Wirelength of 1-fault hamiltonian graphs into wheels and fans

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Abstract In this paper we obtain a fundamental result to find the exact wirelength of 1-fault hamiltonian graphs into wheels and fans. Using this result we compute the exact wirelength of circulant graphs, generalized petersen graphs, augmented cubes, crossed cubes, möbius cubes, locally twisted cubes, twisted cubes, twisted n-cubes, generalized twisted cubes, hierarchical cubic networks, alternating group graphs, arrangement graphs and tori into wheels and fans. In addition, we find the exact wirelength of hypercubes, folded hypercubes, shuffle cubes, cube connected cycles, cyclic-cubes, wrapped butterfly networks and star graphs into fans.

Research highlights ▶ A new technique has been evolved to compute the exact wirelength of embeddings. ▶ We give algorithms to find the exact wirelength of 1-fault hamiltonian graphs into wheels and fans. ▶ Our algorithms produce exact wirelengths in linear time and cover a wide range of interconnection networks.

Keywords : Embedding; congestion; wirelength; f-fault hamiltonian.

1 Introduction

One of the important features of an interconnection network is its ability to efficiently simulate programs or parallel algorithms written for other architectures. Such a simulation problem can be mathematically formulated as a graph embedding problem. So network topology and its embeddable properties are major issues in the design of interconnection networks.

A graph G is an ordered triple (V(G), E(G), Ψ(G)) consisting of a non-empty set V(G) of vertices, a set E(G) of edges and a mapping Ψ(G) : E(G) → V(G) × V(G), called incidence function which maps an edge into a pair of vertices called end-vertices of the edge. For a simple graph G, the mapping Ψ(G) is injective. In other words for each edge e there exists a unique pair of vertices corresponding to the edge. Let G and H be simple graphs with n vertices. An embedding [2, 32] f of G into H is defined by a bijective mapping f : V(G) → V(H) together with a mapping P_f which assigns to each edge (u, v) of G a path between f(u) and f(v) in H. This path is denoted by P_f(f(u), f(v)). Here G is called the guest graph and H, the host graph. The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H. Let EC_f(G, H(e)) denote the number of edges (u, v) of G such that e is in the path P_f(f(u), f(v)) between f(u) and f(v) in H. In other words,

EC_f(G, H(e)) = |{(u, v) ∈ E(G) : e ∈ P_f(f(u), f(v))}|

where P_f(f(u), f(v)) denotes the path between f(u) and f(v) in H with respect to f.

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion EC(G, H) is the minimum, over all embeddings f : V(G) → V(H), of the maximum number of wires that cross any edge of H [3].

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The wirelength [23] of an embedding $f$ of $G$ into $H$ is given by

$$WL_f(G, H) = \sum_{(u,v) \in E(G)} |P_f(f(u), f(v))| = \sum_{e \in E(H)} EC_f(G, H(e)).$$

See Figure 1. Then, the minimum wirelength of $G$ into $H$ is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings $f$ of $G$ into $H$. The wirelength problem of a graph $G$ into $H$ is to find an embedding of $G$ into $H$ that induces the minimum wirelength $WL(G, H)$.

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [32, 21]. See [23, 25] for an overview on embeddings. Even though there are numerous results and discussions on the embedding problems, most of them deal with only approximate results and the estimation of lower bounds [2, 5]. The embeddings discussed in this paper produce exact wirelengths.

Ring or path embedding in the interconnection networks is closely related to the hamiltonian problem [14] which is one of the well known NP-complete problems in graph theory. If an interconnection network has a hamiltonian cycle or a hamiltonian path, ring or linear array can be implemented in this network. Embedding of linear arrays and rings into a faulty interconnection network is one of the central issues in parallel processing. The problem is modeled as finding as long fault-free paths and cycles as possible in the graph with some faulty vertices and/or edges [28]. A graph $G$ is called $f$-fault hamiltonian if there is a cycle which contains all the non-faulty vertices and contains only non-faulty edges when there are $f$ or less faulty vertices and/or edges. Similarly, a graph $G$ is called $f$-fault traceable if for each pair of vertices, $u$ and $v$, there is a path from $u$ to $v$ which contains all the non-faulty vertices and contains only non-faulty edges when there are $f$ or less faulty vertices and/or edges. We note that if a graph $G$ is hypohamiltonian, hyperhamiltonian or almost pancyclic then it is 1-fault hamiltonian.

