. The transitions can be visualized as surfaces in \( \mathcal{D} \) separating the four asymptotic cycles into distinct regions (Figure 7). Knowing which cycle goes in which region allows us to compute the asymptotic velocity of any point in \( \mathcal{D} \) using (32), (37), (38), or (39).

Each transition surface is based on a transition cycle that passes through \( p_1 \) and/or \( p_2 \). Studying the properties of the transition cycle allows us to construct the transition surface. For fixed values of \( \tilde{a}_x \) and \( \tilde{\phi} \), the transition surface reduces to a
transition point that marks the position on the plate separating one type of cycle from the other.

There are five transition surfaces, which are summarized mathematically in Table I. We present a thorough analysis of the AA→AB transition surface. The results pertaining to the other transition surfaces are stated without proof as the mathematics is very similar.

A. The AA→AB Transition Surface

The transition cycle between AA cycles and AB cycles passes above $p_1$ and through $p_2$ (Figure 6(a)). All cycles above this one are of type AA and all cycles below are of type AB. The nine dimensionless equations governing this cycle are:

\[
\begin{align*}
\tilde{v}_B &= \tilde{v}_A - (1 + \tilde{x})\tilde{t}_1 \\
\tilde{v}_D &= \tilde{v}_C + (1 - \tilde{x})\tilde{t}_3 \\
\tilde{v}_E &= \tilde{v}_D - (1 + \tilde{x})\tilde{t}_4 \\
\tilde{v}_A &= \tilde{v}_E \\
\tilde{v}_C &= -\tilde{a}_x \left( \frac{1}{4} - \tilde{\phi} \right) + \tilde{a}_x \tilde{t}_2 \\
1 &= \tilde{a}_x \tilde{t}_1 + \tilde{t}_4 \\
\tilde{\phi} &= \tilde{t}_1.
\end{align*}
\]

Solving this system for $\tilde{x}$ gives:

\[
\tilde{x} = \frac{4\tilde{a}_x \tilde{\phi} - \tilde{a}_x + 1}{4\tilde{a}_x \phi + 1}.
\] (50)

Only a portion of the surface given by (50) actually corresponds to the AA→AB transition surface. This is because the AA→AB transition surface can only contain points with negative values of $\tilde{x}$ (since we have already shown that AA cycles only exist for $\tilde{x} < 0$). Thus, the numerator of (50) must be negative, imposing the following restriction on $\tilde{\phi}$:

\[
\frac{1}{8} - \frac{1}{4\tilde{a}_x} < \tilde{\phi} < \frac{1}{4} - \frac{1}{4\tilde{a}_x}.
\]

In summary, the AA→AB transition surface is given by

\[
\tilde{x}_{\text{AA→AB}} = \frac{4\tilde{a}_x \tilde{\phi} - \tilde{a}_x + 1}{4\tilde{a}_x \phi + 1},
\] (51)

where $\tilde{a}_x$ and $\tilde{\phi}$ must satisfy

\[
\frac{1}{8} - \frac{1}{4\tilde{a}_x} < \tilde{\phi} < \frac{1}{4} - \frac{1}{4\tilde{a}_x}.
\] (52)

As points in $\mathcal{D}$ approach the AA→AB transition surface it is trivial to show that the asymptotic velocities of the AA and AB cycles converge to the same value. This is important, as it ensures that the asymptotic velocity remains continuous when AA cycles transition to AB cycles. The asymptotic velocity remains continuous when crossing any of the other four transition surfaces as well.

VIII. COMPOSITION OF $\mathcal{D}$ AND $\mathcal{D}'$

The five transition surfaces divide $\mathcal{D}$ into four distinct regions. Each region contains one type of asymptotic velocity cycle, as seen in Figure 7.
If we consider what can happen to a part over all locations \((-1 < \tilde{x} < 1)\) for fixed values of \(\tilde{a}_x\) and \(\phi\), Figure 7 indicates that there are five possible scenarios:

1) For \(0 \leq \phi < \frac{1}{8} - \frac{1}{4a_x}\), the part is always in an AB asymptotic cycle. We refer to this region of \(D\) as the AB non-transition region.

2) For \(\frac{1}{8} - \frac{1}{4a_x} < \phi < \frac{1}{4} - \frac{1}{4a_x}\), the part can be in an AA, AB, or BB asymptotic cycle. We refer to this region of \(D\) as the AA→AB→BB transition region.

