Shrinkage Estimation of Threshold Parameter of the Exponential Distribution

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Abstract — This paper studies the usual preliminary test estimator of the threshold parameter of the exponential distribution in censored samples. The optimal levels of significance and their corresponding critical values for the preliminary test are obtained. The optimal values of shrinkage coefficients for a preliminary test shrinkage estimator are also obtained based on the minimax regret criterion.

1. INTRODUCTION

The exponential distribution has been used widely in the field of life testing and reliability theory. Epstein & Sobel [4] obtained the minimum variance unbiased estimator (MVUE) for its scale parameter and threshold parameter respectively. The shrinkage estimators of the scale parameter have been proposed by Bhattacharya & Srivastava [2], and Pandey [8]. In this paper a preliminary test estimator of the threshold parameter is studied. The optimal levels of s-significance and their corresponding critical values for the preliminary test are obtained. The optimal values of shrinkage coefficients for a preliminary test shrinkage estimator are also obtained based on the minimax regret criterion.

2. NOTATION

\( \eta \)  
threshold parameter of exponential distribution  
\( \theta \)  
scale parameter of exponential distribution  
\( X_i \)  
first \( r \) order statistics in sample of size \( n \), \( i = 1, 2, \ldots, r \)  
\( \hat{\eta} \)  
minimum variance unbiased estimator of \( \eta \)  
\( \eta_0 \)  
prior point estimate of \( \eta \)  
\( \hat{\eta}_{pt} \)  
preliminary test estimator of \( \eta \)  
\( C \)  
upper \( \alpha \) Cdf point of \( F \) distribution with 2 and 2\((r-1)\) degrees of freedom  
\( \hat{\eta}_s \)  
preliminary test shrinkage estimator of \( \eta \)  
\( k \)  
shrinkage coefficient  
\( \alpha \)  
s-significance level  
\( B\{\cdot\} \)  
s-bias  
\( \text{MSE}\{\cdot\} \)  
mean square error  
\( \text{REG}\{\cdot;\cdot\} \)  
regret function

3. SHRINKAGE ESTIMATORS OF THRESHOLD PARAMETER

3.1 A preliminary test estimator

Let \( x_i, i = 1, \ldots, r \) denote the first \( r \) ordered observations in a sample of size \( n \) from the 2-parameter exponential distribution with survival function:

\[
S(x) = \exp[-(x-\eta)/\theta], \ x \geq \eta, \ \eta \geq 0, \ \theta > 0.
\]

The \( T = \sum_{i=1}^r (X_i - X_1) + (n-r) (X_r - X_1) \) and \( X_1 \) are \( s \)-independent [4]. Moreover, \( \hat{\eta} = X_1 - T/[n(r-1)] \) is the MUVE of \( \eta \). Since \( 2n(X_1 - \eta)/\theta \) and \( 2T/\theta \) have chi-square distributions with 2 and 2\((r-1)\) degrees of freedom respectively, the statistic \( F = n(r-1)(X_1 - \eta_0)/T \) is used for testing the preliminary hypothesis \( H_p : \eta = \eta_0 \). The preliminary test estimator for \( \eta \) is:

\[
\hat{\eta}_{pt} = \begin{cases} \eta_0, & \text{if } 0 \leq n(r-1)(X_1 - \eta_0)/T \leq C \\ \hat{\eta}, & \text{otherwise.} \end{cases}
\]

(3.1)

The mean of \( \hat{\eta}_{pt} \) is:

\[
E\{\hat{\eta}_{pt}\} = E\{\eta_0 \mid (\eta_0 - \eta) \leq (X_1 - \eta) \leq TC/[n(r-1)] + (\eta_o - \eta)\} + E\{\hat{\eta} \mid (X_1 - \hat{\eta}) < (\eta_o - \eta) \text{ or } (X_1 - \eta) > TC/[n(r-1)] + (\eta_o - \eta)\}.
\]

(3.2)

The value of \( E\{\hat{\eta}_{pt}\} \) depends on whether \( \eta < \eta_0 \) or \( \eta > \eta_0 \).
Case 1: $\eta < \eta_o$. Let $W = X_1 - \eta$.

$$E\{\hat{\eta}_p\} = \eta_o \int_A f(w,t) dt dw$$
$$+ \int_{\pi} \left\{ (w + \eta) - t/[n(r - 1)] \right\} f(w,t) dt dw$$
$$= \eta + \eta_o \int_A f(w,t) dt dw - \int_A \left\{ (w + \eta) - t/[n(r - 1)] \right\} f(w,t) dt dw$$