Now, we consider another interesting NP-complete problem namely the maximum subgraph problem [14] which will be used to solve the wirelength problem. The maximum subgraph problem of a graph is to find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given $m$, if $I_G(m) = \max_{A \subseteq V, |A| = m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$. We call such a set $A$ optimal.

Figure 1: Wiring diagram of a hypercube $G$ into a ladder $H$ with $WL_f(G, H) = 16$. The edge congestions are marked on the edges of $H$. 
Figure 2: Embedding of a graph $G$ on six vertices onto $H$

2 Main Results

For convenience of notation we write $EC_f(e)$ instead of $EC_f(G,H(e))$ in the sequel. For any set $S$ of edges of $H$, $EC_f(S) = \sum_{e \in S} EC_f(e)$.

**Lemma 1 (Congestion Lemma)** [23] Let $G$ be an $r$-regular graph and $f$ be an embedding of $G$ into $H$. Let $S$ be an edge cut of $H$ such that the removal of edges of $S$ leaves $H$ into 2 components $H_1$ and $H_2$ and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also $S$ satisfies the following conditions:

(i) For every edge $(a,b) \in G_i$, $i = 1,2$, $P_f(f(a), f(b))$ has no edges in $S$.

(ii) For every edge $(a,b)$ in $G$ with $a \in G_1$ and $b \in G_2$, $P_f(f(a), f(b))$ has exactly one edge in $S$.

(iii) $G_1$ is an optimal set.

Then $EC_f(S)$ is minimum and $EC_f(S) = r|V(G_1)| - 2|E(G_1)|$. \(\Box\)

**Lemma 2 (Partition Lemma)** [23] Let $f : G \to H$ be an embedding. Let $\{S_1, S_2, ..., S_p\}$ be a partition of $E(H)$ such that each $S_i$ is an edge cut of $H$. Then

$$WL_f(G,H) = \sum_{i=1}^{p} EC_f(S_i).$$ \(\Box\)

The exact wirelength problem of hypercube on a grid has been determined by Manuel et al. [23], using the Congestion Lemma. This strategy is not applicable to all the architectures. There are host graphs in which the edge set cannot be partitioned into edge cuts satisfying the conditions of the Congestion Lemma. This is illustrated in the host graph $H$ shown in Figure 2 for any embedding of a graph $G$ on six vertices onto $H$. This motivates the following result, which is a generalization of the Partition Lemma.

**Lemma 3 (k-Partition Lemma)** Let $f : G \to H$ be an embedding. Let $|kE(H)|$ denote a collection of edges of $H$ with each edge in $H$ repeated exactly $k$ times. Let $\{S_1, S_2, ..., S_p\}$ be a partition of $|kE(H)|$ such that each $S_i$ is an edge cut of $H$. Then

$$WL_f(G,H) = \frac{1}{k} \sum_{i=1}^{p} EC_f(S_i).$$

**Proof.** We have $k \sum_{e \in E(H)} EC_f(e) = \sum_{i=1}^{p} EC_f(S_i) = kWL_f(G,H)$. \(\Box\)

**Remark 1** When $k = 1$, we get the Partition Lemma.
In this section we prove that the Congestion Lemma and the \( k \)-Partition Lemma together solve the minimum wirelength of 1-fault hamiltonian graphs into wheels and fans.

A wheel graph \([26, 30]\) \( W_n \) of order \( n \) is a graph that contains an outer cycle of order \( n - 1 \), and for which every vertex in the cycle is connected to one other vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. See Figure 3(a). This plays an important role in the circuit layout and interconnection network designs.

A fan graph \( F_n \) of order \( n \) is a graph that contains a path of order \( n - 1 \), and for which every vertex in the path is connected to one other vertex (which is known as the core). In other words, a fan graph \( F_n \) is obtained from \( W_n \) by deleting any one of the outer cycle edges. See Figure 3(b).