3) For \(\phi = \frac{1}{4} - \frac{1}{4a_x}\), the part can be in an AA or BB asymptotic cycle. We refer to this region of \(D\) as the AA→BB transition region.

4) For \(\frac{1}{4} - \frac{1}{4a_x} < \phi < \frac{3}{4} - \frac{1}{4a_x}\), the part can be in an AA, BA, or BB asymptotic cycle. We refer to this region of \(D\) as the AA→BA→BB transition region.

5) For \(\frac{3}{4} - \frac{1}{4a_x} < \phi < \frac{1}{2}\), the part is always in a BA asymptotic cycle. We refer to this region of \(D\) as the BA non-transition region.

The projections of the five regions described above onto the \(\tilde{a}_x\)-\(\phi\) plane of \(D\) are shown in Figure 8. We refer to this projected \(\tilde{a}_x\)-\(\phi\) space as \(D'\).

### A. The Three Transition Regions

Let us now examine which of the five regions in \(D'\) correspond to LineSink fields and which correspond to LineSource fields. To do so, we return to the question of whether AA cycles always have positive asymptotic velocity. Since points corresponding to AA cycles must have phases less than \(\frac{3}{8} - \frac{1}{4a_x}\) (i.e., the upper phase limit of the AA→BA transition surface), the restriction on phase given by (36) is clearly satisfied. It follows that all points corresponding to AA cycles have positive asymptotic velocities and are attractive. From (49) we infer that all points corresponding to BB cycles have negative asymptotic velocities and are therefore also attractive. It immediately follows that the AA→BB transition region corresponds to LineSink fields that are symmetric about the nodal line.

We can also conclude that points corresponding to AB and BA cycles in the AA→AB→BB and AA→BA→BB transition regions are attractive. This is because the asymptotic velocity is continuous through the transition points and the asymptotic velocity of AB and BA cycles is an odd function of position. It follows that both the AA→AB→BB transition region and AA→BA→BB transition region correspond to LineSink fields that are symmetric about the nodal line.

### B. The AB Non-Transition Region

In this section we prove that the AB non-transition region is composed of LineSink fields exhibiting a nearly linear relationship between asymptotic speed and distance from the nodal line.

By inspection of (41), \(C_{AB} < 0\) everywhere in \(D'\), indicating that the cubic term of (37) always drives parts towards the center of the plate. It follows that LineSink fields are generated wherever \(L_{AB} < 0\). It is straightforward to show that this occurs only when \(\phi\) satisfies

\[
\frac{1}{4} \tilde{a}_x - \frac{1}{4a_x} \sqrt{1 + \frac{a_x^2}{4}} < \phi < \frac{1}{4} \tilde{a}_x - \frac{1}{4a_x} + \frac{1}{4} \sqrt{1 + \frac{a_x^2}{4}}.
\]

Since the entire AB non-transition region is contained within (54), it must be entirely composed of LineSink fields.

To analyze the relationship between asymptotic speed and position, we define the Q-factor as the ratio of the linear to cubic contribution within the asymptotic velocity:

\[
Q_{AB} = \frac{L_{AB}}{C_{AB}}.
\]

The larger the Q-factor, the more the linear term dominates. Within the AB non-transition region, the minimum value of \(Q_{AB}\) is 11, which occurs at \(\tilde{a} = 2\), \(\phi = 0\). Further,

\[
\lim_{\tilde{a}_x \to \infty} Q_{AB} = \infty.
\]

In general, \(Q_{AB} \gg 1\), and it is reasonable to classify the entire AB non-transition region as LineSink fields governed by the linear relationship

\[
\tilde{v}_{AB}(\tilde{x}) \approx L_{AB}\tilde{x}.
\]
linear term dominates, ensuring that at all positions parts move towards the center. This subregion corresponds to
\[ \frac{3}{8} - \frac{1}{4a_x} < \phi < \frac{1}{2} - \frac{3}{8a_x}. \]

