PROBLEMS AND SOLUTIONS

EDITED BY MURRAY S. KLAMKIN

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All problems and solutions should be sent, typewritten in duplicate, with complete address, to Murray S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta T6G 2G1 Canada. An asterisk placed beside a problem number indicates that the problem was submitted without solution. Proposers and solvers whose solutions are published will receive 5 reprints of the corresponding problem section. Other solvers will receive just one reprint provided a self-addressed stamped (U.S.A. or Canada) envelope is enclosed. Proposers and solvers desiring acknowledgment of their contributions should include a self-addressed stamped postcard. (No stamps necessary for outside the U.S.A. and Canada.) Solutions should be received by September 30, 1991.

PROBLEMS

Variational Problem for Overlapping Laser Pulses

Problem 91-6*, by C. E. CARROLL (University of Rochester).

Transfer of an atom or molecule from its ground state to any desired higher state is often accomplished by applying a succession of laser pulses, because use of only one laser pulse may be difficult or impossible. What shape should the laser pulses have? In a simplified model of this process, the atom or molecule is a three-state quantum system driven by the oscillating electric fields of two laser pulses that overlap in time. We solve the time-dependent Schrödinger equation

\[
\frac{d}{dt} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & A_1(t) & 0 \\ A_1(t) & 0 & A_2(t) \\ 0 & A_2(t) & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},
\]

where \( a_1, a_2, a_3 \) are probability amplitudes. We assume that \( A_1(t) \) and \( A_2(t) \) are real; these two arbitrary functions are proportional to the amplitudes of the two applied oscillating fields. Since we demand complete transfer of the occupation probability from state one to state three, we have the following initial and final conditions:

\[
|a_1| = 1, \quad a_2 = a_3 = 0 \quad \text{for} \quad t = -\infty, \\
a_1 = a_2 = 0, \quad |a_3| = 1 \quad \text{for} \quad t = +\infty.
\]

There are many sets of functions \( a_1, a_2, a_3, A_1(t), A_2(t) \) that satisfy (1) and (2). We cannot use this simple model to find the minimum energy or minimum time for transfer from state one to state three, because \( A_1(t) \) and \( A_2(t) \) could be replaced by \( (1/k)A_1(kt) \) and \( (1/k)A_2(kt) \), where \( k \) is any positive constant. However, some calculations for this model involve eigenvalues of the matrix that appears in (1). We consider the time integral of an eigenvalue or a difference of eigenvalues; we want to minimize

\[
I_{\text{sort}} = \int_{-\infty}^{\infty} \{[A_1(t)]^2 + [A_2(t)]^2\}^{1/2} dt.
\]

It is conjectured that

\[
I_{\text{sort}} \geq \frac{1}{2} \pi \sqrt{3}
\]
with equality if
\[
A_1(t) = \frac{3^{1/2} \pi}{2T} \cos \left( \frac{\pi t}{2T} \right) \quad \text{and} \quad A_2(t) = \frac{3^{1/2} \pi}{2T} \sin \left( \frac{\pi t}{2T} \right) \quad \text{for } 0 \leq t \leq T,
\]
and
\[A_1(t) = A_2(t) = 0\] otherwise.

Prove or disprove.

REFERENCE


A Probability Integral

**Problem 91-7**, by LARRY G. BLAINE (Plymouth State College).

Let \( I(X) = \int_a^b \arccos(X - \cos \theta) \, d\theta \)

where

(i) \( a = \arccos(\lambda) \) and \( b = \frac{\pi}{2} \) if \( 0 < \lambda \leq 1 \),

(ii) \( a = 0 \) and \( b = \arccos(\lambda - 1) \) if \( 1 < \lambda < 2 \).

It is possible to compute \( I(X) \) in closed form?

This integral arises in the following elementary two-dimensional barrier penetration model: a particle hits the edge of a barrier strip of thickness \( \lambda \), \( 0 < \lambda < 2 \), and bounces for 1 unit in a random direction. If the particle is then inside the strip it bounces one more time in the same way. It is easily shown that \( I(\lambda)/\pi^2 \) is the probability that the particle is inside the strip at the end of the first bounce and has passed through after the second.

An Upper Bound

**Problem 91-8**, by PHILIP KORMAN (University of Cincinnati).

For all \( t \geq 1 \) the function \( F(t) \) is continuous, nonnegative, and satisfies

\[
tF(t) \leq c + \int_1^t F(t) \, dt
\]

where \( c \) is a positive constant. Show that \( F(t) \leq c \) for all \( t \geq 1 \).

