PERSISTENT REGIONAL NULL CONTROLLABILITY FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

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Abstract. Motivated by physical models and the so-called Crocco equation, we study the controllability properties of a class of degenerate parabolic equations. Due to degeneracy, classical null controllability results do not hold for this problem in general.

First, we prove that we can drive the solution to rest at time $T$ in a suitable subset of the space domain (regional null controllability). However, unlike for nondegenerate parabolic equations, this property is no more automatically preserved with time. Then, we prove that, given a time interval $(T, T')$, we can control the equation up to $T'$ and remain at rest during all the time interval $(T, T')$ on the same subset of the space domain (persistent regional null controllability). The proofs of these results are obtained via new observability inequalities derived from classical Carleman estimates by an appropriate use of cut-off functions.

With the same method, we also derive results of regional controllability for a Crocco type linearized equation and for the nondegenerate heat equation in unbounded domains.

1. Introduction and Main Results. Motivated by a boundary layer model and the so-called Crocco equation (see section 3.1), we study the controllability properties of a class of degenerate parabolic equations in a bounded domain. Let $c \in L^\infty(0, 1)$ be given and assume that

$$a : [0, 1] \to \mathbb{R}_+ \text{ is } C^1 \text{ on } [0, 1], a(0) = 0 \text{ and } a > 0 \text{ on } (0, 1].$$

Let $0 \leq \alpha < \beta$ be fixed. First, for all $T > 0$, we consider the problem

$$\begin{cases}
  u_t - (a(x)u_x)_x + c(x)u = f(t, x)\chi_{(\alpha, \beta)}(x), & (t, x) \in (0, T) \times (0, 1), \\
  \lim_{x \to 0} a(x)u_x(t, x) = 0, & t \in (0, T), \\
  u(t, 1) = 0, & t \in (0, T), \\
  u(t, 0) = u_0(x), & t \in (0, T), \\
  u(0, x) = u_0(x), & x \in (0, 1),
\end{cases}$$

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where \( u_0 \in L^2(0,1) \) and \( f \in L^2((0,T) \times (0,1)) \). We will see that this problem is well-posed in the sense of semigroup theory, working in appropriate weighted spaces (see section 2.1).

Since \( a(x) \) vanishes at \( x = 0 \), classical null controllability may fail. We recall that (1.2) is null controllable at time \( T > 0 \) if, given \( u_0 \in L^2(0,1), \)

\[
\exists f \in L^2((0,T) \times (0,1)), \quad \text{such that} \quad u(T) = 0. \tag{1.3}
\]

Our first result is the following:

**Theorem 1.1. Regional null controllability.** Given \( T > 0, u_0 \in L^2(0,1), \) and \( \delta > 0 \) such that \( \alpha + \delta < \beta \), there exists \( f \in L^2((0,T) \times (0,1)) \) such that the solution \( u \) of (1.2) satisfies

\[
u(T, x) = 0 \quad \text{for} \quad x \in (\alpha + \delta, 1). \tag{1.4}
\]

**Remarks.** Let us make some comments.

1. We assume that \( a \) vanishes only at \( x = 0 \) in order to clarify the well-posedness of the problem. (Indeed this assumption allows us to describe precisely the weighted spaces that are used). However we could also assume that the equation is 'strongly' degenerate, assuming for example:

\[
a : [0,1] \to \mathbb{R}_+ \text{ is } C^1 \text{ on } [0,1], \quad a \geq 0 \text{ on } [0,1] \text{ and } a > 0 \text{ on } (\beta, 1]. \tag{1.5}
\]

And even in this case, the result of Theorem 1.1 still holds.

2. Of course, in the previous case of a strongly degenerate equation, if \( a \equiv 0 \) on a subinterval of \( (0,\alpha) \), (1.3) is false. But even for a weaker degeneracy satisfying (1.1), classical null controllability may be false. Indeed, in section 3.2, we present a simple example for which \( a(0) = 0, a(x) > 0 \) if \( x \in (0,1] \), and (1.3) is false. Hence regional controllability is a relevant notion for degenerate problems.

3. Property (1.3) is strong (in the sense that it is automatically preserved with time) : under (1.3), if we stop controlling the system at time \( T \), then for all \( t \geq T \), \( u(t) \equiv 0 \) in \( (0,1) \). Regional null controllability is a weaker property, since due to the uncontrolled part on \( (0,\alpha+\delta) \), (1.4) is no more preserved with time if we stop controlling at time \( T \).

In view of our last remark, it is important to improve Theorem 1.1 in order to obtain a persistent regional null controllability result, that will hold during a given time interval \((T,T')\).

Hence, we now consider the following problem for all \( T' > T > 0 \):

\[
\begin{aligned}
&u_t - (a(x)u_x)_x + c(x)u = f(t,x)\chi_{(\alpha,\beta)}(x), \quad (t,x) \in (0,T') \times (0,1), \\
&\lim_{x \to 0} a(x)u_x(t,x) = 0, \quad t \in (0,T'), \\
&u(t,1) = 0, \quad t \in (0,T'), \\
&u(0,x) = u_0(x), \quad x \in (0,1),
\end{aligned} \tag{1.6}
\]

where \( u_0 \in L^2(0,1) \) and \( f \in L^2((0,T') \times (0,1)) \). Then we prove the following persistent regional property of null controllability:

**Theorem 1.2. Persistent regional null controllability.** Given \( T' > T > 0, u_0 \in L^2(0,1), \) and \( \delta > 0 \) such that \( \alpha + \delta < \beta \), there exists \( f \in L^2((0,T') \times (0,1)) \) such that the solution \( u \) of (1.6) satisfies

\[
u(t, x) = 0 \quad \text{for} \quad (t, x) \in (T,T') \times (\alpha + \delta, 1). \tag{1.7}
\]
The proofs of Theorems 1.1 and 1.2 are based on some suitable penalized problems, and on appropriate observability inequalities for the adjoint problem (see Theorems 2.4 and 2.5 in section 2). Such inequalities are derived from classical Carleman estimates for nondegenerate equations by an appropriate use of cut-off functions.

In equation (1.2), we have considered the Neumann boundary condition at \( x = 0 \), i.e. \( a(x)u_x(t, x) \to 0 \) as \( x \to 0 \), since it is the only one which allows a unified treatment of well-posedness, independent of the degree of degeneracy of \( a \). Moreover such a condition is automatically satisfied if \( a \) is highly degenerate (see Campiti, Metafune and Pallara [6]). However, the controllability results given in Theorems 1.1 and 1.2 are independent of this boundary condition.

Finally, using the new observability inequalities derived to prove Theorems 1.1 and 1.2, we also deduce regional null controllability properties for a Crocco type linearized equation (see section 4).

Moreover our method can also be applied to unbounded domains. In section 5, we study the null controllability of the (nondegenerate) heat equation in an unbounded domain. In particular, we obtain results of regional and persistent regional null controllability that extend some recent results of Micu and Zuazua [30] and of Cabanillas, De Menezes and Zuazua [5].

The paper is organized as follows:
- in section 2, we give explicit statements of our results for degenerate problems;
- in section 3, we make some further comments;
- in section 4, we give the results for the Crocco type equation;
- in section 5, we state our results in the case of an unbounded domain;
- in section 6, we prove the results concerning the degenerate equation;
- in section 7, we give the proofs for the Crocco type equation;
- in section 8, we give the proofs in the case of an unbounded domain;
- in the appendix, we give the proof of well-posedness.

2. Controllability of Degenerate Parabolic Equations.

2.1. Well-posedness of the problem. First, we discuss the well-posedness of (1.2), assuming that \( a \) satisfies (1.1). Without loss of generality, we assume throughout this paper that \( c \geq 0 \). (Otherwise one can reduce the problem to this case introducing \( \tilde{u} = e^{-\lambda t}u \) for a suitable \( \lambda > 0 \).

We define the operator \((A, D(A))\) by
\[
\begin{align*}
D(A) &:= \{ u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1), \\
& \quad a u \in H^1_0(0, 1), a u_x \in H^1(0, 1) \text{ and } \lim_{x \to 0} a u_x(x) = 0 \}, \\
\forall u \in D(A), \quad Au &:= (a u_x)_x - cu.
\end{align*}
\] (2.1)

Note that, for all \( u \in D(A) \), \((a u)(0) = 0\) and \( u(1) = 0\). We also introduce the following weighted spaces that will be useful to characterize \( D(A) \):
\[
H^1_{a}(0, 1) := \{ u \in L^2(0, 1) \mid u \text{ locally absolutely continuous in } (0, 1), \\
\sqrt{a} u_x \in L^2(0, 1) \text{ and } u(1) = 0 \}.
\]

We first prove

**Proposition 2.1**. \( D(A) := \{ u \in H^1_{a}(0, 1) \mid a u_x \in H^1(0, 1) \} \).

Then we deduce
Proposition 2.2. A : \( D(A) \rightarrow L^2(0,1) \) is a closed self-adjoint negative operator with dense domain.

Hence A is the infinitesimal generator of a strongly continuous semigroup \( e^{At} \) on \( L^2(0,1) \). Thus we obtain

**Theorem 2.3.** Let \( f \) be given in \( L^2((0,T) \times (0,1)) \). Then for all \( u_0 \in D(A) \), there exists a unique strong solution \( u \in C^0([0, +\infty); D(A)) \) of (1.2); and for all \( u_0 \in L^2(0,1), \) there exists a unique weak solution \( u \in C^0([0, +\infty); L^2(0,1)) \) of (1.2).

For this result, we refer to Campiti, Metafune and Pallara [6] : they study the case where \( c \equiv 0 \) and \( a \in C^1([0,1]), \) \( a(0) = a(1) = 0, \) under the boundary conditions \( (au_x)(x) = 0 \) at \( x = 0 \) and \( x = 1. \) Here, we assume that \( a(x) \) vanishes only at \( x = 0. \) Thus, the spaces \( D(A) \) and \( H^1_0 \) are not exactly the same but the proof is based on similar arguments. We give the proof in Appendix for the reader’s convenience.

2.2. First observability inequality. First we consider the adjoint equation:

\[
\begin{aligned}
&v_t + (a(x)v_x)_x - c(x)v = 0, \quad (t,x) \in (0,T) \times (0,1), \\
&\lim_{x \to 0} a(x)v_x(t,x) = 0, \quad t \in (0,T), \\
&v(t,1) = 0, \quad t \in (0,T), \\
&v(T, \cdot) \in L^2(0,1).
\end{aligned}
\]

(2.2)

And we prove the following observability inequality:

**Theorem 2.4.** Choose \( \delta > 0 \) such that \( \alpha + \delta < \beta. \) Then, for any \( T > 0, \) there exists \( C > 0 \) such that the solution \( v \) of (2.2) satisfies

\[
\int_0^1 v(0,x)^2 \, dx \leq C \int_0^T \int_\alpha^\beta v(t,x)^2 \, dx dt + C \int_0^{\alpha+\delta} v(T,x)^2 \, dx.
\]

Note that this is just the classical observability inequality with an additional term (the last term). And this new term leads to a regional controllability result instead of the classical one.

Note also that the result still holds if we assume (1.5) instead of (1.1).

We prove this estimate using some cut-off functions in order to treat separately the region where \( a \) may vanish from the region where \( a \) does not vanish. In particular, in the second region, this leads to a non degenerate equation for which we can apply Carleman estimates.

**Consequence: regional null controllability.** Theorem 1.1 follows from Theorem 2.4 considering the penalized minimization problem:

\[
\inf_{f \in L^2((0,T) \times (0,1))} \frac{1}{2} \int_0^T \int_0^1 f(t,x)^2 \, dx dt + \frac{1}{2\varepsilon} \int_{\alpha+\delta}^1 u^f(T,x)^2 \, dx,
\]

(2.3)

where \( u^f \) is the solution of (1.2) associated to \( f. \) Note that this is the classical penalized minimization problem where the last term had been modified in order to lead to a regional null controllability result.

**Remarks.**

1. If \( \alpha = 0, \) the null controllability holds, in fact, on the whole interval \((0,1)\) (and not only on \((\delta,1)\)).