Let us focus on subregion 2 where there are ambiguous transitions from LineSource to LineSink fields. We note that the upper and lower limits of (56) are virtually indistinguishable, implying that subregion 2 is nearly one-dimensional. Indeed, the maximum difference between the upper and lower limits of (56) is less than 0.004 (i.e., less than four thousandths of a period), occurring when \( \tilde{a}_x = 2 \). Thus, we arbitrarily choose the lower limit of (56) to mark the transition phase \( \phi_o \) separating LineSource from LineSink fields:
\[ \tilde{\phi}_o = \frac{1}{2} - \frac{3}{8a_x}. \]
At \( \tilde{\phi} = \tilde{\phi}_o \), the asymptotic velocity is identically zero at the outer edges of the plate (i.e., at \( |\tilde{x}| = 1 \)). Though it is possible that the asymptotic velocity reaches large values at other positions, the maximum dimensionless asymptotic speed at \( \tilde{\phi} = \tilde{\phi}_o \) is only
\[ \frac{1}{48a_x^3\sqrt{3}}. \]

Fig. 8. The three transition and two non-transition regions of \( \mathcal{D}' \). The BA non-transition region is subdivided by \( \phi_o \) (dotted), which is the phase that yields no part motion. Points above \( \phi_o \) correspond to LineSource fields and points below \( \phi_o \) correspond to LineSink fields. There is a linear relationship between asymptotic speed and distance from the nodal line in both non-transition regions.

C. The BA Non-Transition Region

In this section we show that both LineSink and LineSource fields can be generated in the BA non-transition region. In both cases the asymptotic speed is approximately proportional to the part’s distance from the nodal line.

By inspection of (43), \( C_{BA} \geq 0 \) everywhere in \( \mathcal{D}' \), indicating that the cubic term of (38) drives parts away from the center of the plate or makes no contribution to part motion. Since \( L_{BA} \) can be positive or negative, we divide the BA non-transition region into three subregions based on the Q-factor:

1) \( Q_{BA} \geq 0 \): LineSource fields.
Both the linear and cubic terms of (38) drive parts away from the center of the plate. This subregion corresponds to
\[ \frac{1}{2} - \frac{1}{4a_x} + \frac{1}{4} \tilde{a}_x - \frac{\sqrt{1 + \tilde{a}_x^2}}{4} \leq \phi < \frac{1}{2}. \]

2) \(-1 \leq Q_{BA} < 0\): Transition from LineSource to LineSink fields.
The linear term of (38) drives parts towards the center of the plate in opposition to the cubic term. Since the cubic term dominates, parts near the center of the plate move towards the center whereas parts farther away move away from the center; LineSink and LineSource classifications are ambiguous, although the behavior is closer to a LineSink field as \( Q_{BA} \) approaches \(-1\) and closer to a LineSource field as \( Q_{BA} \) approaches 0. This subregion corresponds to
\[ \frac{1}{2} - \frac{3}{8a_x} \leq \phi < \frac{1}{2} - \frac{1}{4a_x} + \frac{1}{4} \tilde{a}_x - \frac{\sqrt{1 + \tilde{a}_x^2}}{4}. \] (56)

3) \( Q_{BA} < -1 \): LineSink fields.
The linear term of (38) drives parts towards the center of the plate in opposition to the cubic term; however, the

\[ \frac{1}{2} - \frac{1}{4a_x} + \frac{1}{4} \tilde{a}_x - \frac{\sqrt{1 + \tilde{a}_x^2}}{4} \leq \phi < \frac{1}{2}. \]
IX. Optimal Nodal Line Motion

Let us define an optimal trajectory through $D$ as one which maximizes the asymptotic speed at all positions, possibly subject to constraints. The two problems we address are:

1) Optimal Linear LineSink Field: Given a fixed value of $	ilde{a}_x$ and the constraint that $\tilde{v}(\tilde{x}) \propto \tilde{x}$, find the phase $\phi_{\text{linear}}^{\text{opt}}$ that maximizes $|\tilde{v}(\tilde{x})|$ for all $\tilde{x}$ and the corresponding asymptotic velocity field, $\tilde{v}_{\text{linear}}^{\text{opt}}$.

2) Optimal Nonlinear LineSink Field: Given a fixed value of $\tilde{a}_x$, find the phase $\phi_{\text{opt}}$ that maximizes $|\tilde{v}(\tilde{x})|$ for all $\tilde{x}$ and the corresponding asymptotic velocity field, $\tilde{v}_{\text{opt}}$.

A. Optimal Linear LineSink Field

All linear LineSink fields are confined to the AB or BA non-transition regions of $D'$ and have the form of (55) or (58). In the AB non-transition region, the phase that maximizes part speed towards the origin is the one that minimizes $L_{AB}$. Analogously, the phase that maximizes part speed towards the origin in the BA non-transition region minimizes $L_{BA}$. In either case, the phase is not a function of $\tilde{x}$ since neither $L_{AB}$ nor $L_{BA}$ depend on position.