Even Minus Odd Involutions in the Symmetric Group

**Problem 91-9**, by FRANK SCHMIDT (Bryn Mawr College) AND RODICA SIMION (George Washington University).

Let \( a_n \) count the excess of even over odd involutions in the symmetric group \( S_n \). The sequence \( (a_n) \) can be computed from the recurrence relation \( a_n = a_{n-1} - \)
(n - 1)a_{n-2} starting with a_0 = a_1 = 1. The exponential generating function is
\[ \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \exp \left( x - \frac{x^2}{2} \right). \]

(a) Show that a_n ≠ 0 for n > 2. (b) Show that |a_n| \to \infty as n \to \infty.

Editorial note. There are elementary proofs of both (a) and (b), but the problem of proving a "good" lower bound for the growth of |a_n| as n \to \infty may require more effort. The proposer remarks that his purpose in posing this problem is to elicit alternate proofs of (a) and "good" lower bounds for (b).

Convexity for the Self-Circumference of a Unit Disk

Problem 91-10*, by IGNACE I. KOLODNER (Carnegie-Mellon University).

The unit disk in the two-dimensional normed space \( l^p(2) \) is given parametrically by
\[ x(t) = \cos^{2s} t, \quad y(t) = \sin^{2s} t, \]
where \( s = 1/p, \) \( t \in [0, 2\pi), \) \( p \in [1, \infty). \) Let \( F(s) \) be the \( l^{1/s}(2) \)-length of the boundary (the "self-circumference") of the unit disk of \( l^{1/s}(2) \), i.e.,
\[ F(s) = 16s \int_0^{\pi/4} \left( |\cos^{2s-1} t \sin t|^{1/s} + |\sin^{2s-1} t \cos t|^{1/s} \right) dt. \]

It is known that \( F(s) = F(1 - s) \) for all \( s \in [0, 1] \) and there are geometric and analytical proofs for this. Prove also that \( F \) is strictly convex.

Editorial note. The proposer notes that he has a complicated proof of the strict convexity of \( F \) (which he will supply on request) but expects that there should be an easier one.

SOLUTIONS

Optimization of a Stability Bound

Problem 90-6, by P. TH. L. M. VAN WOERKOM (Naarden, the Netherlands).

Consider a dynamic system described by
\[ \dot{\vec{x}} = A\vec{x} + \vec{f}(\vec{x}, t) \]
where \( \vec{x} \) is an \( n \)-dimensional state vector, \( A \) an \( n \times n \) constant matrix, and \( \vec{f} \) an \( n \)-dimensional disturbance vector, \( \vec{f}(\vec{0}, t) = \vec{0}. \) It is assumed that the system is uniformly asymptotically stable in the absence of disturbances (\( \vec{f} = 0 \)). Find an upper bound for the norm \( ||\vec{f}|| \) for which the disturbed system remains uniformly asymptotically stable. Lyapunov theory is to be used, with the Lyapunov function
\[ V = \vec{x}^T B \vec{x} \]
where \( B \) is some constant, symmetric, positive definite matrix.
(i) Prove that, for given $B$, uniform asymptotic stability is assured when

$$
\| \tilde{f} \| < \frac{\lambda_{\min}}{2} \left( \frac{\lambda_{\min}}{\lambda_{\max}} \right)^{1/2} \| \bar{x} \|
$$

where $\lambda_{\min}$ is the smallest eigenvalue of $B^{-1}C$ with

$$
C \triangleq A^T B - BA \quad (C = C^T > 0)
$$

and where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest eigenvalue, respectively, of $B$.

(ii) Result (3) is a function of the Lyapunov matrix $B$. Find an expression for the optimal matrix $B$ that maximizes the right-hand side of (3).

This problem arose in a study on robustness of control algorithms applied to incorrectly modeled dynamic systems.

Solution by P. Th. L. M. Hanau (Naarden, the Netherlands).

(i) Derivation of the bound on $\tilde{f}$. For the indicated dynamic system we derive for the time-derivative of the Lyapunov function $V$,

$$
V = -\dot{x}^T C \bar{x} + 2\tilde{f}^T B \bar{x}.
$$

To ensure uniform asymptotic stability of the system we require that

$$
-(\dot{x}^T C \bar{x})_{\min} + 2(\tilde{f}^T B \bar{x})_{\max} < 0
$$

for all $\bar{x}$ and all admissible $\tilde{f}$.

Minimizing $\dot{x}^T C \bar{x}$ over all $\bar{x}$, subject to the constraint $V = \dot{x}^T B \bar{x}$, gives

$$
(\dot{x}^T C \bar{x})_{\min} = \lambda_{\min} V,
$$

where $\lambda_{\min} (>0)$ is the smallest eigenvalue of $B^{-1}C$.