2. **Approximate controllability.** Note also that, assuming (1.1), approximate controllability holds for the degenerate problem (1.6) : given \( T > 0, \) \( \varepsilon > 0, \) \( u_0 \in L^2(0,1) \) and \( \varepsilon^T \in L^2(0,1), \) there exists \( f \in L^2((0,T) \times (0,1)) \) such that \( \| u(T) -
z^T \|L^2(0,1) \leq \epsilon$. Indeed approximate controllability reduces to proving a unique continuation property. And this property follows from local Carleman estimates, (see for example V. Isakov [21]).

2.3. Second observability inequality. Now we consider the following non homogeneous adjoint equation:

\[
\begin{cases}
  w_t + (a(x)w_x)_x - c(x)w = G(t,x)\chi_{(T,T')} (t), & (t,x) \in (0,T') \times (0,1), \\
  \lim_{x \to 0} a(x)w_x(t,x) = 0, & t \in (0,T'), \\
  w(t,1) = 0, & t \in (0,T'), \\
  w(T',\cdot) \in L^2(0,1),
\end{cases}
\]

where $G \in L^2((0,1) \times (T,T'))$. And we prove the following observability inequality:

**Theorem 2.5.** Choose $\delta > 0$ such that $\alpha + \delta < \beta$. Then for any $T' > T > 0$, there exists $C > 0$ (independent of $G$) such that the solution $w$ of (2.4) satisfies

\[
\int_0^1 w(0,x)^2 \, dx \leq C \int_0^{T'} \int_0^\beta w(t,x)^2 \, dx \, dt + C \int_0^{T'} \int_0^{\alpha+\delta} w(T',x)^2 \, dx + C \int_T^{T'} \int_0^{\alpha+\delta} G(t,x)^2 \, dx \, dt.
\]

The proof of Theorem 2.5 follows from Theorem 2.4.

**Consequence: persistent regional null controllability.** The proof of Theorem 1.2 follows from Theorem 1.1 using an appropriate cut-off function. However, in an earlier version of this paper [7], we also noted that it can be proved by introducing the appropriate penalized minimization problem:

\[
\inf_{f \in L^2((0,T') \times (0,1))} \frac{1}{2} \int_0^{T'} \int_0^1 f(t,x)^2 \, dx dt + \frac{1}{2\epsilon} \int_T^{T'} \int_0^{\alpha+\delta} u^f(t,x)^2 \, dx dt,
\]

where $u^f$ is the solution of (1.6) associated to $f$. Then the proof is based on the observability estimate stated in Theorem 2.5. In this paper, we give the two methods for the proof of Theorem 1.2:

- the first method is a direct proof of Theorem 1.2 : we apply Theorem 1.1 using appropriate cut-off functions;
- the second method is more general : we prove Theorem 1.2 via the penalized problem (2.6) and using the observability inequality given in Theorem 2.5. Even if the second method is longer in this context, it is more natural and the authors think that this method is of interest in its own right, since the tools that we introduce could be useful for other problems (see for example the open questions presented in section 3.3).

**Remark.** Using the idea of the proof of Theorem 1.2, we can also consider the case of the nondegenerate heat equation with an initial datum that is equal to zero on the part $(\gamma,1)$ for some $\gamma < \beta$. Then we can construct a control such that the solution $u$ of

\[
\begin{cases}
  u_t(t,x) - u_{xx}(t,x) = f(t,x)\chi_{(\alpha,\beta)}(x), & (t,x) \in (0,T) \times (0,1), \\
  u(t,0) = u(t,1) = 0, & t \in (0,T), \\
  u(0,x) = u_0(x), & x \in (0,1),
\end{cases}
\]

satisfies $u_{xx}(T,x) = 0$ for all $x \in (0,1)$.
satisfies
\[ u(t, x) = 0 \quad \text{for} \quad (t, x) \in (0, T) \times (\max(\gamma, \alpha) + \delta, 1) \quad \text{and} \quad u(T) \equiv 0. \]

3. Further Comments.

3.1. Motivations. First, let us recall the main results for nondegenerate parabolic equations, and let us explain why we are interested in degenerate problems, and why we study only regional null controllability properties.

In the non-degenerate case (i.e. \( a > 0 \) on \([0, 1]\)), (global) null controllability is by now well-known: for all \( T > 0 \), there exists \( f \in L^2((0, T) \times (0, 1)) \) such that the solution of (1.2) satisfies \( u(T) \equiv 0 \) in the whole domain \((0, 1)\). This result is in general obtained via Carleman estimates (see, for example, [19, 17, 1]). More generally, several results were obtained for non-degenerate parabolic equations possibly with a semilinear term. See, in particular, Fattorini and Russell [14, 15] for results in one space dimension or under special geometric conditions, Lebeau and Robbiano [25] for a general result without geometric conditions, Fursikov and Imanuvilov [19] for similar results in a more general context, including heat equations with time-dependent coefficients, Fernández-Cara [16], Fernández-Cara and Zuazua [17, 18] for similar results for semilinear heat equations.

However, many problems that are relevant for applications are described by degenerate parabolic equations where degeneracy occurs at the boundary of the space domain. Therefore it is interesting to study controllability for this kind of problems as well.

For example, the velocity field of a laminar flow on a flat plate can be described by the Prandtl equations (see Oleinik [32]). By using the so-called "Crocco" transformation, these equations are transformed into a nonlinear degenerate parabolic equation (the Crocco equation, see [32]) which is stated in a bounded domain \( \Omega = (0, L) \times (0, 1) \). The linearization of the Crocco equation around a stationary solution is an equation of the form:

\[
\begin{cases}
u t + b \nu_x - a \nu_{yy} + c \nu = f, & (x, y, t) \in \Omega \times (0, T), \\
u_0(x, 0, t) = u(x, 1, t) = 0, & (x, t) \in (0, L) \times (0, T), \\
u(0, y, t) = u_1(y, t), & (y, t) \in (0, 1) \times (0, T), \\
u(x, y, 0) = u_0(x, y), & (x, y) \in \Omega,
\end{cases}
\]

(3.1)

where \( f \) and \( u_1 \) depend on the incident velocity of the flow, and where the coefficients \( a, b \) and \( c \) are regular, but degenerate, and have the following behavior (see Buchot and Villedieu [4], Buchot and Raymond [3]):

\[ 0 < b_1 \leq \frac{b(y)}{y} \leq b_2, \quad 0 < a_1 \leq \frac{a(x, y)}{-(y-1)^2 \ln(\mu(1-y))} \leq a_2, \quad c(x, y) \geq 0, \]

where \( 0 < \mu < 1 \). Note that (3.1) is also degenerate because there is no term in \( u_{xx} \). See [28, 29] for the study of null controllability of the solutions of (3.1) when the coefficients \( a, b \) are constants.

3.2. Notion of regional controllability. It is interesting to note that even if \( a \) vanishes only at \( x = 0 \), then global null controllability may not hold. Indeed, if we consider \( a(x) = x^p \) with \( p > 2 \) in problem (1.2), and

\[ c(x) = - \frac{p}{4} \left( \frac{3p}{4} - 1 \right) x^{p-2}, \quad x \in [0, 1], \]
then using the following classical change of variables (see [9] p. 292)

\[ X := \int_x^1 \frac{1}{\sqrt{a}}, \quad \text{and} \quad U(t, X) := a(x)^{1/4} u(t, x), \]

problem (1.2) is transformed into the heat equation on the half-line:

\[
\begin{cases}
U_t(t, X) - U_{XX}(t, X) = \tilde{f}(t, X) \chi(\tilde{\alpha}, \tilde{\beta})(X), \\
U(t, 0) = 0,
\end{cases}
\]

Then, we can apply a result by Micu and Zuazua [30]: they proved that, within the class of solutions defined by transposition, no compactly supported initial datum different from 0 can be driven to zero in finite time by a control acting at the extremity \( x = 0 \). Similar results hold when \( \Omega \) is the half space \( \mathbb{R}^n_+ \) and when the control is distributed in a bounded control region \( \omega \) (see Micu and Zuazua [31]). The lack of null controllability comes from the fact that the controlled heat equation holds in an unbounded domain, while the control is localized on a bounded domain, thus an unbounded region is left without control. Hence, if \( \alpha > 0 \), the control region \( (\tilde{\alpha}, \tilde{\beta}) \) of the above problem is bounded and we cannot expect global null controllability results. So, even in this weakly degenerate case, global null controllability does not hold, and regional null controllability is a relevant notion. Note also that, if \( p = 4/3 \), then the transformed problem is the classical heat equation set in a bounded domain, for which global null controllability holds.

### 3.3. Open questions

The main extension of these results would be to enlarge the part of the domain where we can control the equation, or to prove in some cases that (global) null controllability holds.

**Domain of controllability.** As a consequence of our method (based on classical global Carleman estimates and on cut-off functions), the regional and persistent regional null controllability results only hold in the interval \((\alpha + \delta, 1)\). It would be interesting to improve these two results by enlarging this domain.

Under assumption (1.1) (respectively (1.5)), it would be natural to prove controllability on the domain \((\delta, 1)\) (respectively on the domain \((\alpha_0 + \delta, 1)\) where \(\alpha_0 = \inf\{x \in [0, 1] : a(x) > 0\}\)).

For the moment, these questions remain open. Progress could be made trying to improve the two observability inequalities given in Theorems 2.4 and 2.5. For this, it could be useful to obtain new Carleman estimates for degenerate parabolic problems like (1.2).

However, even if we conjecture that Theorems 1.1 and 1.2 may be improved in this direction, this work is a first step in the study of degenerate parabolic equation like (1.2). Indeed we provide here some tools that may be useful for this study (like the notions of regional controllability, the associated penalized problems and the form of the observability inequalities that are required to solve these questions).

Note also that, for the moment, the possibility of using a boundary control supported at \( x = 1 \) is another open question. This problem is connected to the previous one.

**Global null controllability.** In some "weakly degenerate" case, it seems that the domain of controllability is \((0, 1)\), i.e. that (global) null controllability holds. In section 3.2, we provided an example for which (global) null controllability is true \((a(x) = x^{4/3})\) and another example for which it is false \((a(x) = x^p\) with \(p > 2\)).
result seems to be related to the 'degree of degeneracy' of \( a \) i.e. to the exponent \( p \) when \( a(x) = x^p \). We plan to investigate this question in a forthcoming paper.

4. Application to a Crocco Type Equation. Let \( \Omega = (0, L) \times (0, 1) \) and let 
\( \omega = \omega_x \times \omega_y \) where \( \omega_x \) and \( \omega_y \) are two non empty open subsets of \( (0, L) \) and \( (0, 1) \), respectively. For \( u_0 \in L^2(\Omega) \), \( u_1 \in L^2((0, 1) \times (0, T)) \) and \( f \in L^2(\omega \times (0, T)) \), we consider the following initial-boundary value problem:

\[
\begin{align*}
& u_t + u_x - (a(y)u_y)_y + c(y)u = f(x, y, t) \chi_\omega, \quad (x, y, t) \in \Omega \times (0, T), \\
& u(x, 0, t) = 0, \quad (x, t) \in (0, L) \times (0, T), \\
& (au_y)(x, 1, t) = 0, \quad (x, t) \in (0, L) \times (0, T), \\
& u(0, y, t) = u_1(y, t), \quad (y, t) \in (0, 1) \times (0, T), \\
& u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega,
\end{align*}
\]

(4.1)

In [29], the authors study this model for \( a(\cdot) \) equal to a positive constant and \( c \equiv 0 \), with \( \omega_x := \langle x_0, x_1 \rangle \). They prove that for all \( T > 0 \), \( \delta > 0 \) such that \( \delta < (x_1 - x_0)/2 \), \( u_0 \in L^2(\Omega) \), \( u_1 \in L^2((0, 1) \times (0, T)) \), there exists \( f \in L^2(\omega \times (0, T)) \) such that \( u(x, y, T) = 0 \) for \( (x, y) \in \Omega_C(T, \delta) \), where \( \Omega_C(T, \delta) \) is defined by

\[
\Omega_C(T, \delta) := \begin{cases} 
(x_0 + \delta, x_1 + T - \delta) \times (0, 1) & \text{if } 0 < T < L - x_1 + \delta, \\
(x_0 + \delta, L) \times (0, 1) & \text{if } T > L - x_1 + \delta.
\end{cases}
\]

Indeed, the above domain \( \Omega_C(T, \delta) \) represents the domain of influence of \( \chi_\omega f \) at time \( T \), see [29]. (Due to diffusion in the direction \( y \), the region of influence in \( y \) at time \( T \) of a control supported in \( y \) in \( \omega_y \) is the whole interval \( (0, 1) \). On the other hand, due to the transport phenomenon at speed equal to 1 in the \( x \)-direction, at time \( T \), the region of influence in \( x \) by a control supported in \( \langle x_0, x_1 \rangle \) is only \( \langle x_0, x_1 + T \rangle \) in the case \( T < L - x_1 \), and \( \langle x_0, L \rangle \) in the case \( T > L - x_1 \).)