**Embedding Algorithm A**

**Input** : An \( r \)-regular 1-fault hamiltonian graph \( G \) with \( n \) vertices and a wheel \( W_n \).

**Algorithm** : Let \( v \) be a vertex of \( G \) such that \( G - v \) is hamiltonian. Label the vertices of a hamiltonian cycle in \( G - v \) as \( 0, 1, \ldots, n - 2 \) and the vertex \( v \) as \( n - 1 \). Label the outer cycle vertices of the wheel as \( 1, 2, \ldots, n - 2 \) and the hub vertex as \( n - 1 \).

**Output** : An embedding \( f \) of \( G \) into \( W_n \) given by \( f(x) = x \) with minimum wirelength.

**Proof of correctness** : Let \( S_i = \{(i-1,i),(i+1,i+2),(n-1,i),(n-1,i+1)\} \), \( 1 \leq i \leq n-1 \), where the labels are taken mod \((n - 1)\), except the label of hub vertex. The edge set \( \{(i-1,i),(n-1,i) : 1 \leq i \leq n-1\} \) constitutes all the edges of \( W_n \) exactly once. Similarly the edge set \( \{(i+1,i+2),(n-1,i+1) : 1 \leq i \leq n-1\} \) constitutes all the edges of \( W_n \) exactly once. Thus \( \{S_1,S_2,\ldots,S_{n-1}\} \) is a partition of \( 2E(W_n) \). See Figure 4(a). For each \( i \), \( E(W_n) \setminus S_i \) has two components \( H_{i1} \) and \( H_{i2} \). Without loss of generality, let \( H_{i1} = \{i,i+1\} \). Let \( G_{i1} = f^{-1}(H_{i1}) \) and \( G_{i2} = f^{-1}(H_{i2}) \). Then \( G_{i1} \) induces an edge of \( G \). Thus each \( S_i \) satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore \( EC_f(S_i) \) is minimum. The 2-Partition Lemma implies that the wirelength is minimum. \( \square \)

**Theorem 1** The minimum wirelength of an \( r \)-regular 1-fault hamiltonian graph \( G \) with \( n \) vertices into \( W_n, n \geq 5 \) is given by

\[
WL(G, W_n) = (r - 1)(n - 1).
\]

**Proof.** Following the notations in the proof of Embedding Algorithm A, we have by Lemma 1, \( EC_f(S_i) = 2r - 2 = 2(r - 1) \). But \( W_n \setminus S_i \) is isomorphic to \( W_n \setminus S_j \) for \( i \neq j, 1 \leq i, j \leq n - 1 \). Therefore, \( WL(G, W_n) = \frac{2r-2}{2}EC_f(S_i) = (r - 1)(n - 1) \). \( \square \)

**Embedding Algorithm B**

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Figure 3: (a) Wheel graph \( W_9 \) (b) Fan graph \( F_9 \)
Let $v$ be a vertex of $G$ such that $G - v$ has a Hamiltonian path. Label the vertices of a Hamiltonian path in $G - v$ as $0, 1, \ldots, n - 2$ and the vertex $v$ as $n - 1$. Label the outer path vertices of the Fan as $0, 1, \ldots, n - 2$ and the core vertex as $n - 1$.

Output: An embedding $f$ of $G$ into $F_n$ given by $f(x) = x$ with minimum wirelength.