In the AB non-transition region, $L_{AB}$ decreases as $\phi$ increases for all $\tilde{a}_x$. Thus, the optimal phase within the AB non-transition region is the largest one possible:

$$\phi_{AB}^{\text{opt}} = \frac{3}{8} - \frac{1}{4\tilde{a}_x}.$$ 

At this phase, the linear approximation of the asymptotic velocity is

$$\tilde{v}_{AB}^{\text{opt}}(\tilde{x}) \approx -\frac{1}{8} \left( \tilde{a}_x - \frac{3}{4} \right) \tilde{x}.$$ 

In the BA non-transition region, $L_{BA}$ decreases as $\phi$ decreases for all $\tilde{a}_x$. Thus, the optimal phase within the BA non-transition region is the smallest one possible:

$$\phi_{BA}^{\text{opt}} = \frac{1}{8} - \frac{1}{4\tilde{a}_x}.$$ 

At this phase, the linear approximation of the asymptotic velocity is

$$\tilde{v}_{BA}^{\text{opt}}(\tilde{x}) \approx -\frac{1}{8} \left( \tilde{a}_x + \frac{3}{4} \right) \tilde{x}.$$ 

Since $\tilde{v}_{BA}^{\text{opt}}$ clearly corresponds to larger speeds than $\tilde{v}_{AB}^{\text{opt}}$ at all positions, we conclude that

$$\phi_{\text{linear}}^{\text{opt}} = \phi_{BA}^{\text{opt}} = \frac{1}{8} - \frac{1}{4\tilde{a}_x}.$$ 

The corresponding asymptotic velocity field is

$$\tilde{v}_{\text{linear}}^{\text{opt}}(\tilde{x}) = \tilde{v}_{BA}^{\text{opt}}(\tilde{x}) = -\frac{1}{8} \left( \tilde{a}_x + \frac{3}{4} \right) \tilde{x}.$$ 

B. Optimal Nonlinear LineSink Field

The second problem is more general than the first since we place no restrictions on the trajectory through $D$ other than requiring $\tilde{a}_x$ to remain fixed. We expect the solution to correspond to LineSink fields with greater speeds at all positions than (60).

When $\tilde{a}_x$ is fixed, the optimal solution is a path confined to an $\tilde{x}$-$\phi$ slice of $D$, such as shown in Figure 9. The search for the optimal trajectory can be immediately simplified in three ways: all points on the slice that are repulsive can be disregarded (since we are looking for an optimal LineSink field), all points on the slice corresponding to BA cycles can be disregarded (since for all positions we have already shown that AB cycles can generate larger speeds than BA cycles), and all points on the slice with positions $\tilde{x} > 0$ can be disregarded (due to the symmetry of the asymptotic velocity about the origin of the plate). The remaining points all have positive asymptotic velocities and consist of either AA or AB cycles.

First, let us consider the AA region of the slice in which the asymptotic velocity is given by (32). From (34), the nonlinear
coefficient $N_{AA}$ of the asymptotic velocity is always positive, indicating that the nonlinear term of (32) drives parts towards the center of the plate. The linear term also drives parts towards the center of the plate if $L_{AA} < 0$, which only occurs in the AA$\rightarrow$AB$\rightarrow$BB transition region. Within this region, the optimal phase is the one that minimizes $L_{AA}$. From (33), $L_{AA}$ is minimized when $\phi$ is as small as possible, which occurs along the AA$\rightarrow$AB transition surface. Thus, solving (51) for $\phi$ gives the optimal phase for AA cycles:

$$\hat{\phi}_{AA}^{opt}(\tilde{x}) = \frac{1 - \tilde{a}_x - \tilde{x}}{4\tilde{a}_x (\tilde{x} - 1)}.$$  

Let us now look for the optimal phase in the AB region of the plate. We have already shown that within the AB non-transition region the optimal phase is the largest one possible. This result extends to all AB cycles—i.e., the asymptotic velocity for AB cycles at all positions increases with phase. The maximum phase for AB cycles lies along the AA$\rightarrow$AB transition surface. Thus, on the left-hand side of the plate, $\hat{\phi}_{AB}^{opt} = \hat{\phi}_{AA}^{opt} = \hat{\phi}_{AB}^{opt}$. Because $\tilde{x}_{BB\rightarrow AB} = -\tilde{x}_{AA\rightarrow AB}$ it follows that the optimal phase is:

$$\hat{\phi}_{opt}(\tilde{x}) = \begin{cases} 
1 - \tilde{a}_x - \tilde{x} \quad & -1 < \tilde{x} < 0; \\
1 - \tilde{a}_x + \tilde{x} \quad & 0 < \tilde{x} < 1.
\end{cases}$$  