Maximizing $(\tilde{f}^T B \bar{x})^2$ over all $\bar{x}$, subject to the same constraint, gives

$$
(\tilde{f}^T B \bar{x})_{\max} = \left( (\tilde{f}^T B \bar{f}) V \right)^{1/2}.
$$

To ensure uniform asymptotic stability we therefore require, for all admissible $\tilde{f}$,

$$
-\lambda_{\min} V + 2 \left( (\tilde{f}^T B \bar{f}) V \right)^{1/2} < 0,
$$

or

$$
(\tilde{f}^T B \bar{f})^{1/2} < \frac{\lambda_{\min}}{2} V^{1/2}.
$$

This condition is certainly satisfied when

$$
\{ \lambda(B)_{\max} \tilde{f}^T \tilde{f} \}^{1/2} < \frac{\lambda_{\min}}{2} \{ \lambda(B)_{\min} \tilde{x}^T \tilde{x} \}^{1/2}.
$$

This gives

$$
\| \tilde{f} \| < \alpha \| \tilde{x} \|, \quad \alpha \triangleq \frac{\lambda_{\min}}{2} \left( \frac{\lambda(B)_{\min}}{\lambda(B)_{\max}} \right)^{1/2},
$$

which was to be proven.

(ii) Maximization of the bound on $f$. The second task is to maximize $\alpha$ with respect to $B$. This task is facilitated if we first transform the system equation to real modal form [2]. Let the system matrix $A$ possess distinct real eigenvalues $-\sigma_{\min}$ and distinct complex
conjugate eigenvalues $-\sigma_j \pm i\omega_j \ (i = \sqrt{-1})$. Then there exists a real $n \times n$ modal matrix $M$ that transforms the system equation to the real modal form

(9) \[ \dot{\bar{y}} = \Lambda \bar{y} + \bar{g}, \]

where $\bar{y} = M^{-1} \bar{x}$, $\bar{g} = M^{-1} \bar{f}$, and $\Lambda$ is a block-diagonal matrix. For each real eigenvalue $-\sigma_j$ we find on the diagonal of $\Lambda$ the scalar $-\sigma_j$; for each complex conjugate pair of eigenvalues $-\sigma_j \pm i\omega_j$ we find on the diagonal of $\Lambda$ a $2 \times 2$ block of the form

(10) \[ \begin{bmatrix} -\sigma_j & \omega_j \\ -\omega_j & -\sigma_j \end{bmatrix}. \]

Define

(11) \[ B_m \triangleq M^T B M, \quad C_m \triangleq M^T C M. \]

There follows

(12) \[ V = \bar{y}^T B_m \bar{y}, \quad \dot{V} = -\bar{y}^T C_m \bar{y} + 2 \bar{g}^T B_m \bar{y}, \]

(13) \[ \Lambda^T B_m + B_m \Lambda = -C_m. \]

Applying the bound in (8) to the (real) modal system gives

(14) \[ \| \bar{g} \| < \alpha_m \| \bar{y} \|, \quad \alpha_m \triangleq \frac{\lambda_{\min}}{2} \left( \frac{\lambda_{(B_m)^{\min}}}{\lambda_{(B_m)^{\max}}} \right)^{1/2}, \]

where $\lambda_{\min}$ is the smallest eigenvalue of $B_m^{-1} C_m$, and where $\lambda_{(B_m)^{\min}}$ is an eigenvalue of $B_m$.

Before maximizing $\alpha_m$, we shall first maximize $\lambda_{\min}$ separately. We readily show that (see also (3))

(15) \[ \lambda_{\min} = \min_{\bar{y}} \left\{ \bar{y}^T C_m \bar{y} / \bar{y}^T B_m \bar{y} \right\}. \]

The relation between $B_m$ and $C_m$ may also be expressed in integral form [1]

(16) \[ B_m = \int_0^\infty e^{\Lambda t} C_m e^{\Lambda t} \, dt. \]

Let $c$ be the smallest eigenvalue of $C_m$. Define

(17) \[ C_m^* \triangleq c I_n \times n \quad (c > 0), \]

where $I_n \times n$ denotes the $n \times n$ unit matrix.