Proof of correctness: Let $S_i = \{(i - 1, i), (i + 1, i + 2), (n - 1, i), (n - 1, i + 1)\}, 1 \leq i \leq n - 4$, $S_{n-3} = \{(n - 4, n - 3), (n - 1, n - 3), (n - 1, n - 2)\}$, $S_{n-2} = \{(1, 2), (n - 1, 0), (n - 1, 1)\}$, $S_{n-1} = \{(n - 3, n - 2), (n - 1, n - 2)\}$ and $S_n = \{(1, 1)\}$. The edge set $\{(i - 1, i), (n - 1, i) : 1 \leq i \leq n - 4\} \cup S_{n-3} \cup \{(n - 3, n - 2), (n - 1, 0)\}$ constitutes all the edges of $F_n$ exactly once. Similarly the edge set $\{(i + 1, i + 2), (n - 1, i + 1) : 1 \leq i \leq n - 4\} \cup S_{n-2} \cup \{(n - 1, n - 2), (0, 1)\}$ constitutes all the edges of $F_n$ exactly once. Thus $\{S_1, S_2, \ldots, S_n\}$ is a partition of $[2E(F_n)]$. See Figure 4(b). For each $i$, $1 \leq i \leq n - 3$, $E(F_n) \setminus S_i$ has two components $H_{i1}$ and $H_{i2}$. Without loss of generality, let $H_{i1} = \{i, i + 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Then $G_{i1}$ induces an edge of $G$. For $i = n - 2$, $E(F_n) \setminus S_i$ has two components $H_{i1}$ and $H_{i2}$, let $H_{i1} = \{0, 1\}$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Then $G_{i1}$ induces an edge of $G$. For $i = n - 1, n$, $E(F_n) \setminus S_i$ has two components $H_{i1}$ and $H_{i2}$, let $H_{i1} = \{i - 1\}$, label is taken mod$(n - 1)$. Let $G_{i1} = f^{-1}(H_{i1})$ and $G_{i2} = f^{-1}(H_{i2})$. Then $G_{i1}$ is a vertex of $G$. Thus each $S_i$ satisfies conditions (i), (ii) and (iii) of the Congestion Lemma. Therefore $EC_f(S_i)$ is minimum. The 2-Partition Lemma implies that the wirelength is minimum. $\square$

Theorem 2 The minimum wirelength of an $r$-regular 1-fault traceable graph $G$ with $n$ vertices into $F_n$, $n \geq 5$ is given by

$$WL(G, F_n) = (r - 1)(n - 1) + 1.$$ 

Proof. Following the notations in the proof of Embedding Algorithm B, we have by Lemma 1, for each $1 \leq i \leq n - 2$, $EC_f(S_i) = 2(r - 1)$. But for $i = n - 1, n$, $EC_f(S_i) = r$. Therefore, $WL(G, F_n) = \frac{1}{2} \{2(r - 1)(n - 2) + 2r\} = (r - 1)(n - 1) + 1$. $\square$

3 Conclusion

The guest graphs in the Embedding Algorithm A and the Embedding Algorithm B cover a wide range of graphs. This motivates us to identify interconnection networks which fall into this category, namely the circulant graphs $G(n; \pm S), \{1, 2\} \subseteq S \subseteq \{1, 2, \ldots, \lceil n/2 \rceil\}$ and generalized Petersen graphs $P(n, m)$-hypohamiltonian/hyperhamiltonian [1, 24, 31, 32], augmented cubes $AQ_n$-pancyclic [4], crossed cubes $CQ_n$-almost pancyclic [9], Möbius cubes $MQ_n$-$(n - 2)$-fault almost pancyclic [7, 28], locally twisted
cubes $LTQ_n$ - almost pancyclic [33], twisted cubes $TQ_n$ - (n - 2)-fault almost pancyclic [10, 11, 16, 28], twisted n-cubes $T_nQ$ - 1-fault hamiltonian [27], generalized twisted cubes $GQ_n$ - (n - 2)-fault almost pancyclic [28], hierarchical cubic networks $HCN(n)$ - almost pancyclic [13, 15], alternating group graphs $AG_n$ - (n - 2)-fault hamiltonian [6], arrangement graphs $A_n,k$ - pancyclic [8], tori $T(d_1,d_2,...,d_n)$ - fault hamiltonian [19, 20]. Further, hypercubes $Q_n$ [32], folded hypercubes $FQ_n$ [32], shuffle cubes $SQ_n$ [22], cube connected cycles $CCC(n)$ [29], cyclic-cubes $G_n^k$ [12], wrapped butterfly networks $WB(n,k)$ [17], star graphs $S_n$ [18] are hamiltonian. As a consequence of the above literature and by theorems 1 and 2, we have the following results.

**Theorem 3** (a) Let $G(n;\pm S)$ be the circulant graph with \{1, 2\} \(\subseteq S \subseteq \{1, 2, ..., [n/2]\}\), \(n \geq 5\). Then $WL(G(n;\pm S), W_n) = (2 \mid S \mid - 1)(n - 1)$ and $WL(G(n;\pm S), F_n) = (2 \mid S \mid - 1)(n - 1) + 1$. (b) For \(n \geq 5\), $WL(K_n, W_n) = (n - 2)(n - 1)$ and $WL(K_n, F_n) = (n - 2)(n - 1) + 1$.