The optimal phase as a function of position is plotted in Figure 9 for the case when $\tilde{a}_x = 5$. For other values of $\tilde{a}_x$ the plot would look similar, with the optimal phase always lying on the AA$\rightarrow$AB and BB$\rightarrow$AB transition surfaces.

Substituting (62) into (37) gives the optimal nonlinear asymptotic velocity field:

$$\tilde{v}_{opt}^{non}(\tilde{x}) = \begin{cases} 
\tilde{x}^2 + \frac{\tilde{a}_x}{4} \tilde{x} - 1 \quad & -1 < \tilde{x} < 0; \\
\tilde{x}^2 - \frac{\tilde{a}_x}{4} \tilde{x} + 1 \quad & 0 < \tilde{x} < 1.
\end{cases}$$  

Optimal linear and nonlinear asymptotic velocity fields for two values of $\tilde{a}_x$ are shown in Figure 10. For comparison, we have also plotted the asymptotic velocity given by (46) which corresponds to $\tilde{a}_x = 0$ and $\tilde{a}_x \gg 1$.

A control system is needed to implement $\tilde{v}_{opt}^{non}(\tilde{x})$ on a physical system—the position of the part must be sensed and the phasing of the plate’s translational and rotational accelerations must be actuated. Additionally, the plate’s motion is no longer periodic because the phase changes as the part moves. It is currently unknown whether the part will continue to undergo approximate asymptotic velocity cycles for non-periodic plate motion. Nonetheless, (63) provides a theoretical upper bound on asymptotic speed for the entire class of bang-bang nodal line motion discussed in this paper.

In the absence of a control system to sense the location of the part, (62) indicates that the fixed phase generating the strongest field in the vicinity of the plate’s origin is $\tilde{\phi} = \frac{1}{4} - \frac{1}{4\tilde{a}_x}$, which approaches $\frac{1}{4}$ when $\tilde{a}_x \gg 1$.

\section{Part Speed and Sensitivity}

In this section we reintroduce dimensional quantities to address two practical questions: how fast do parts move in optimized LineSink fields, and how sensitive are LineSink fields to variations in $\mu$? Throughout this section we assume $T = 0.03$ s and $g = 9.8$ m/s$^2$.

\subsection{Part Speed}

For the case $\tilde{a}_x = 15$ shown in Figure 10, let us choose $\tilde{a}_x = 60$ m/s$^2$, $\mu = 0.4$ and $\alpha = 100$ rad/s$^2$. Then the maximum asymptotic speeds for the linear and nonlinear solutions are more than five times greater than (46) at all locations. Stronger fields with even greater asymptotic speeds can be obtained by further increasing $\tilde{a}_x$. When $\tilde{a}_x \gg 1$, both the optimal linear and nonlinear asymptotic velocities scale nearly linearly with $\tilde{a}_x$.

\subsection{\(\mu\)-Independent Fields}

In general, $\mu$ is a parameter in the asymptotic velocity field. In extreme cases, such as when $\tilde{\phi} = 0$, the asymptotic velocity at every position on the plate is almost directly proportional to $\mu$, as in (1). In other cases, such as when $\tilde{\phi} = \tilde{\phi}_{opt}^{linear}$ or $\tilde{\phi} = \tilde{\phi}_{opt}^{non}$, the asymptotic velocity becomes virtually independent of $\mu$ for large values of $\alpha$ (Figure 11). A very useful consequence of this is that all parts with identical geometries and mass distributions should behave almost identically within the field after an initial transient, regardless of variations in surface texture. Asymptotic velocity fields that are independent of $\mu$ are also beneficial from a practical point of view since precisely measuring the coefficient of friction is notoriously difficult due to fluctuations caused by wear, dust, and dirt.