Notice that the above result still holds (with the same proof) in the case of non constant coefficients as soon as \( a \) is non degenerate: for example, we may assume \( c \in L^\infty(0, 1), \ a \in C^1([0, 1]), \ a > 0 \) on \( [0, 1] \).

In this paper, we consider the case of a degenerate coefficient \( a \): we assume

\[
c \in L^\infty(0, 1), \ a \in C^1([0, 1]), \ a > 0 \text{ on } [0, 1] \text{ and } a(1) = 0.
\]

Moreover, we assume that the control is uniformly distributed in the direction \( x \), i.e., we consider \( \omega_x := (0, L) \). On the other hand, we take \( \omega_y = (\alpha, \beta) \) with \( 0 \leq \alpha < \beta \leq 1 \). Hence the region of influence in \( x \) at any time \( T \) of \( \chi_\omega f \) will be \( (0, L) \). But now, due to the degeneracy of \( a \) at \( y = 1 \), we expect that the region of influence in \( y \) at any time \( T \) of \( \chi_\omega f \) will (at least) contain \( (0, \beta - \delta) \). Thus it is natural to define for all \( \delta > 0 \) such that \( \delta < \beta \),

\[
\Omega_C(\delta) := (\delta, L) \times (0, \beta - \delta),
\]

and to prove a regional controllability property in such a domain:

**Theorem 4.1.** Under the previous assumptions, for all \( u_0 \in L^2(\Omega) \), \( u_1 \in L^2((0, 1) \times (0, T)) \), there exists \( f \in L^2(\omega \times (0, T)) \) such that the solution \( u^f \) of (4.1) satisfies

\[
u^f(x, y, T) = 0 \quad \text{for} \ (x, y) \in \Omega_C(\delta).
\]
From this result, we can also deduce a result of persistent regional controllability in the domain $\Omega C(\delta) \times (T, T')$. (The proof of this fact is left to the reader). The proof of Theorem 4.1 is based, as in [29], on the following penalized problem

$$\inf_{f \in L^2(\Omega \times (0, T))} \left( \frac{1}{2} \iint_{\Omega \times (0, T)} f^2 \, dt \, dy \, dx + \frac{1}{2\varepsilon} \iint_{\Omega C(\delta)} u^f(x, y, T)^2 \, dy \, dx \right),$$

where $u^f$ is the solution of (4.1) associated with $f$. As in [29], the proof of Theorem 4.1 reduces to proving the following observability estimate:

**Theorem 4.2.** Under the previous assumptions, there exists $C = C(\delta, \omega_y) > 0$ such that the solution $v$ of the adjoint equation

$$\begin{cases}
v_t + v_x + (a(y)v_y)_y - c(y)v = 0, & (x, y, t) \in \Omega \times (0, T), \\
v(x, 0, t) = 0, & (x, t) \in (0, L) \times (0, T), \\
(\alpha v_y)(x, 1, t) = 0, & (x, t) \in (0, L) \times (0, T), \\
v(0, y, t) \in L^2((0, 1) \times (0, T)), \\
v(x, y, T) \in L^2(\Omega),
\end{cases}$$

(4.2)

satisfies

$$\iint_{(0, L) \times (0, 1)} v(x, y, 0)^2 \, dy \, dx + \iint_{(0, 1) \times (0, T)} v(0, y, t)^2 \, dt \, dy \leq C \left( \iint_{\Omega \times (0, T)} v(x, y, t)^2 \, dt \, dy \, dx \\
+ \iint_{\Omega C(\delta)} v(x, y, T)^2 \, dy \, dx + \iint_{(0, 1) \times (0, T)} v(L, y, t)^2 \, dt \, dy \right).$$

(4.3)

The proof of Theorem 4.2 is based on a decomposition of the domain $\Omega \times (0, T)$ and on the observation that, along the characteristics, $v$ is solution of a degenerate parabolic equation like (2.2). Then, on each subdomains and along characteristics, we can apply the regional observability inequality of Theorem 2.4.

**5. Heat Equation in Unbounded Domain.** Now we consider the (non degenerate) heat equation in the half-line $\mathbb{R}^+$: for all $T' > T > 0$ and $u_0 \in L^2(0, +\infty)$,

$$\begin{cases} 
  u_t - u_{xx} = f(t, x)\chi_{(\alpha, \beta)}(x), & (t, x) \in (0, T') \times (0, +\infty), \\
  u(t, 0) = 0, & t \in (0, T'), \\
  u(0, x) = u_0(x), & x \in (0, +\infty).
\end{cases}$$

(5.1)

First, let us note that null controllability in an unbounded domain is quite different from the same property in a bounded domain. For problem (5.1), Micu and Zuazua [30, 31] proved that no compactly supported initial datum different from 0 can be driven to zero in finite time by a control acting at the extremity $x = 0$.

More recently Cabanillas, De Menezes and Zuazua [5] obtained a positive null controllability result: for the heat equation in an unbounded domain $\Omega$ of $\mathbb{R}^N$, (global) null controllability is true if $f$ is supported by a domain $\omega$ such that $\Omega \setminus \omega$ is bounded. For example, in the case of $\Omega = (0, +\infty)$, one can choose $\omega = (\alpha, +\infty)$.

Using the same methods as in the degenerate case, we prove the following intermediate results:
Theorem 5.1.

(i) Regional null controllability. Given $T > 0$, $u_0 \in L^2(0, +\infty)$ and $\delta > 0$ such that $\beta - \delta > \alpha$, there exists $f \in L^2((0, T) \times (0, +\infty))$ such that the solution $u$ of (5.1) satisfies

$$u(T, x) = 0 \quad \text{for} \quad x \in (0, \beta - \delta).$$

(ii) Persistent regional null controllability. Given $T' > T > 0$, $u_0 \in L^2(0, +\infty)$ and $\delta > 0$ such that $\beta - \delta > \alpha$, there exists $f \in L^2((0, T') \times (0, +\infty))$ such that the solution $u$ of (5.1) satisfies

$$u(t, x) = 0 \quad \text{for} \quad (t, x) \in (T, T') \times (0, \beta - \delta).$$

Remarks. 1. In particular, if $\beta = +\infty$, we find again null controllability in the whole domain $(0, +\infty)$ as in [5].

2. In [8] we give another (global) null controllability result for controls supported by unbounded domain of finite measure. More precisely, we consider the problem

$$\begin{align*}
  u_t - u_{xx} &= f(t, x) \chi_\omega(x), \quad t \in (0, T), x \in (0, \infty), \\
  u(t, 0) &= 0, \quad t \in (0, T), \\
  u(0, x) &= u_0(x), \quad x \in (0, +\infty).
\end{align*}$$

We take control regions of the form $\omega := \cup_n (a_n, b_n)$, where the intervals $(a_n, b_n)$ are disjoint and $a_n \to \infty$ as $n \to \infty$. In particular, we allow $|\omega| < \infty$. We obtain controllability results in weighted spaces.

6. Proofs in the Degenerate Case.

6.1. Proof of Theorem 2.4. For the proof of Theorem 2.4, we will use the following Lemma:

Lemma 6.1. Let $v$ be a strong solution of (2.2) and let $\psi$ be given in $C^\infty(\mathbb{R})$. If we set $w(t, x) := \psi(x)\psi(t, x)$, then $w$ is a solution of

$$\begin{align*}
  w_t + (a(x)w_x)_x - c(x)w &= h(t, x), \quad (t, x) \in (0, T) \times (0, 1), \\
  \lim_{x \to 0} a(x)w_x(t, x) &= 0, \quad t \in (0, T), \\
  w(t, 1) &= 0, \quad t \in (0, T),
\end{align*}$$

where $h(t, x) = (\text{av}_x v)_x + a\text{av}_x v_x$. Moreover, for every $0 \leq t_0 \leq t_1 \leq T$, we have

$$\begin{align*}
  \frac{1}{2} \int_0^1 \psi(x)^2 v(t_1, x)^2 \, dx - \int_{t_0}^{t_1} \int_0^1 c(x)\psi(x)^2 v(t, x)^2 \, dx \, dt \\
  &= \frac{1}{2} \int_0^1 \psi(x)^2 v(t_0, x)^2 \, dx + \int_{t_0}^{t_1} \int_0^1 a\psi_x^2 v_x^2 \, dx \, dt + 2 \int_{t_0}^{t_1} \int_0^1 \psi \psi_x v v_x \, dx \, dt.
\end{align*}$$

Proof of Lemma 6.1. The only point to check in order to show that $w$ satisfies (6.1) is the boundary condition at $x = 0$: for each $t$, $v(t) \in D(A)$, and so

$$(aw_x)(x = 0) = \psi_x(0)(av)(0) + \psi(0)(av_x)(0) = 0,$$

since $(av)(0) = 0$ and $(av_x)(0) = 0$. Then (6.2) is obtained by direct computations. \qed
Now, we turn to the proof of Theorem 2.4 proceeding with several steps. (Note that it is sufficient to prove Theorem 2.4 for strong solutions $v$ of (2.2). Then the result follows for weak solutions by density arguments).

**Step 1.** First we apply (6.2) with $t_0 = 0$ and $t_1 = t \in [0, T]$ for $\psi \equiv 1$ to obtain

$$\forall t \in [0, T], \quad \int_0^1 v(0, x)^2 \, dx \leq \int_0^1 v(t, x)^2 \, dx.$$  

(We used that $c \geq 0$). Thus taking the integral over $t \in (T/4, 3T/4)$,

$$\frac{T}{2} \int_0^1 v(0, x)^2 \, dx \leq \int_{T/4}^{3T/4} \int_0^1 v(t, x)^2 \, dx \, dt.$$  

(6.3)

**Step 2.** Fix $\beta'$ and $\beta''$ such that $0 < \beta'' < \beta' < \beta$ and let $\psi$ be given in $C^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$ and

$$\left\{ \begin{array}{l} \psi(x) = 0, \quad 0 \leq x \leq \beta'', \\
 \psi(x) = 1, \quad \beta' \leq x \leq 1. \end{array} \right.$$  

(In the case of assumption (1.5), we define $\alpha_0 := \inf\{x \in [0, 1] : a(x) > 0\}$ and we notice that $0 \leq \alpha_0 \leq \beta$. Then the proof is the same choosing $\beta'$ and $\beta''$ such that $\max\{\alpha_0, \alpha\} < \beta'' < \beta' < \beta$).

Now, we define $w(t, x) := \psi(x) v(t, x)$. Then $w$ is a solution of

$$\left\{ \begin{array}{l} w_t + (a(x)w_x)_x - c(x)w = h(t, x), \quad (t, x) \in (0, T) \times (\beta'', 1), \\
w(t, \beta'') = \psi(t, 1) = 0, \quad t \in (0, T), \end{array} \right.$$  

(6.4)

where $h(t, x) = (a\psi_x v)_x + a\psi_x v_x$.