The matrix $B_m^*$ corresponding to $C_m^*$ is obtained from (16),

(18) \[ B_m^* = c \int_0^\infty e^{\Lambda t} e^{\Lambda t} \, dt. \]

From (16)--(18) we derive

(19) \[ \bar{y}^T B_m \bar{y} \geq \bar{y}^T \bar{B}_m^* \bar{y}. \]

Maximizing $\lambda_{\min}$ with respect to $B_m$, and therefore also with respect to $C_m$, gives

(20) \[ \max_{C_m} \lambda_{\min} = \min_{\bar{y}} \left\{ c \bar{y}^T \bar{y} / \bar{y}^T B_m^* \bar{y} \right\}. \]
Substitute $C_m = C^*_m$ into (13) and solve for $B_m = B^*_m$. One obtains (cf. [2]) the result

$$B^*_m = \frac{C}{2} \left[ \mathrm{diag} \left( \frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n} \right) \right].$$

From (20)–(21),

$$\max_{C_m} \lambda_{sm} = 2\sigma_{\text{min}},$$

where $\sigma_{\text{min}}$ denotes the smallest of all $\sigma_j$ ($\sigma_j > 0$). It now follows that the second term in $\alpha_m$ attains the value

$$\frac{\lambda(B^*_m)_{\text{min}}}{\lambda(B^*_m)_{\text{max}}} = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} < 1,$$

where $\sigma_{\text{max}}$ denotes the largest of all $\sigma$. Therefore,

$$\alpha_m = \sigma_{\text{min}} \left[ \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} \right]^{1/2}.$$

However, this estimate can be improved by noting from inspection of (13) that the optimum for $\lambda_{sm}$ can also be obtained by choosing

$$B^*_m = bI_n \times n \quad (b > 0),$$

which implies

$$C^*_m = 2b \left[ \mathrm{diag} \left( \sigma_1, \ldots, \sigma_n \right) \right].$$

With the choice $B_m = B^*_m$, we obtain

$$\lambda_{sm} = 2\sigma_{\text{min}}, \quad \frac{\lambda(B^*_m)_{\text{min}}}{\lambda(B^*_m)_{\text{max}}} = 1,$$

whence

$$\alpha_m = \sigma_{\text{min}}.$$

This is indeed a larger bound than the one given by (24).

With $B_m = B^*_m$, we finally obtain for the modal system

$$\|\ddot{y}\| < \sigma_{\text{min}} \|\ddot{x}\|.$$

From (11) and (25) one obtains (for $b = 1$, without loss of generality)

$$B_{\text{optimum}} = (MM^T)^{-1}.$$

This is the desired result. Using the relations $\ddot{x} = M\ddot{y}$ and $\ddot{y} = M\ddot{\bar{y}}$, one obtains from (29)

$$\|\ddot{\bar{y}}\| < \sigma_{\text{min}} \left[ \frac{\lambda(MM^T)_{\text{min}}}{\lambda(MM^T)_{\text{max}}} \right]^{1/2} \|\ddot{x}\|.$$

This same result can also be obtained directly from (8), when making use of the fact that the eigenvalues of $(MM^T)^{-1}$ are the inverses of the eigenvalues of $M^TM$. 
Convexity of Functions Which Are Generalizations of the Erlang Loss Function and the Erlang Delay Function

Problem 90-8, by A. A. JAGERS AND E. A. VAN DOORN (Universiteit Twente, Enschede, the Netherlands).

For real $\lambda$, $a$, $x$ with $\lambda$, $a > 0$, let

$$f_\lambda(x, a) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-at} t^{\lambda-1} (1+t)^{-\lambda+x+1} dt,$$

a confluent hypergeometric or Kummer function. Prove that, for $\lambda \geq 1$, $\log f_\lambda(x, a)$ and $f_\lambda(x, a)^{-1}$ are convex functions of $x$, for $x \geq 0$. Or, equivalently, prove that for $\lambda \geq 1$, $x \geq 0$

$$(f_\lambda'(x, a))^2 \leq f_\lambda(x, a) f_\lambda''(x, a) \leq 2(f_\lambda'(x, a))^2,$$

where a prime denotes differentiation with respect to $x$.

For $\lambda = 1, 2$ two celebrated functions from teletraffic theory are obtained, viz., for $x$ a nonnegative integer

$$af_1(x, a) = \left(\frac{a^x}{x!}\right)^{-1} \sum_{i=0}^x \frac{a^i}{i!} = B(x, a)^{-1},$$

the reciprocal of the Erlang loss function, while for $x$ a nonnegative integer and $x > a$

$$af_2(x, a) = 1 + \left(\frac{a^x}{x! (1-a/x)}\right)^{-1} \sum_{i=0}^{x-1} \frac{a^i}{i!} = C(x, a)^{-1},$$

the reciprocal of the Erlang delay function (cf. [1]).

For $\lambda = 1$ the problem has been solved in [3] (the error on line 4 of p. 45 may be corrected by replacing $a^2$ by $(d/dt)^2$). For $\lambda = 2$ only a weaker result is available in the literature. It has been shown in [2], and in a more general context in [4], that $C(x, a)/(x-a)$ is a convex function of $x$ for $x > a$ and $x$ a nonnegative integer. Clearly, the latter result is implied by the convexity of $f_2(x, a)^{-1}$ since $C(x, a)$ is decreasing in $x$, $x > a$.

REFERENCES