\section{Simulation Results}

For a wide range of practical parameters (e.g, $\tilde{a}_x < 50$ m/s$^2$, $\mu > 0.05$, $T < 0.1$ s) numerical simulations show that the
simplified asymptotic velocity model accurately predicts the motion of the part after a short transient (Figure 12(a)-(c)). However, if $\mu$ becomes very small the duration of the transient tends to increase (compare Figure 12(d) to Figure 12(a)). The fact that it requires more time for the part to get up to asymptotic speed is because the part’s instantaneous acceleration is proportional to $\mu$.

For large values of $T$ the part’s motion is better approximated by the asymptotic velocity computed numerically without assuming simplified dynamics, as in Figure 12(e). This is most likely because large values of $T$ nullify the assumptions that $p \approx 0$ and $R \approx I$, as well as increase Coriolis and centripetal accelerations.

By far the most significant deviation between the simulated part motion and the simplified asymptotic velocity model occurs when $a_x$ is very large, as in Figure 12(f). There are four potential explanations: (1) the displacement of the plate becomes substantial, invalidating our assumption that $p \approx 0$; (2) Coriolis accelerations become substantial, invalidating our assumption that $||x|| \gg \frac{a_x T^2}{4||p||}$; (3) the translational velocity of the part may become substantial, invalidating our assumption that the change in the position of the part during a cycle is negligible; (4) the part may not converge quickly enough to the asymptotic velocity state corresponding to its initial location as suggested by (25), possibly invalidating our assumption that the change in the position of the part during the convergence is negligible. We can most likely rule out points (1) and (2) because the numerically computed asymptotic velocity incorporating the full dynamics so closely matches the asymptotic velocity from the simplified model. In reference to point (3), simulations show that for very large values of $a_x$ the speed of the part must decrease so much over even a single cycle that it is no longer valid to assume the velocity at the beginning and end of each cycle are approximately equal. Furthermore, simulations show that the rate of convergence to the asymptotic state tends to decrease for large values of $a_x$.

Because large values of $a_x$ generate large average part speeds over a cycle, it becomes unreasonable to assume that the part does not displace from its initial location during the period of convergence.

XII. Conclusions

We have shown that simultaneous periodic small amplitude translation and rotation of a rigid plate induces parts on the plate to move towards or away from a nodal line aligned with the rotation axis. In most cases, the motion of an isolated point part can be accurately described using the asymptotic velocity model with simplified dynamics. When the phasing between the translational and rotational accelerations is properly timed, strong asymptotic velocity fields that are nearly independent of the coefficient of friction can be created.

The analytical results presented in this paper are for bang-bang plate accelerations; however, simulations and the accompanying videos show that sinusoidal plate motions can also induce velocity fields with nodal lines. These fields are not as strong as those created with bang-bang motion, but they exhibit similar dependencies on $\mu$, $T$, $\phi$, $a_x$, and $\alpha$.

In future work, we plan to build a new PPOD that can better approximate bang-bang motion with large enough translational accelerations to experimentally test the accuracy of the asymptotic velocity model of part motion. We also plan to investigate the behavior of parts with planar extent. The motion of planar parts on the PPOD appears to be closely related to the asymptotic velocity fields derived for point parts. A fundamental question is whether equilibrium configurations of two-dimensional parts can be determined using asymptotic velocity fields derived for point parts.

References


Centripetal accelerations are fully modeled and it is not assumed that transient period. For larger values of $a$, as in (a)–(e), the part approaches the velocity predicted by the simplified asymptotic velocity model after a short transient period. For larger values of $a$, as in (f), the part does not approach the predicted asymptotic velocity until it settles at the origin. Only in (e), where the period is large, does the asymptotic velocity derived from the full dynamics differ noticeably from the simplified asymptotic velocity model.

Fig. 12. Each plot shows a simulation of a point part beginning from rest at $\tilde{x} = 0.9$ and the corresponding asymptotic velocities based on the simplified analytical model and a full dynamics numerical model. The simulated part motion also incorporates the full dynamics of the system (i.e., Coriolis and centripetal accelerations are fully modeled and it is not assumed that $p = 0$ or $R = I$). Each asterisk corresponds to the average velocity over one cycle of plate motion. For smaller values of $a$, as in (a)–(e), the part approaches the velocity predicted by the simplified asymptotic velocity model after a short transient period. For larger values of $a$, as in (f), the part does not approach the predicted asymptotic velocity until it settles at the origin. Only in (e), where the period is large, does the asymptotic velocity derived from the full dynamics differ noticeably from the simplified asymptotic velocity model.


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