Notice that (6.4) is a nondegenerate equation. Hence, we claim that

$$\int_{T/4}^{3T/4} \int_{\beta'}^1 v(t, x)^2 \, dx \, dt \leq C \int_0^T \int_0^\beta v(t, x)^2 \, dx \, dt.$$  

(6.5)

In order to prove (6.5), we will apply Carleman estimates to $w$ on the domain $(0, T) \times (\beta'', 1)$ where equation (6.4) is nondegenerate. For this, we need to introduce further notations: first, choose a function $\phi : \mathcal{O} := (\beta'', 1) \to \mathbb{R}$ of class $C^2$ such that $\phi'(x) < 0$ for all $x \in \mathcal{O}$. Then, introduce

$$\theta(t) := \frac{1}{t(T-t)}, \quad \sigma(t, x) := \theta(t)(e^{2S\|\phi\|_\infty} - e^{S\phi(x)}), \quad \text{and} \quad r(t, x) := R S \theta(t) e^{S\phi(x)},$$  

where $R$ and $S$ are constants. Then, if $R$ and $S$ are sufficiently large, the solution $w$ of (6.4) satisfies the following Carleman estimate (see for example [19, 17, 1]):

$$S \| w \|_{L^2_x}^2 \leq C \| e^{-R\sigma} w \|_{L^2((0, T) \times \mathcal{O})}^2 + C \int_0^T \int_{\partial\mathcal{O}} e^{-2R\sigma} \partial_x \phi |\partial_x w|^2,$$

where $C$ is a constant and

$$\|z\|_{L^2}^2 = \int_0^T \int_{\mathcal{O}} \left( \psi^2 z^2 + z_t^2 + z_{xx}^2 + (\psi z_x)^2 \right) \, dx \, dt.$$

Note that $\partial_x w = 0$ at $x = \beta''$, and that

$$\| w \|_{L^2_x}^2 \geq C \int_{T/4}^{3T/4} \int_{\beta'}^1 w^2 \, dx \, dt.$$
with \( C = \min\{r^3e^{-2R\sigma}, t \in [T/4, 3T/4], x \in [\beta', 1]\} > 0 \). This implies that

\[
\int_{T/4}^{3T/4} \int_{\beta'}^{1} w(t, x)^2 \, dx \, dt \\
\leq C \int_{0}^{T} \int_{\beta'}^{1} e^{-2R\sigma} h(t, x)^2 \, dx \, dt + C \int_{0}^{T} r(t, 1)e^{-2R\sigma} \phi'(1)|\partial_v w(1, t)|^2 \, dt \\
\leq C \int_{0}^{T} \int_{\beta'}^{1} e^{-2R\sigma} h(t, x)^2 \, dx \, dt,
\]

since \( \phi'(1) < 0 \). Using \( w = v \) on \((\beta', 1)\) and the definition of \( h \), we deduce that

\[
\int_{T/4}^{3T/4} \int_{\beta'}^{1} v(t, x)^2 \, dx \, dt = \int_{T/4}^{3T/4} \int_{\beta'}^{1} w(t, x)^2 \, dx \, dt \\
\leq C \int_{0}^{T} \int_{0}^{1} e^{-2R\sigma} h(t, x)^2 \, dx \, dt \\
\leq C \int_{0}^{T} \int_{\beta'}^{1} v(t, x)^2 \, dx \, dt + C \int_{0}^{T} \int_{\beta'}^{1} e^{-2R\sigma} v_x(t, x)^2 \, dx \, dt, \quad (6.6)
\]

since \( \psi_x \) and \( \psi_{xx} \) are supported in \((\beta'', \beta')\).

Then it remains to prove the following Caccioppoli-type inequality for \( v \):

\[
\int_{0}^{T} \int_{\beta'}^{1} e^{-2R\sigma} v_x(t, x)^2 \, dx \, dt \leq C \int_{0}^{T} \int_{\alpha}^{\beta} v(t, x)^2 \, dx \, dt. \quad (6.7)
\]

For this, we choose \( \xi \in C^{\infty}(\mathbb{R}) \) such that \( 0 \leq \xi \leq 1 \) and

\[
\begin{cases}
\xi(x) = 0, & x \in (0, 1) \setminus (\alpha, \beta), \\
\xi(x) = 1, & x \in (\beta'', \beta').
\end{cases}
\]

Then defining \( \eta(t, x) := \xi(x)e^{-R\sigma(t, x)} \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{0}^{1} \eta^2 v^2 \, dx = \int_{0}^{1} \eta \eta_t v^2 + \eta^2 v v_t \, dx = \int_{0}^{1} \eta \eta_t v^2 + \eta^2 v[-(av)_x] + c(x)v \, dx \\
= \int_{0}^{1} \eta \eta_t v^2 + a\eta^2 v_x^2 + 2a\eta_x v v_x \, dx + \int_{0}^{1} \eta^2 c v_x^2 \, dx.
\]

Therefore, recalling that \( c \geq 0 \) and \( \eta \equiv 0 \) for \( t = 0 \) and for \( t = T \), we have

\[
\int_{0}^{T} \int_{0}^{1} a\eta^2 v_x^2 \, dx \\
\leq - \int_{0}^{T} \int_{0}^{1} (\eta \eta_t v^2 + 2a\eta_x v v_x) \, dx \, dt + \frac{1}{2} \left[ \int_{0}^{1} \eta^2 v_x^2 \, dx \right]_{0}^{T} \\
\leq \int_{0}^{T} \int_{0}^{1} (|\eta \eta_t| v^2 + 2a\eta_x v v_x + \frac{1}{2} a\eta^2 v_x^2) \, dx \, dt \\
\leq C \int_{0}^{T} \int_{\alpha}^{\beta} v^2 \, dx \, dt + \frac{1}{2} \int_{0}^{T} \int_{0}^{1} a\eta^2 v_x^2 \, dx \, dt,
\]
since \( \eta \) is supported in \((\alpha, \beta)\). Then, (6.7) follows from
\[
\min_{[\beta', \beta]} \left( a \right) \int_{0}^{T} \int_{\beta'}^{\beta} e^{-2Re(t, x)} dx dt \leq \int_{0}^{T} \int_{0}^{1} a\eta^2 v_x^2 dx.
\]

Finally, from (6.6) and (6.7), we obtain (6.5).

**Step 3.** Now we choose \( \psi \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \psi \leq 1 \) and
\[
\begin{cases}
\psi(x) = 1, & 0 \leq x \leq \alpha, \\
\psi(x) = 0, & \alpha + \delta \leq x \leq 1.
\end{cases}
\]
Then, (6.2) gives, for any \( 0 \leq t \leq T \),
\[
\frac{1}{2} \int_{0}^{1} \psi(x)^2 v(T, x)^2 dx - \int_{t}^{T} \int_{0}^{1} c(x) \psi(x)^2 v(s, x)^2 dx ds \\
= \frac{1}{2} \int_{0}^{1} \psi(x)^2 v(t, x)^2 dx + \int_{t}^{T} \int_{0}^{1} a\psi^2 v_x^2 dx ds + 2 \int_{t}^{T} \int_{0}^{1} a\psi_x v v_x dx ds \\
\geq \frac{1}{2} \int_{0}^{1} \psi(x)^2 v(t, x)^2 dx - \int_{t}^{T} \int_{0}^{1} a\psi^2 v^2 dx ds.
\]
Hence
\[
\int_{0}^{1} \psi(x)^2 v(t, x)^2 dx \leq \int_{0}^{1} \psi(x)^2 v(T, x)^2 dx + 2 \int_{t}^{T} \int_{0}^{1} a\psi^2 v^2 dx ds.
\]
Finally, using the definition of \( \psi \), and taking the integral over \( t \in (0, T) \), we obtain
\[
\int_{t}^{T} \int_{0}^{1} v^2 dx dt \leq T \int_{0}^{1} \psi(T, x)^2 dx + C \int_{0}^{T} \int_{\alpha}^{\alpha + \delta} v^2 dx dt. \tag{6.8}
\]

**Conclusion.** From (6.3), (6.5) and (6.8), we finally obtain
\[
\frac{T}{2} \int_{0}^{1} v(0, x)^2 dx \leq \int_{T/4}^{3T/4} \int_{0}^{1} v(t, x)^2 dx dt \\
\leq \int_{T/4}^{3T/4} \int_{0}^{1} v(t, x)^2 dx dt + \int_{T/4}^{3T/4} \int_{\alpha}^{\beta} v(t, x)^2 dx dt + \int_{T/4}^{3T/4} \int_{\beta'}^{1} v(t, x)^2 dx dt \\
\leq C \int_{0}^{T} \int_{0}^{1} v(t, x)^2 dx dt + C \int_{0}^{\alpha + \delta} v(T, x)^2 dx,
\]
which concludes the proof of Theorem 2.4.

\[\square\]

6.2. **Proof of Theorem 2.5.** Since \( w \) is solution of (2.2) in \((0, T)\), we can apply Theorem 2.4, which gives the first observability estimate:
\[
\int_{0}^{1} w(0, x)^2 dx \leq C \int_{0}^{T} \int_{0}^{1} w(t, x)^2 dx dt + C \int_{0}^{\alpha + \delta/2} w(T, x)^2 dx. \tag{6.9}
\]

Now, we need an estimate for the last term of (6.9). Let \( \psi \in C^\infty(\mathbb{R}) \) be such that \( 0 \leq \psi \leq 1 \) and
\[
\begin{cases}
\psi(x) = 1, & 0 \leq x \leq \alpha + \delta/2, \\
\psi(x) = 0, & \alpha + \delta \leq x \leq 1.
\end{cases}
\]
Then, \( z(t, x) := \psi(x)w(t, x) \) is a solution of

\[
\begin{cases}
  z_t + (a(x)z_x)_x - c(x)z = h(t, x), & (t, x) \in (T, T') \times (0, 1), \\
  \lim_{x \to 0} a(x)z_x(t, x) = 0, & t \in (T, T'), \\
  z(t, 1) = 0, & t \in (T, T'),
\end{cases}
\]

where

\[
h(t, x) = \psi G + a\psi_x w_x + (a\psi_x w)_x.
\]

Moreover, direct computations (similar to Lemma 6.1) yield

\[
\frac{1}{2} \frac{d}{dt} \int_0^1 z(t, x)^2 \, dx - \int_0^1 cz^2 \, dx = \int_0^1 a z_x^2 \, dx + \int_0^1 \psi G z \, dx + \int_0^1 a\psi_x w_x z \, dx + \int_0^1 (a\psi_x w)_x z \, dx
\]

\[
= \int_0^1 a z_x^2 \, dx + \int_0^1 \psi G w \, dx + \int_0^1 a\psi_x w_x \psi w \, dx - \int_0^1 a\psi_x w(\psi w)_x \, dx
\]

\[
\geq -\frac{1}{2} \int_0^1 (\psi G)^2 \, dx - \frac{1}{2} \int_0^1 (\psi w)^2 \, dx - \int_0^1 a\psi_x^2 w^2 \, dx.
\]

Set

\[
F(t) := \frac{1}{2} \int_0^1 z(t, x)^2 \, dx = \frac{1}{2} \int_0^1 \psi(x)^2 w(t, x)^2 \, dx.
\]

Then, since \( c \geq 0 \), \( F \) satisfies the differential inequality:

\[
F(t) + F'(t) \geq H(t) := -\frac{1}{2} \int_0^1 (\psi G)^2 \, dx - \int_0^1 a\psi_x^2 w^2 \, dx.
\]

This yields, for all \( t \in (T, T') \),

\[
F(t)e^t \geq F(T)e^T + \int_T^t H(\tau)e^\tau \, d\tau.
\]

Taking \( t = T' \), we obtain

\[
F(T) \leq F(T')e^{T'-T} - e^{-T} \int_T^{T'} H(\tau)e^\tau \, d\tau.
\]

Consequently,

\[
\int_0^1 \psi(x)^2 w(T, x)^2 \, dx \leq C \int_0^1 \psi(x)^2 w(T', x)^2 \, dx
\]

\[
+ C \int_T^{T'} \int_0^1 (a\psi_x^2 w^2 + (\psi G)^2) \, dx \, dt.
\]
Using the definition of $\psi$ and the fact that $\text{supp}(\psi_x) \subset (\alpha + \delta/2, \alpha + \delta) \subset (\alpha, \beta)$, we have
\[
\int_0^{\alpha+\delta/2} w(T, x)^2 \, dx \leq \int_0^{\alpha+\delta} w(T', x)^2 \, dx
\]
\[+ C \int_T^{T'} \int_0^\beta w(t, x)^2 \, dx \, dt + C \int_T^{T'} \int_0^{\alpha+\delta} G(t, x)^2 \, dx \, dt. \quad (6.10)
\]
Finally, combining (6.9) and (6.10), we obtain exactly (2.5). \qed

6.3. Proof of Theorem 1.1. For all $\varepsilon > 0$, consider the penalized problem
\[
\text{Min } \{ J_\varepsilon(f) \mid f \in L^2((0, T) \times (0, 1)) \}, \quad (6.11)
\]
where
\[J_\varepsilon(f) := \frac{1}{2} \int_0^T \int_0^1 f(t, x)^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_0^1 u^f(T, x)^2 \, dx,
\]
$u^f$ being the solution of (1.2) associated with $f$.