(region $B = \{(w,t) \mid w \geq 0, t \geq n(r-1)(w + (\eta - \eta_o))/C]\}$.

$$f(w,t) = \begin{cases} (n/\theta) \exp[-nw/\theta][1/T(r - 1)[r^{-2}/\theta r^{-1}]\exp[-t/\theta], & w \geq 0, t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

After a straightforward evaluation we obtain (see appendix):

$$E\{\hat{\eta}_p\} = \eta + (a\theta/n) (Cd' + d'^{-1} - d')$$

$$\approx = \eta + \eta_o \int_B f(w,t) dt dw$$

$$- \int_B \left\{ (w + \eta) - t/[n(r - 1)] \right\} f(w,t) dt dw$$

$$(3.3a)$$

$$E\{\hat{\eta}_p\} = \eta + \eta_o \int_B f(w,t) dt dw$$

$$(3.3b)$$

$$(3.4)$$

$$b = \exp[-n(r - 1)(\eta - \eta_o)/(C\theta)]$$

$$S(j) = \sum_{i=0}^{j} \left[ n(C+r-1)(\eta - \eta_o)/(C\theta) \right]^{j/i}$$

$$E\{\hat{\eta}_p\} = \eta + (b\theta/n) \left[ (1-d) \sum_{j=0}^{r-1} d^j S(j) \right]$$

$$- (1-d) \sum_{j=0}^{r-2} d^j (j+1) S(j+1)$$

$$(3.5)$$

$$(3.6)$$

$$(3.7)$$

The first term on the r.h.s. of (3.7) is the mean square error of the MVUE.

Case 2: $\eta > \eta_o$

$$E\{\hat{\eta}_p\} = \eta_o \int_B f(w,t) dt dw$$

$$+ \int_{\pi} \left\{ (w + \eta) - t/[n(r - 1)] \right\} f(w,t) dt dw$$

$$(3.8a)$$

The $s$-bias and mean square error are:

$$B\{\hat{\eta}_p\} = E\{\hat{\eta}_p\} - \eta$$

$$(3.12)$$

$$\text{MSE}\{\hat{\eta}_p\} = E\{\hat{\eta}_p^2\} + [B\{\hat{\eta}_p\}]^2 - [E\{\hat{\eta}_p\}]^2$$

$$(3.13)$$
MSE\{\hat{\eta}_p\}/\theta^2 can be considered a risk function:

\[ R_1\{\psi;\alpha\} \equiv \text{MSE}\{\hat{\eta}_p\}/\theta^2 \]  \hspace{1cm} (3.14)

\[ \psi = (\eta - \eta_o)/\theta \]

If \( \psi \to -\infty \) or \( \infty \), then \( R_1\{\psi;\alpha\} \) converges to \( R_1\{\psi;1\} \) which is the risk for \( \hat{\eta} \). The general shapes of \( R_1\{\psi;\alpha\} \) are shown in figure 1. An optimal value is \( \alpha = 1 \) if \( \psi \leq \psi_1 \) or \( \psi \geq \psi_2 \); it is \( \alpha = 0 \) otherwise; where \( \psi_1 \) and \( \psi_2 \) are intersections of \( R_1\{\psi;0\} = \psi^2 \) and \( R_1\{\psi;1\} = \psi/(n^2(r-1)) \):

Figure 1. Risk Function

\[
\psi_1 = -(1/n)\sqrt{r/(r-1)} \quad \text{and} \quad \psi_2 = (1/n)\sqrt{r/(r-1)}.
\]

Since \( \psi \) is unknown we seek an optimal value \( \alpha = \alpha^* \) which gives a reasonable risk for all values of \( \psi \). Along the lines of Sawa & Hiromatsu [9], the regret function is

\[
\text{REG}\{\psi;\alpha\} = R_1\{\psi;\alpha\} - \inf_{\alpha'} R_1\{\psi;\alpha'\} \hspace{1cm} (3.15)
\]

\[
\inf_{\alpha'} R_1\{\psi;\alpha'\} = \begin{cases} R_1\{\psi;1\}, & \psi \leq \psi_1 \text{ or } \psi \geq \psi_2 \\ R_1\{\psi;0\}, & \text{otherwise} \end{cases} \hspace{1cm} (3.16)
\]