**Theorem 4** For \(n \geq 5\), let $G$ be the generalized petersen graph (i) $P(n, 1)$, $n$ is odd (ii) $P(n, 2)$, $n \equiv 1, 3(\text{mod } 6)$, $n \equiv 5(\text{mod } 6)$ (iii) $P(n, 3)$, $n$ is odd (iv) $P(n, 4)$, $n \neq 12$ (v) $P(n, m)$, $m$ is even with $m \geq 6$ and $n \geq 2m + 2 + (4m - 1)(4m + 1)$ (vi) $P(n, m)$, $m$ and $n$ are odd integers with $m \geq 5$ and $n \geq 6m - 3 + (2m)(6m - 2)$. Then $WL(G, W_{2n}) = 4n - 2$ and $WL(G, F_{2n}) = 4n - 1$.

**Theorem 5** $WL(Q_n, F_{2^n}) = (n - 1)(2^n - 1) + 1$, \(n \geq 3\).

**Theorem 6** $WL(FQ_n, F_{2^n}) = n(2^n - 1) + 1$, \(n \geq 3\).

**Theorem 7** $WL(AQ_n, W_{2^n}) = (2n - 1)(2^n - 1)$ and $WL(AQ_n, F_{2^n}) = (2n - 1)(2^n - 1) + 1$, \(n \geq 3\).

**Theorem 8** Let $G$ be the graph $CQ_n$, $MQ_n$, $LTQ_n$, $TQ_n$, $T_nQ$ or $GQ_n$ for $n \geq 3$. Then $WL(G, W_{2^n}) = \left(\frac{n}{2}\right)(2^n - 1)$ and $WL(G, F_{2^n}) = \left(\frac{n}{2}\right)(2^n - 1) + 1$.

**Theorem 9** $WL(SQ_n, F_{2^n}) = (n - 1)(2^n - 1) + 1$, \(n \geq 3\).

**Theorem 10** $WL(CCC(n), F_{2^n}) = 2n2^n - 1$, \(n \geq 2\).

**Theorem 11** $WL(G^k_n, F_{nk^n}) = WL(WB(n, k), F_{nk^n}) = (2k - 1)(nk^n - 1) + 1$ for $n \geq 3, k \geq 2$.

**Theorem 12** $WL(HCN(n), W_{2^n}) = n2^{2n} - n$ and $WL(HCN(n), F_{2^n}) = n2^{2n} - n + 1$, \(n \geq 2\).

**Theorem 13** $WL(S_n, F_{ni}) = (n - 2)(n! - 1) + 1$, \(n \geq 3\).

**Theorem 14** $WL(AG_n, W_{n^{i/2}}) = (2n - 4 - 1)(n^{i/2} - 1)$ and $WL(AG_n, F_{n^{i/2}}) = (2n - 4 - 1)(n^{i/2} - 1) + 1$, \(n > 4\).

**Theorem 15** $WL(A_n,k, W_{\frac{n^i}{(n-k)^i}}) = (k(n - k) - 1)(\frac{n^i}{(n-k)^i} - 1)$ and $WL(A_{n,k}, F_{\frac{n^i}{(n-k)^i}}) = (k(n - k) - 1)(\frac{n^i}{(n-k)^i} - 1) + 1$, \(n - k > 1\).

**Theorem 16** (a) For $m, n \geq 3$ and $n$ odd, $WL(T(m, n), W_{mn}) = 3mn - 3$ and $WL(T(m, n), F_{mn}) = 3mn - 2$. (b) Let $G$ be a non-bipartite $n$-dimensional tori $T(d_1,d_2,...,d_n)$ with $d_i \geq 3$ for each $i (1 \leq i \leq n)$. Then $WL(G, W_{d_1,d_2...d_n}) = (2n - 1)(d_1d_2...d_n - 1)$ and $WL(G, F_{d_1,d_2...d_n}) = (2n - 1)(d_1d_2...d_n - 1) + 1$.

**References**