Functional $J_\varepsilon$ is continuous on $L^2((0, T) \times (0, 1))$, strictly convex and $J_\varepsilon(f) \to \infty$ as $\|f\|_{L^2((0, T) \times (0, 1))} \to \infty$. Thus, for all $\varepsilon > 0$, problem (6.11) has a unique solution $f^\varepsilon$. Moreover,
\[
f^\varepsilon(t, x) = -v^\varepsilon(t, x) \chi_{(\alpha, \beta)}(x), \quad (6.12)
\]
where $v^\varepsilon$ is the solution of the adjoint problem
\[
\begin{aligned}
&v^\varepsilon_t + (a(x)v^\varepsilon_x)_x - c(x)v^\varepsilon = 0, \\
&\lim_{x \to 0} a(x)v^\varepsilon_x(t, x) = 0, \\
&v^\varepsilon(t, 1) = 0, \\
&v^\varepsilon(T, x) = \frac{1}{\varepsilon} u^f(T, x) \chi_{(\alpha, \delta, 1)}(x),
\end{aligned} \quad (x, t) \in (0, 1), \quad t \in (0, T), \quad (6.13)
\]
and where $u^f$ satisfies
\[
\begin{aligned}
&u^f_t - (a(x)u^f_x)_x + c(x)u^f = f^\varepsilon \chi_{(\alpha, \beta)}(x), \\
&\lim_{x \to 0} a(x)u^f_x(t, x) = 0, \\
&u^f(t, 1) = 0, \\
&u^f(0, x) = u_0(x),
\end{aligned} \quad (x, t) \in (0, 1), \quad t \in (0, T). \quad (6.14)
\]
Indeed, $f^\varepsilon$ is characterized by $DJ_\varepsilon(f^\varepsilon) \cdot h = 0$ for all $h \in L^2((0, T) \times (0, 1))$. (As usual, $DJ_\varepsilon(f^\varepsilon) \cdot h$ denotes the differential of the functional $J_\varepsilon$ computed at the point $f^\varepsilon$ and applied to the element $h$.) In order to compute $DJ_\varepsilon(f^\varepsilon) \cdot h$, we recast problem (1.2) into an abstract form:
\[
\begin{aligned}
&u_t = Au + \chi_{(\alpha, \beta)}f, \\
&u(0) = u_0,
\end{aligned} \quad (6.15)
\]
where $D(A)$ and $Au = (au_x)_x - cu$ are defined by (2.1). The solution of (6.15) is
\[
u^f(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} \chi_{(\alpha, \beta)}f(s) \, ds =: e^{tA}u_0 + L_t f.
\]
Now we can compute $DJ_\varepsilon(f) \cdot h$ :

$$J_\varepsilon(f + h) - J_\varepsilon(f) = \frac{1}{2} \int_0^T \int_0^1 (2fh + h^2) \, dx \, dt$$

$$+ \frac{1}{2\varepsilon} \int_{\alpha+\delta}^1 (u^{f+h}(T)^2 - u^f(T)^2) \, dx.$$ 

Note that $u^{f+h} = u^f + z^h$, where $z^h$ is the solution of

$$\begin{cases}
    z_t^h = A z^h + \chi_{(\alpha,\beta)} h, & t \in (0, T), \\
    z^h(0) = 0.
\end{cases}$$

(6.16)

Hence

$$z^h(T) = L_T h = \int_0^T e^{(T-t)A} \chi_{(\alpha,\beta)} h(t) \, dt.$$ 

Therefore

$$J_\varepsilon(f + h) - J_\varepsilon(f) = \int_0^T \int_0^1 fh \, dx \, dt + \frac{1}{2} \int_0^T \int_0^1 h^2 \, dx \, dt$$

$$+ \frac{1}{\varepsilon} \int_{\alpha+\delta}^1 u^f(T)z^h(T) \, dx + \frac{1}{2\varepsilon} \int_{\alpha+\delta}^1 z^h(T)^2 \, dx.$$ 

Using the notation $\langle \cdot, \cdot \rangle$ for the scalar product of $L^2(0, 1)$, we deduce that

$$DJ_\varepsilon(f) \cdot h = \int_0^T \int_0^1 fh \, dx \, dt + \frac{1}{\varepsilon} \int_{\alpha+\delta}^1 u^f(T)z^h(T) \, dx$$

$$+ \frac{1}{\varepsilon} \int_{\alpha+\delta}^1 \langle \chi_{(\alpha,\beta)} h, u^f(T)z^h(T) \rangle \, dx.$$ 

$$= \int_0^T \langle f, h \rangle \, dt + \frac{1}{\varepsilon} \int_{\alpha+\delta}^1 \chi_{(\alpha,\beta)} h(T) \, dt$$

$$= \int_0^T \langle f, h \rangle \, dt + \frac{1}{\varepsilon} \int_{\alpha+\delta}^1 \chi_{(\alpha,\beta)} e^{(T-t)A} \chi_{(\alpha,\beta)} h(T) \, dt.$$ 

Thus we obtain that $J_\varepsilon$ reaches its minimum at $f^\varepsilon$ characterized by

$$f^\varepsilon = e^{-(T-t)A} \chi_{(\alpha,\beta)} u^f(T),$$

which yields to (6.12).

Now, we need suitable a priori estimates to let $\varepsilon \to 0$. We multiply (6.14) by $v^\varepsilon$ and (6.13) by $u^f$. Then, we add these identities and integrate over $(0, 1)$ to obtain

$$\frac{d}{dt} \int_0^1 u^{f^\varepsilon} (t, x) v^\varepsilon(t, x) \, dx = \int_0^1 \chi_{(\alpha,\beta)} f^{\varepsilon}(t, x) v^\varepsilon(t, x) \, dx = -\int_\alpha^\beta f^\varepsilon(t, x)^2 \, dx.$$ 

Hence, taking the integral over $(0, T)$, we deduce

$$\int_0^1 u^{f^\varepsilon} (T, x) v^\varepsilon(T, x) \, dx + \int_0^T \int_\alpha^\beta f^\varepsilon(t, x)^2 \, dx \, dt = \int_0^1 u^f(0, x) v^\varepsilon(0, x) \, dx.$$ 

So, recalling $v^\varepsilon(T, x) = \frac{1}{\varepsilon} u^{f^\varepsilon}(T, x) \chi_{(\alpha,\beta+1)}(x)$, we have, for all $\gamma > 0$,

$$\frac{1}{\varepsilon} \int_{\alpha+\delta}^1 u^{f^\varepsilon}(T, x)^2 \, dx + \int_0^T \int_\alpha^\beta f^\varepsilon(t, x)^2 \, dx \, dt = \int_0^1 u_0(x) v^\varepsilon(0, x) \, dx$$

$$\leq \frac{1}{4\gamma} \int_0^1 u_0(x)^2 \, dx + \gamma \int_0^1 v^\varepsilon(0, x)^2 \, dx.$$
Thus, applying Theorem 2.4 and choosing \( \gamma \) appropriately, we obtain

\[
\frac{1}{\varepsilon} \int_{\alpha + \delta}^{1} u^{\varepsilon}(T, x)^2 \, dx + \int_{0}^{T} \int_{\alpha}^{\beta} f^{\varepsilon}(t, x)^2 \, dx \, dt \\
\leq C \int_{0}^{1} u_0(x)^2 \, dx + \frac{1}{2} \int_{0}^{T} \int_{\alpha}^{\beta} v^{\varepsilon}(t, x)^2 \, dx \, dt + C \int_{0}^{\alpha + \delta} v^{\varepsilon}(T, x)^2 \, dx \\
= C \int_{0}^{1} u_0(x)^2 \, dx + \frac{1}{2} \int_{0}^{T} \int_{\alpha}^{\beta} v^{\varepsilon}(t, x)^2 \, dx \, dt
\]

since \( v^{\varepsilon}(T, x) = 0 \) on \( (0, \alpha + \delta) \). Since \( f^{\varepsilon} = -v^{\varepsilon} \) on \( (0, T) \times (\alpha, \beta) \), we obtain that

\[
\frac{1}{\varepsilon} \int_{\alpha + \delta}^{1} u^{\varepsilon}(T, x)^2 \, dx + \frac{1}{2} \int_{0}^{T} \int_{\alpha}^{\beta} f^{\varepsilon}(t, x)^2 \, dx \, dt \leq C \int_{0}^{1} u_0(x)^2 \, dx.
\]

This gives the a priori estimates that allows us to pass to the limit in (6.14) as \( \varepsilon \to 0 \), providing a solution to the null controllability problem. \( \square \)

**Proof in the case** \( \alpha = 0 \). In this case, we prove (global) null controllability. We consider the classical penalized problem

\[
\min_{f \in L^2((0,T) \times (0,1))} \frac{1}{2} \int_{0}^{T} \int_{0}^{1} f(t, x)^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_{0}^{1} u^{f}(T, x)^2 \, dx,
\]

with \( u^{f} \) the solution of (1.2) associated to \( f \). This problem has a unique solution \( f^{\varepsilon} \) still characterized by (6.12), where \( v^{\varepsilon} \) is the solution of the adjoint problem (6.13) but now with the condition

\[
v^{\varepsilon}(T, x) = \frac{1}{\varepsilon} u^{f^{\varepsilon}}(T, x), \quad \text{for } x \in (0, 1),
\]

and where \( u^{f^{\varepsilon}} \) satisfies (6.14). Thus (global) null controllability reduces to proving the classical observability estimate:

\[
\int_{0}^{1} v(0, x)^2 \, dx \leq \int_{0}^{T} \int_{0}^{\beta} v(t, x)^2 \, dx \, dt.
\]

Using (6.3) and (6.5), we obtain

\[
\frac{T}{2} \int_{0}^{1} v(0, x)^2 \, dx \leq \int_{T/4}^{3T/4} \int_{0}^{\beta} v(t, x)^2 \, dx \, dt + \int_{T/4}^{3T/4} \int_{0}^{1} v(t, x)^2 \, dx \, dt \\
\leq C \int_{0}^{T} \int_{0}^{\beta} v(t, x)^2 \, dx \, dt.
\]

**6.4. Proof of Theorem 1.2.**

**First method.** First, we apply the previous regional null controllability result at time \( T \): for all \( T > 0, \delta > 0 \) and \( u_0 \in L^2(0,1) \), it follows from Theorem 1.1 that there exists \( f_1 \in L^2((0,T) \times (0,1)) \) such that the solution \( u \) of

\[
\begin{align*}
&u_t - (a(x)u_x)_x + c(x)u = f_1(t, x)\chi_{(\alpha, \beta)}(x), \quad (t, x) \in (0, T) \times (0, 1), \\
&\lim_{x \to 0} u_x(t, x) = 0, \quad t \in (0, T), \\
u(t, 1) = 0, \quad t \in (0, T), \\
u(0, x) = u_0(x), \quad x \in (0, 1),
\end{align*}
\]

satisfies \( u(T, x) = 0 \) for \( x \in (\alpha + \delta/2, 1) \).
Thus, for all $\varepsilon > 0$ we choose a cut-off function $\psi \in C^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$ and
\[
\begin{cases}
\psi(x) = 1, & 0 \leq x \leq \alpha + \delta/2, \\
\psi(x) = 0, & \alpha + \delta \leq x \leq 1.
\end{cases}
\]

Then $u := \psi v$ is solution of
\[
\begin{cases}
u_t - (a(x)\nu_x)_x + c(x)v = 0, & (t, x) \in (T, T') \times (0, 1), \\
\lim_{x \to 0} a(x)\nu_x(t, x) = 0, & t \in (T, T'), \\
u(t, 1) = 0, & t \in (T, T'), \\
u(T, x) = u^T(x), & x \in (0, 1).
\end{cases}
\]

Now we choose $f(t, \cdot) := f_2(t, \cdot)$ for $t \in (0, T')$ and $f(t, \cdot) = f_2(t, \cdot)$ for $t \in (T, T')$, then $u$ is solution of (1.6) and satisfies $u(t, x) = 0$ for $(t, x) \in (T, T') \times (\alpha + \delta, 1)$.