For an estimator \( \hat{\eta} \) of \( \eta_o \), let \( (\psi_1',\psi_2') \) be the largest interval of \( \psi \) such that for all \( \psi \in (\psi_1',\psi_2') \), the risk of \( \hat{\eta} \) is smaller than \( R_1\{\psi;1\} \), the risk of \( \hat{\eta} \). \( (\psi_1',\psi_2') \) is the effective interval of \( \hat{\eta} \) when compared with \( \hat{\eta} \). For \( \psi \leq \psi_2 \), \( \text{REG}\{\psi;\alpha\} \) takes a maximum value at \( \psi_2 \) and this value of \( \text{REG}\{\psi;\alpha\} \) is labeled \( \delta_l \) in figure 1. For \( \psi \geq \psi_2 \), \( \text{REG}\{\psi;\alpha\} \) takes a maximum value at \( \psi_2 \) and this value of \( \text{REG}\{\psi;\alpha\} \) is labeled \( \delta_u \) in figure 1. Thus, the minimax regret criterion determines \( \alpha^* \) such that \( \text{REG}\{\psi_l;\alpha^*\} = \text{REG}\{\psi_u;\alpha^*\} \). The risk of \( \hat{\eta} \) and the length of the effective interval decrease rapidly as \( n \) increases so that only small sample sizes are considered. Table 1 gives the optimal levels of \( s \)-significance \( \alpha^* \), and the corresponding critical values \( C \) for \( n = 2(1)5 \) and \( r = 2(1)n \). The \( \alpha^* \) for a given \( r \) does not depend on \( n \) for \( r \leq n \).

3.2 A preliminary test shrinkage estimator

The preliminary test estimator given in section 3.1 uses the prior estimate \( \eta_o \) when the preliminary test accepts the null hypothesis. Instead of using \( \eta_o \), we use a linear combination of \( \eta_o \) and \( \hat{\eta} \) when the preliminary test accepts \( H_p \); this gives a preliminary test shrinkage estimator which assigns suitable weights to \( \eta_o \) and \( \hat{\eta} \) rather than assigns weight 1 to \( \eta_o \) and 0 to \( \hat{\eta} \) when the preliminary test accepts \( H_p \). The estimator is:

\[
\hat{\eta}_t = \begin{cases} k\eta_o + (1-k)\hat{\eta}, & \text{if } 0 \leq n(r-1)(X_l - \eta_o)/T \leq C \\ \hat{\eta}, & \text{otherwise} \end{cases}
\]

(3.17)

Hirano [5] has applied AIC (Akaike's Information Criterion [1]) to determine the optimal level of \( s \)-significance for the preliminary test. Inada [6] has proposed a shrinkage estimator for the mean of the normal distribution by use of the AIC optimal level of \( s \)-significance for the preliminary test. In this section we study the preliminary test shrinkage estimator \( \hat{\eta}_t \) following the same estimation procedure set forth by Inada [6] which recommends \( \alpha = 0.16 \).

The risk of \( \hat{\eta}_t \), denoted by \( R_2\{\psi;k\} = \text{MSE}\{\hat{\eta}_t\}/\theta^2 \), is calculated in the same fashion as section 3.1.

Case 1: \( \eta < \eta_o \)

\[
R_2\{\psi;k\} = r/[n^2(r-1)] - (2k\psi a/n) [Cd^* + d^{* - 1} - d^*] - [k(k-2)a/n^2] [(r/(r-1))[C^2d^{* + 1} - 2Cd^{* + 1}] - 1 + d^{* + 1}] + 2[Cd' + d^{* + 1} - d^*] \]

(3.18)

Case 2: \( \eta > \eta_o \)

\[
R_2\{\psi;k\} = r/[n^2(r-1)] - 2(k\psi b/n) \left[ (1-d) \sum_{j=0}^{r-1} d^j S(j) \right] - (1-d)^2 \sum_{j=0}^{r-2} d^j (j+1) S(j+1) \]

- \([k(k-2)b/n^2] \left[ 2(1-d)^2 \sum_{j=0}^{r-1} d^j (j+1) S(j+1) \right] \]

Table 1

<table>
<thead>
<tr>
<th>r</th>
<th>( \alpha^* )</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>.31</td>
<td>2.23</td>
</tr>
<tr>
<td>3</td>
<td>.19</td>
<td>2.59</td>
</tr>
<tr>
<td>4</td>
<td>.12</td>
<td>3.08</td>
</tr>
<tr>
<td>5</td>
<td>.08</td>
<td>3.52</td>
</tr>
</tbody>
</table>
\[-(1-d)^3 \sum_{j=0}^{r-2} d^j(j+1)(j+2)S(j+2)\]

\[-\left[\frac{r}{r-1}\right](1-d) \sum_{j=0}^{r} d^jS(j)\]  