**Second method.** For all $\varepsilon > 0$, consider the penalized problem
\[
\min \{ J_\varepsilon(f) \mid f \in L^2((0, T') \times (0, 1)) \},
\]
(6.17)
where
\[
J_\varepsilon(f) := \frac{1}{2} \int_0^T \int_0^1 f(t, x)^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_T^{T'} \int_{\alpha+\delta}^{1+\alpha} u^f(t, x)^2 \, dx \, dt,
\]
$u^f$ being the solution of (1.6) associated with $f$. Functional $J_\varepsilon$ is continuous on $L^2((0, T') \times (0, 1))$, strictly convex and $J_\varepsilon(f) \to \infty$ as $\|f\|_{L^2((0, T') \times (0, 1))} \to \infty$.

Thus, for all $\varepsilon > 0$, problem (6.17) has a unique solution $f^\varepsilon$ characterized by
\[
f^\varepsilon(t, x) = -w^\varepsilon(t, x)\chi_{(\alpha, \beta)}(x),
\]
(6.18)
where $w^\varepsilon$ is the solution of the nonhomogeneous adjoint problem
\[
\begin{cases}
w^\varepsilon + (aw^\varepsilon_x)_x - cw^\varepsilon = -\frac{1}{\varepsilon} u^f \chi_{(T', T)}(t)\chi_{(\alpha+\delta, 1)}(x), & (t, x) \in (0, T') \times (0, 1), \\
\lim_{x \to 0} a(x)w^\varepsilon_x(t, x) = 0, & t \in (0, T'), \\
w^\varepsilon(t, 1) = 0, & t \in (0, T'), \\
w^\varepsilon(T', x) = 0, & x \in (0, 1),
\end{cases}
\]
(6.19)
and where $u^f$ satisfies
\[
\begin{cases}
u^f - (au^f_x)_x + cu^f = \chi_{(\alpha, \beta)}(x)f^\varepsilon(t, x), & (t, x) \in (0, T') \times (0, 1), \\
\lim_{x \to 0} a(x)u^f_x(t, x) = 0, & t \in (0, T'), \\
u^f(t, 1) = 0, & t \in (0, T'), \\
u^f(0, x) = u_0(x), & x \in (0, 1).
\end{cases}
\]
(6.20)
Thus we obtain that

$$J_\varepsilon(f + h) - J_\varepsilon(f) = \frac{1}{2} \int_0^{T'} \int_0^1 (2fh + h^2) \, dx \, dt$$

$$+ \frac{1}{2\varepsilon} \int_T^{T'} \int_{\alpha + \delta}^1 (u^{f+h}(t)^2 - u^f(t)^2) \, dx \, dt.$$ 

Defining \( z^h \) as in (6.16) (with \( T' \) instead of \( T \)), we obtain

$$J_\varepsilon(f + h) - J_\varepsilon(f) = \frac{1}{2} \int_0^{T'} \int_0^1 fh \, dx \, dt + \frac{1}{2} \int_0^{T'} \int_0^1 h^2 \, dx \, dt$$

$$+ \frac{1}{\varepsilon} \int_T^{T'} \int_{\alpha + \delta}^1 u^f(t)z^h(t) \, dx \, dt + \frac{1}{2\varepsilon} \int_T^{T'} \int_{\alpha + \delta}^1 z^h(t)^2 \, dx \, dt.$$ 

Denoting by \( \langle \cdot, \cdot \rangle \) the scalar product of \( L^2(0,1) \), we deduce that

$$DJ_\varepsilon(f) \cdot h = \int_0^{T'} \int_0^1 fh \, dx \, ds + \frac{1}{\varepsilon} \int_T^{T'} \int_{\alpha + \delta}^1 u^f(t)z^h(t) \, dx \, dt$$

$$= \int_0^{T'} \langle f, h \rangle \, ds + \frac{1}{\varepsilon} \int_T^{T'} \langle \chi_{(\alpha + \delta, 1)}u^f(t), e^{(t-s)A}\chi_{(\alpha, \beta)}h(s) \rangle \, ds \, dt$$

$$= \int_0^{T'} \langle f, h \rangle \, ds + \frac{1}{\varepsilon} \int_T^{T'} \int_0^t \langle \chi_{(\alpha + \delta, 1)}u^f(t), e^{(t-s)A}\chi_{(\alpha, \beta)}h(s) \rangle \, ds \, dt$$

$$= \int_0^{T'} \langle f, h \rangle \, ds + \frac{1}{\varepsilon} \int_T^{T'} \int_0^t \langle \chi_{(\alpha, \beta)}e^{(t-s)A^*}(\chi_{(\alpha + \delta, 1)}u^f(t)), h(s) \rangle \, ds \, dt$$

$$= \int_0^{T'} \langle f, h \rangle \, ds + \frac{1}{\varepsilon} \int_T^{T'} \int_{\max(s, T)}^T \langle \chi_{(\alpha, \beta)}e^{(t-s)A}u^f(t) \rangle \, dt \, h(s) \rangle \, ds.$$ 

Thus we obtain that \( J_\varepsilon \) reaches its minimum at \( f^\varepsilon \) characterized by:

$$f^\varepsilon = -\frac{1}{\varepsilon} \chi_{(\alpha, \beta)} \int_{\max(s, T)}^{T'} e^{(t-s)A^*}(\chi_{(\alpha + \delta, 1)}u^f(t)) \, dt,$$

i.e.,

$$f^\varepsilon(s, x) = -\chi_{(\alpha, \beta)}(x) \int_{\max(s, T)}^{T'} v_{t,x}^\varepsilon(s, x) \, dt,$$ 

for \( (s, x) \in (0, T') \times (0, 1), \) (6.21)

where, for \( t \in (T, T') \), \( v_{t,x}^\varepsilon \) are the solutions of the family of the adjoint problems:

\[
\begin{align*}
&v_{t,x}^\varepsilon(s, x) + (av_{t,x}^\varepsilon(s, x))_x - cv_{t,x}^\varepsilon(s, x) = 0, \quad (s, x) \in (0, t) \times (0, 1), \\
&\lim_{s \to 0} a(x)v_{s,x}^\varepsilon(s, 0) = 0, \quad s \in (0, t), \\
v_{t,x}^\varepsilon(s, 1) = 0, \quad s \in (0, t), \\
v_{t,x}^\varepsilon(t, x) = \frac{1}{2}u^{f^\varepsilon}(t, x)\chi_{(\alpha + \delta, 1)}(x). \quad x \in (0, 1),
\end{align*}
\]
But (6.21) implies (6.18). Indeed let us introduce
\[ w^\varepsilon(s, x) := \int_{\max(s,T)}^{T'} \max(s,T) v^{t,\varepsilon}(s, x) \, dt \]
and let us write the equation satisfied by \( w^\varepsilon \). Note that
\[ w^\varepsilon(0, x) = \int_0^{T'} v^{t,\varepsilon}(0, x) \, dt, \quad \text{and} \quad w^\varepsilon(T', x) = 0. \]
Note, also, that
\[ w^\varepsilon(s, x) = \begin{cases} \int_0^T v^{t,\varepsilon}(s, x) \, dt & \text{for } 0 \leq s < T, \\ \int_T^s v^{t,\varepsilon}(s, x) \, dt - v^{s,\varepsilon}(s, x) & \text{for } T < s < T'. \end{cases} \]
Thus, for \( s \in (0, T) \), \( w^\varepsilon \) is solution of
\[ \begin{cases} w^\varepsilon + (aw^\varepsilon_x) - cw^\varepsilon = 0, & (s, x) \in (0, T) \times (0, 1), \\ \lim_{x \to 0} a(x)w^\varepsilon_x(s, x) = 0, & s \in (0, T), \\ w^\varepsilon(s, 1) = 0, & s \in (0, T), \end{cases} \]
and for \( s \in (T, T') \), \( w^\varepsilon \) is solution of
\[ \begin{cases} w^\varepsilon + (aw^\varepsilon_x) - cw^\varepsilon = -v^{s,\varepsilon}, & (s, x) \in (T, T') \times (0, 1), \\ \lim_{x \to 0} a(x)w^\varepsilon_x(s, x) = 0, & s \in (0, T), \\ w^\varepsilon(s, 1) = 0, & s \in (T, T'). \end{cases} \]
Thus \( w^\varepsilon \) is solution of (6.19).

Now, we need suitable a priori estimates to let \( \varepsilon \to 0 \). We multiply (6.20) by \( w^\varepsilon \) and (6.19) by \( u^{f,\varepsilon} \). Then, we add these identities and integrate over \((0, 1)\) to obtain:
\[ \frac{d}{dt} \int_0^1 u^{f,\varepsilon} w^\varepsilon \, dx = \int_0^1 \chi_{(\alpha, \beta)}(x) f^\varepsilon \, w^\varepsilon \, dx - \frac{1}{\varepsilon} \int_{\alpha+\delta}^1 (u^{f,\varepsilon})^2 \chi(T, T')(t) \, dx. \]
Hence, taking the integral over \((0, T')\) and using \( w^\varepsilon = -f^\varepsilon \) on \((\alpha, \beta)\), and \( w^\varepsilon(T', x) = 0 \), we have for all \( \eta > 0 \),
\[ \frac{1}{\varepsilon} \int_T^{T'} \int_{\alpha+\delta}^1 u^{f,\varepsilon}(t, x)^2 \, dx \, dt + \int_0^{T'} \int_{\alpha}^{\beta} f^\varepsilon(t, x)^2 \, dx \, dt = \int_0^1 u^0(x)w^\varepsilon(0, x) \, dx \leq \frac{1}{4\gamma} \int_0^1 u^0(x)^2 \, dx + \gamma \int_0^1 w^\varepsilon(0, x)^2 \, dx. \]
Thus applying Theorem 2.5 and choosing $\gamma$ appropriately, we obtain
\[
\frac{1}{\varepsilon} \int_0^{T'} \int_0^1 w^\varepsilon(t,x)^2 \, dx \, dt + \int_0^{T'} \int_0^\beta f^\varepsilon(t,x)^2 \, dx \, dt
\leq C \int_0^1 u_0(x)^2 \, dx + \frac{1}{2} \int_0^{T'} \int_0^\beta w^\varepsilon(t,x)^2 \, dx \, dt
\]
\[
+ C \int_0^{\alpha+\delta} w^\varepsilon(T',x) \, dx + C \int_0^{T'} \int_0^{\alpha+\delta} \left( -\frac{1}{\varepsilon} w^\varepsilon(t,x) \chi_{(\alpha+\delta,1)}(x) \right)^2 \, dx \, dt
\leq C \int_0^1 u_0(x)^2 \, dx + \frac{1}{2} \int_0^{T'} \int_0^\beta f^\varepsilon(t,x)^2 \, dx \, dt
\]
where we used the fact that $w^\varepsilon = f^\varepsilon$ on $(\alpha, \beta)$ and $w^\varepsilon(T',x) = 0$. Hence
\[
\frac{1}{\varepsilon} \int_0^{T'} \int_0^1 w^\varepsilon(t,x)^2 \, dx \, dt + \frac{1}{2} \int_0^{T'} \int_0^\beta f^\varepsilon(s,x)^2 \, dx \, ds \leq C \int_0^1 u_0(x)^2 \, dx.
\]
This gives the a priori estimates that allows us to pass to the limit in (6.20) as $\varepsilon \to 0$, providing a solution to the null controllability problem. \hfill \Box

7. Proof for the Crocco Type Equation. Proof of Theorem 4.2. We note that, as in [29], it suffices to establish (4.3) for regular solutions $v$ of (4.2). Also, we only treat the case $T < L$, the other one being similar. Let us decompose the first term of (4.3) as follows
\[
\int\int_{(0,L) \times (0,1)} v(x,y,0)^2 \, dy \, dx + \int\int_{(0,1) \times (0,T)} v(0,y,t)^2 \, dt \, dy
\]
\[
= \int\int_{(0,L-T) \times (0,1)} v(x,y,0)^2 \, dy \, dx + \int\int_{(L-T,L) \times (0,1)} v(x,y,0)^2 \, dy \, dx
\]
\[
+ \int\int_{(0,1) \times (0,T-\delta)} v(0,y,t)^2 \, dt \, dy + \int\int_{(0,1) \times (T-\delta,T)} v(0,y,t)^2 \, dt \, dy. \quad (7.1)
\]
We claim that the following inequalities hold:
\[
\int\int_{(0,L-T) \times (0,1)} v(x,y,0)^2 \, dy \, dx \leq C \int_0^T \int_0^L v(x,y,t)^2 \, dy \, dxdt
\]
\[
+ C \int_T^L \int_{\beta-\delta} v(x,y,T)^2 \, dy \, dx, \quad (7.2)
\]
\[
\int\int_{(L-T,L) \times (0,1)} v(x,y,0)^2 \, dy \, dx \leq \int_0^T \int_0^1 v(L,y,t)^2 \, dy \, dt, \quad (7.3)
\]
\[
\int\int_{(0,T-\delta) \times (0,1)} v(0,y,t)^2 \, dy \, dt \leq C \int_0^T \int_0^T v(x,y,t)^2 \, dy \, dt \]
\[
+ C \int_0^T \int_{\beta-\delta} v(x,y,T)^2 \, dy \, dx, \quad (7.4)
\]
\[
\int\int_{(T-\delta,T) \times (0,1)} v(0,y,t)^2 \, dy \, dt \leq \int_0^\delta \int_0^1 v(x,y,T)^2 \, dy \, dx. \quad (7.5)
\]
Clearly, Theorem 4.2 follows from (7.1)-(7.5). To explain the idea underneath the proof, we observe that, along the characteristics 
\[ x = \xi + t, \]
the solution \( v \) of (4.2) solves a degenerate parabolic equation like (2.2). Then (7.3) and (7.5) will follow from the fact that the energy of solutions is increasing along the characteristics. Similarly, (7.2) and (7.4) will follow from the regional observability inequality of Theorem 2.4.

Getting down to the technical details, let \( v \) be a solution of (4.2). For all \((\xi, y, t) \in (-T, L) \times (0,1) \times (0,T)\) such that \(\xi + t \in (0,L)\), we introduce \(w(\xi, y, t) := v(\xi + t, y, t)\) and we verify that
\[
w_t + (aw_y)_y - cw = v_t + v_x + (av_y)_y - cv = 0.
\]

In particular, for all fixed \(\xi \in (-T, L)\), \(w^\xi(y, t) := w(\xi, y, t)\) turns out to be a solution of the degenerate parabolic equation
\[
\begin{cases}
  w^\xi_t + (aw^\xi_y)_y - cw^\xi = 0, & (y, t) \in (0,1) \times (t_0^\xi, t_1^\xi), \\
  w^\xi(0, t) = 0, & t \in (t_0^\xi, t_1^\xi), \\
  (aw^\xi_y)(1, t) = 0, & t \in (t_0^\xi, t_1^\xi),
\end{cases}
\]

where \(t_0^\xi = \max(0, -\xi), \ t_1^\xi = \min(T, L - \xi)\). Note that, for all \(\xi \in (-T, L)\), the energy of \(w^\xi\) is increasing, i.e.
\[
\text{for all } t_0^\xi \leq T_0 \leq T_1 \leq t_1^\xi, \quad \int_0^1 w^\xi(y, T_0)^2 \, dy \leq \int_0^1 w^\xi(y, T_1)^2 \, dy. \tag{7.6}
\]

Moreover, \(w^\xi\) satisfies the regional observability inequality (see Theorem 2.4):

\[
\text{for all } t_0^\xi \leq T_0' \leq T_1' \leq t_1^\xi, \quad \int_0^1 w^\xi(y, T_0')^2 \, dy \leq C \int_{T_0'}^{T_1'} \int_0^\beta w^\xi(y, t)^2 \, dy \, dt + C \int_{\beta-\delta}^1 w^\xi(y, T_1')^2 \, dy, \tag{7.7}
\]

where \(C\) depends only on \((\alpha, \beta), \delta\) and \(T_1' - T_0'\).

**Proof of (7.2).** From the definition of \(w\), it follows that
\[
\int_0^{L-T} \int_0^1 v(x, y, 0)^2 \, dy \, dx = \int_0^{L-T} \int_0^1 w(\xi, y, 0)^2 \, dy \, d\xi,
\]

For all \(\xi \in (0, L - T)\), we apply (7.7) to \(w^\xi\) between \(T_0' = 0\) and \(T_1' = T\):
\[
\int_0^1 w(\xi, y, 0)^2 \, dy \leq C \int_0^T \int_0^\beta w(\xi, y, t)^2 \, dy \, dt + C \int_{\beta-\delta}^1 w(\xi, y, T)^2 \, dy,
\]
where $C$ is independent of $\xi$ (it depends on $(\alpha, \beta), \delta$ and $T'_1 - T'_0 = T$). Next, taking the integral over $\xi \in (0, L - T)$, we obtain
\[
\int_0^{T - \delta} \int_{T - \delta}^1 v(0, y, t)^2 \, dy \, dt = \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, 0)^2 \, dy \, dt \leq C \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, t)^2 \, dy \, dt + C \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, T)^2 \, dy \, dt.
\]

**Proof of (7.4).** From the definition of $w$, it follows that
\[
\int_0^{T - \delta} \int_{T - \delta}^1 v(0, y, t)^2 \, dy \, dt = \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, T - \delta)^2 \, dy \, dt.
\]

For all $\xi \in (0, T - \delta)$, we apply (7.6) to $w^{-\xi}$ between $T_0 = \xi$ and $T_1 = T - \delta$ and we apply (7.7) between $T_0' = T - \delta$ and $T_1' = T$:
\[
\int_0^1 w(0, y, \xi)^2 \, dy \leq \int_0^1 w(0, y, T - \delta)^2 \, dy \leq C \int_{T - \delta}^T \int_{\alpha}^T w(0, y, T - \delta)^2 \, dy \, dt + C \int_{T - \delta}^T \int_{\alpha}^T w(0, y, T)^2 \, dy \, dt,
\]
where $C$ is independent of $\xi$ (it depends on $(\alpha, \beta), \delta$ and $T'_1 - T'_0 = \delta$). Next, taking the integral over $\xi \in (0, T - \delta)$, we obtain
\[
\int_0^{T - \delta} \int_{T - \delta}^1 v(0, y, t)^2 \, dy \, dt = \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, \xi)^2 \, dy \, dt \leq C \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, t)^2 \, dy \, dt + C \int_0^{T - \delta} \int_{T - \delta}^1 w(0, y, T)^2 \, dy \, dt.
\]

**Proof of (7.3) and (7.5).** To prove (7.3), we write
\[
\int_{L - T}^L \int_{0}^1 v(x, y, 0)^2 \, dy \, dx = \int_{L - T}^L \int_0^1 w(\xi, y, 0)^2 \, dy \, dx.
\]

For all $\xi \in (L - T, L)$, we apply (7.6) to $w^\xi$ between $T_0 = 0$ and $T_1 = L - \xi$ and we take the integral over $\xi$. Then we deduce (7.3) from the definition of $w$. The proof of (7.5) is similar.
8. Proofs in the Case of an Unbounded Domain. We only give a sketch of the proof of part (i) in Theorem 5.1. Part (ii) follows from part (i) (using a similar cut-off argument, as used in Theorem 1.2).

**Step 1 : observability estimate.** First, we consider the adjoint problem:

\[
\begin{aligned}
\begin{cases}
v_t(t, x) + v_{xx}(t, x) = 0, & (t, x) \in (0, T) \times (0, +\infty), \\
v(t, 0) = 0, & t \in (0, T).
\end{cases}
\end{aligned}
\]  

(8.1)

And we prove the following observability estimate: for all \( \delta > 0 \) (such that \( \beta - \delta > \alpha \)), there exists \( C > 0 \) such that all solution \( v \) of (8.1) satisfies

\[
\int_0^{+\infty} v(0, x)^2 \, dx \leq C \int_0^T \int_{\alpha}^{\beta} v(t, x)^2 \, dx \, dt + C \int_{\beta - \delta}^{+\infty} v(T, x)^2 \, dx.
\]  

(8.2)

Let \( \alpha' \in (\alpha, \beta) \). As for Theorem 2.4, (8.2) follows from the following estimates:

\[
\begin{aligned}
\frac{T}{2} \int_0^{+\infty} v(0, x)^2 \, dx &\leq \int_{T/4}^{3T/4} \int_{\alpha}^{\beta} v(t, x)^2 \, dx \, dt, \\
\int_{T/4}^{3T/4} \int_{\alpha}^{\alpha'} v(t, x)^2 \, dx \, dt &\leq C \int_0^T \int_{\alpha}^{\beta} v(t, x)^2 \, dx \, dt, \\
\int_0^T \int_{\beta - \delta}^{+\infty} v^2 \, dx \, dt &\leq T \int_{\beta - \delta}^{+\infty} v(T, x)^2 \, dx + C \int_0^T \int_{\beta - \delta}^{+\infty} v^2 \, dx \, dt.
\end{aligned}
\]  

(8.3)

The proof of (8.3) is the same as Step 1 of Theorem 2.4. In order to prove (8.4), we introduce \( \alpha'' \) such that \( \alpha < \alpha' < \alpha'' < \beta \) and we choose a cut-off function \( \psi \) such that \( \psi = 0 \) on \((\alpha'', +\infty)\) and \( \psi = 1 \) on \((0, \alpha')\). Then as in Step 2 of Theorem 2.4, we apply Carleman estimates to \( v(t, x) := \psi(x) v(t, x) \) (since \( w \) is solution of a parabolic equation in a bounded domain). Finally, to obtain (8.5), we introduce a cut-off function \( \psi \) such that \( \psi = 1 \) on \((\beta, +\infty)\) and \( \psi = 0 \) on \((0, \beta - \delta)\), and we proceed as in Step 3 of Theorem 2.4.

**Step 2 : penalized problem.** For all \( \varepsilon > 0 \), consider the penalized problem

\[
\begin{aligned}
\min_{f \in L^2((0, T) \times (0, 1))} \frac{1}{2} \int_0^T \int_0^1 f(t, x)^2 \, dx \, dt + \frac{1}{2\varepsilon} \int_0^{\beta - \delta} u^f(T, x)^2 \, dx,
\end{aligned}
\]

with \( u^f \) the solution of (5.1) associated to \( f \). It has a unique solution \( f^\varepsilon \) :

\[
f^\varepsilon(t, x) = -v^\varepsilon(t, x) \chi_{(\alpha, \beta)}(x),
\]

(8.6)

where \( v^\varepsilon \) is the solution of the adjoint problem (8.1) with the condition

\[
v^\varepsilon(T, x) = \frac{1}{\varepsilon} u^f(T, x) \chi_{(0, \beta - \delta)}(x), \quad \text{for} \quad x \in (0, +\infty),
\]

(8.7)

and where \( u^f \) is the solution of (5.1) associated to \( f^\varepsilon \). Now as in the proof of Theorem 1.1, we compute, for all \( \gamma > 0 \),

\[
\begin{aligned}
\frac{1}{\varepsilon} \int_0^{\beta - \delta} u^f(T, x)^2 \, dx + \int_0^T \int_{\alpha}^{\beta} f^\varepsilon(t, x)^2 \, dx \, dt &= \int_0^{+\infty} u_0(x) v^\varepsilon(0, x) \, dx \\
&\leq \frac{1}{4\gamma} \int_0^{+\infty} u_0(x)^2 \, dx + \gamma \int_0^{+\infty} v^\varepsilon(0, x)^2 \, dx \\
&\leq C \int_0^{+\infty} u_0(x)^2 \, dx + \frac{1}{2} \int_0^T \int_{\alpha}^{\beta} f^\varepsilon(t, x)^2 \, dx \, dt,
\end{aligned}
\]
where we used (8.2) (for γ small) and (8.6), (8.7). Finally we deduce the a priori estimates that allows us to pass to the limit in (5.1) as ε → 0.