(3.19)

\[a\] and \[d\] are given in (3.6), \[b\] and \[S(j)\] are given in (3.10), \[\psi = (\eta - \eta_o)/\theta,\]

\[k = \text{shrinkage coefficient. The regret function is:}\]

\[\text{REG}\{\psi;k\} = R_2\{\psi;k\} - \inf_k R_2\{\psi;k\}\]  

(3.20)

After rather extensive numerical investigation, the values of \[k\] which attain the infinum of sup \[\text{REG}\{\psi;k\}\] were obtained. Table 2 gives the optimal values of shrinkage coefficients \[k^*\] and the critical values \[C\] for \[n = 2(1)5\] and \[r = 2(1)n\]. The value \[k^*\] for a given \[r\] does not depend on \[n\] for \[r \leq n\].

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Optimal Shrinkage Coefficients (k^*) and Critical Values (C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r)</td>
<td>(k^*)</td>
</tr>
<tr>
<td>2</td>
<td>.08</td>
</tr>
<tr>
<td>3</td>
<td>.27</td>
</tr>
<tr>
<td>4</td>
<td>.46</td>
</tr>
<tr>
<td>5</td>
<td>.64</td>
</tr>
</tbody>
</table>

4. COMPARISON AND CONCLUSION

The risks of \(\hat{\eta}_{pt}\) and \(\hat{\eta}_s\) are depicted in figure 1. The curves indicate that:

a. The maximum possible value of the risk is smaller for \(\hat{\eta}_s\) than for \(\hat{\eta}_{pt}\),

b. the length of the effective interval is greater for \(\hat{\eta}_s\) than for \(\hat{\eta}_{pt}\),

c. for larger values of \(|\psi|\), \(\hat{\eta}_s\) is better than \(\hat{\eta}_{pt}\),

d. for smaller values of \(|\psi|\), \(\hat{\eta}_{pt}\) is better than \(\hat{\eta}_s\).

Therefore we conclude that if we use the mean square error as a criterion for goodness of estimators, \(\hat{\eta}_s\) is appropriate when a prior point estimate is available.

5. EXAMPLE

Consider the following simulated censored sample from an exponential distribution with threshold parameter \(\eta = 5.0\) and scale parameter \(\theta = 25\) for the weeks to failure of a certain component.
\[
- \int_A \{(w + \eta) - t/[n(r - 1)]\} f(w, t) dw dt
\]
\[
= \eta + \eta_o \int_A f(w, t) dw dt
\]
\[
- \int_A \{(w + \eta) - t/[n(r - 1)]\} f(w, t) dw dt
\]
region \(A\)
\[
= \{(w, t) \mid (\eta_o - \eta) \leq w \leq t (C/[n(r - 1)]) + (\eta_o - \eta), t \geq 0\}
\]
Let \(\gamma(u; r) = [1/\Gamma(r)] u^{r-1} \exp[-u]
\]
\[
E\{\bar{\eta}_m\}
\]
\[
= \eta + \eta_o \int_0^\infty \{(\eta_o - \eta) + t (C/[n(r - 1)])
\]
\[
\cdot \gamma(nw/\theta; 1) \gamma(t/\theta; r - 1) d(nw/\theta) d(t/\theta)
\]
\[
- \int_0^\infty \{(\eta_o - \eta) + t (C/[n(r - 1)])
\]
\[
\cdot \gamma(nw/\theta; 1) \gamma(t/\theta; r - 1) d(nw/\theta) d(t/\theta)
\]
\[
= \eta + \eta_o \exp[-(\eta_o - \eta)n/\theta]
\]
\[
\cdot \int_0^\infty \{1 - \exp[-Ct/((r - 1)\theta)]\} \gamma(t/\theta; r - 1) d(t/\theta)
\]
\[
- \exp[-(\eta_o - \eta)n/\theta] \int_0^\infty \{\eta_o \{1 - \exp[-Ct/((r - 1)\theta)]\}
\]
\[
- Ct/[n(r - 1)] \exp[-Ct/((r - 1)\theta)]
\]
\[
+ \theta \{1 - \exp[-Ct/((r - 1)\theta)]\}/n - t/[n(r - 1)]
\]
\[
\cdot \{1 - \{1 - t/(C + r - 1)\}^{r-1}\}
\]
\[
- \exp[-(\eta_o - \eta)n/\theta] \{\eta_o \{1 - \{1 - t/(C + r - 1)\}^{r-1}\}
\]
\[
- \theta C \{t/(C + r - 1)\}^r/n
\]
\[
+ \theta \{1 - \{1 - t/(C + r - 1)\}^{r-1}\}/n
\]

REFERENCES


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