9. Appendix: Well-posedness of the Degenerate Equation. In this part, we prove the well-posedness of (1.2) assuming that a satisfies (1.1). For the proofs of Propositions 2.1 and 2.2, we will need the following result:

**Lemma 9.1.** For u given in D(A), we have

\[ \forall x \in [0,1], \ |a(x)u(x)| \leq \|(au_x)\|_{L^2(0,1)} \sqrt{x}, \]  

(9.1)

and

\[ \forall x \in [0,1], \ |a(x)u_x(x)| \leq \|(au_x)\|_{L^2(0,1)} \sqrt{x}. \]  

(9.2)

**Proof of Lemma 9.1.** Let u be given in D(A). Then

\[ |(au)(x)| = |(au)(0) + \int_0^x (a(au)_x(s) \, ds) | \leq \|a(au)\|_{L^2(0,1)} \sqrt{x}. \]

Similarly,

\[ |(au_x)(x)| = |(au_x)(0) + \int_0^x (a(au)_x(s) \, ds) | \leq \|a(au)_x\|_{L^2(0,1)} \sqrt{x}. \quad \square \]

**Proof of Proposition 2.1.** We denote by G the space

\[ G := \{ u \in H^1_a(0,1) \mid au_x \in H^1(0,1) \}, \]

and we prove that G = D(A).

In order to prove that D(A) ⊆ G, we consider u given in D(A) and it is sufficient to prove that \( \sqrt{au}_x \in L^2(0,1) \). For s ∈ (0, 1), we write

\[ \int_s^1 (au_x) u \, dx = \left[ au_x u \right]_s^1 - \int_s^1 au_x^2 \, dx. \]

Since \( au_x \in H^1(0,1) \) and \( u(1) = 0 \), we have \( (au_x u)(1) = 0 \). Hence

\[ (au_x u)(s) = -\int_s^1 (au_x) u \, dx - \int_s^1 au_x^2 \, dx. \]  

(9.3)

Since \( u \in D(A) \), we have \( (au_x)_x u \in L^1(0,1) \), thus there exists \( L \in [-\infty, +\infty) \) such that

\[ (au_x u)(s) \to L \quad \text{as } s \to 0. \]

If \( L \neq 0 \), then, there exists \( C > 0 \) such that, for s small enough,

\[ |(au_x u)(s)| \geq C. \]

Using (9.2), we deduce that there exists \( C' > 0 \) such that, for s small enough,

\[ |u(s)| \geq \frac{C'}{\sqrt{s}}, \]

which implies \( u \notin L^2(0,1) \). Finally \( L = 0 \) and (9.3) implies

\[ \int_0^1 (au_x)_x u \, dx = -\int_0^1 au_x^2 \, dx. \]

Since \( (au_x)_x u \in L^1(0,1) \), we deduce \( \sqrt{au}_x \in L^2(0,1) \). This proves \( D(A) \subseteq G \).
In order to prove that $G \subset D(A)$, we consider $u$ given in $G$ and will prove that $au \in H^1_0(0,1)$ and $(au_x)(0) = 0$. Since $au_x \in H^1(0,1)$, there exists $L \in \mathbb{R}$ such that 

$$(au_x)(x) \to L \text{ as } x \to 0.$$ 

If $L \neq 0$, then, for $x$ small enough,

$$|a(x)u_x(x)^2| \geq \frac{L^2}{2a(x)}.$$ 

Hence $\sqrt{a}u_x \notin L^2(0,1)$. Indeed $1/a \notin L^1(0,1)$ since $a(0) = 0$ and $a \in C^1([0,1])$. This implies $L = 0$ and $(au_x)(0) = 0$. Moreover $au \in H^1(0,1)$ since $(au)_x = a_xu + au_x \in L^2(0,1)$ and $(au)(1) = 0$ since $u(1) = 0$. It remains to prove that $(au)(0) = 0$. Thus for all $x \in D(A)$, there exists $L \in \mathbb{R}$ such that 

$$(au)(x) \to L \text{ as } x \to 0.$$ 

If $L \neq 0$, then $u \notin L^2(0,1)$. Indeed it implies, for $x \to 0$,

$$u(x)^2 \sim \frac{L^2}{a(x)^2} \geq \frac{C}{a(x)} \notin L^1(0,1).$$

Thus we obtain $L = 0$. This proves $G \subset D(A)$. 

**Proof of Proposition 2.2.**

(i) First we prove that $(A, D(A))$ is closed in $L^2(0,1)$. Consider $(u^n)_n$ such that $\forall n \in \mathbb{N}$, $u^n \in D(A)$ and

$$u^n \to u \text{ and } U^n := Au^n \to U \text{ in } L^2(0,1) \text{ as } n \to \infty,$$

and prove that $u \in D(A)$ and $Au = U$. We define

$$v^n(x) := \int_0^x U^n(y) \, dy = \int_0^x Au^n(y) \, dy$$

$$= \int_0^x (au^n_x)(y) \, dy - \int_0^x c(y)u^n(y) \, dy = (au^n_x)(x) - \int_0^x c(y)u^n(y) \, dy,$$

and

$$v(x) := \int_0^x U(y) \, dy.$$

Since $U^n \to U$ in $L^2(0,1)$, we deduce that $v^n \to v$ uniformly on $[0,1]$. Using $u^n \to u$ in $L^2(0,1)$, we deduce that

$$au^n_x \to v + \int_0^x cu \text{ uniformly on } [0,1].$$

Thus for all $\eta > 0$,

$$u^n_x = \frac{v^n}{a} + \frac{1}{a} \int_0^x cu^n \to \frac{v}{a} + \frac{1}{a} \int_0^x cu \text{ uniformly on } [\eta, 1].$$

Since we also have $u^n(1) = 0$ for all $n \in \mathbb{N}$, we deduce that $u^n \to u$ in $C^1([\eta, 1])$. Passing to the limit in $(au^n_x)(x) = v^n(x) + \int_0^x cu^n$, we obtain $(au_x)(x) = v(x) + \int_0^x cu$ for all $x \in (0,1)$. This implies $au_x \in H^1(0,1)$ and $au \in H^1(0,1)$ since $(au)_x = a_xu + au_x \in L^2(0,1)$. Moreover, since $(au^n)(1) = 0$, we also have $(au)(1) = 0$. From Lemma 9.1, we deduce passing to the limit in (9.1) and (9.2) : for all $x \in (0,1)$,

$$|a(x)u(x)| \leq \|v + \int_0^x cu\|_{L^2(0,1)} \sqrt{x} \quad \text{and} \quad |a(x)u_x(x)| \leq \|U + cu\|_{L^2(0,1)} \sqrt{x}.$$
We deduce \((au)(0) = 0\) and \((au_u)(0) = 0\). Thus we have proved \(u \in D(A)\). Moreover \((au_u)(x) - \int_0^x cu = v(x) = \int_0^x U(y) dy\) for all \(x \in [0,1]\) implies \(Au = U\).

(ii) Now we prove that \(A\) is negative. For this, we first prove the following property:

\[
\forall u, v \in D(A), \quad a(x)u_x(x)v(x) \to 0 \text{ as } x \to 0.
\]

(9.4)

Indeed, we write

\[
\int_x^1 (au_u)_x v \, dy = -(au_u)(x) - \int_x^1 au_x v_x \, dy.
\]

Using \(u, v \in D(A)\) and Proposition 2.1, we know that \((au_u)_x v\) and \(au_x v_x\) belong to \(L^1(0,1)\). Consequently, there exists \(L \in \mathbb{R}\) such that

\[
a(x)u_x(x)v(x) \to L \text{ as } x \to 0.
\]

If \(L \neq 0\), then we deduce that \(v \notin L^2(0,1)\) since, using (9.2), we have

\[
|v(x)| \geq \frac{L}{2|a(x)u_x(x)|} \geq \frac{L}{2C\sqrt{x}}.
\]

Finally \(L = 0\), which proves (9.4). Using (9.4), we conclude

\[
\forall u \in D(A), \quad \langle Au, u \rangle = \int_0^1 (au_u)_x u \, dx - \int_0^1 cu^2 \, dx = -\int_0^1 au_x^2 \, dx - \int_0^1 cu^2 \, dx \leq 0,
\]

since \(c \geq 0\). Hence \(A\) is negative.

(iii) Finally, we prove that \(A\) is self-adjoint. We define

\[
D(A^*) := \{ v \in L^2(0,1) \mid \exists C > 0, \forall u \in D(A), \quad |\langle v, Au \rangle| \leq C\|u\|_{L^2(0,1)} \},
\]

and

\[
\forall v \in D(A^*), \quad \forall u \in D(A), \quad \langle A^* v, u \rangle = \langle v, Au \rangle.
\]

First we prove that \(D(A) \subset D(A^*)\) and \(A^*|_{D(A)} = A\). Consider \(v \in D(A)\) and prove that \(v\) belongs to \(D(A^*)\). Using (9.4), we compute

\[
\forall u \in D(A), \quad \langle v, Au \rangle = \int_0^1 v(au_u)_x \, dx - \int_0^1 cuv \, dx
\]

\[
= \int_0^1 (au_u)_x u \, dx - \int_0^1 cuv \, dx = \langle Av, u \rangle.
\]

We deduce that \(v \in D(A^*)\) and that \(A^* v = Av\).

Next we prove that \(D(A^*) \subset D(A)\). For this, we note that \(H^1_a(0,1)\) is an Hilbert space with the scalar product

\[
\langle u, \phi \rangle_1 := \int_0^1 (u\phi + au_x \phi_x + cu\phi) \, dx.
\]

Consequently, for all \(f \in L^2(0,1)\), there exists a unique \(u \in H^1_a(0,1)\) such that

\[
\forall \phi \in H^1_a(0,1), \quad \langle u, \phi \rangle_1 = \int_0^1 f\phi \, dx.
\]

This implies that \((au_u)_x = u + cu - f\) and so \(au_x \in H^1(0,1)\), which yields in turn

\[
u \in D(A) \quad \text{and} \quad u - Au = f.
\]

(9.5)
Now, for $v \in D(A^*)$, let us show that $v$ belongs to $D(A)$. There exists $w \in L^2(0, 1)$ such that

$$\forall \phi \in D(A), \quad \int_0^1 A\phi v \, dx = \int_0^1 w\phi \, dx.$$ 

Let $u$ be the solution of (9.5) with $f = v - w$. Then $w = v - u + Au$ and

$$\int_0^1 A\phi v \, dx = \int_0^1 (v - u)\phi \, dx + \int_0^1 Au \phi \, dx = \int_0^1 (v - u)\phi \, dx + \int_0^1 uA\phi \, dx,$$

or

$$\forall \phi \in D(A), \quad \int_0^1 (A\phi - \phi)(v - u) \, dx = 0.$$ 

Now take $\phi$ to be the solution of (9.5) with $f = -(v - u)$, i.e. such that $A\phi - \phi = v - u$, to obtain that $v \equiv u$. Thus, $v \in D(A)$. This shows that $D(A^*) \subset D(A)$. □

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