COMPARING FB AND PS SCHEDULING POLICIES

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ABSTRACT. In this paper we obtain new results concerning the expected response time of the foreground-background (FB) scheduling discipline and its comparison with processor sharing (PS). Some results previously derived for job sizes with finite second moment or bounded sizes, are extended to infinite second moments. New bounds and asymptotic results are also derived. We show that for job sizes with infinite second moment large jobs may benefit from the FB scheduling discipline although this discipline favors short jobs. For certain distributions all jobs sizes may even benefit from FB with respect to PS showing that the performance benefits obtained by some job sizes need not be obtained at the expense of others.

1. INTRODUCTION

In this paper we study the performances obtained with foreground/background (FB) scheduling discipline in terms of expected response times conditioned on the job sizes. Practical bounds and comparisons with processor sharing (PS) have recently be obtained [BHB01, HBSW02, RUKB03, WHB03] most often derived in the context of bounded job sizes or job sizes with finite second moments. In certain applications such as data networks this is too restrictive, as flow sizes are known to be well modelled by infinite second moment distributions. We characterize the expected conditional sojourn time of the FB queue in this context and we show through some results that qualitatively different and non intuitive behaviors may appear when considering infinite second moments. Some previously obtained results are extended to the case of infinite second moment job size distributions. We derive a new simple criteria for which FB performs better than PS. Finally we show that when job sizes have infinite second moments, all job sizes may have smaller expected sojourn times with FB than with PS. This is a property which may not be obtained with finite second moments according to Kleinrock’s conservation law [Kle76, O’D74, BB03]. This law states that for a work conserving non anticipative systems:

\[
\lambda \int_0^{\infty} T(x)(1 - F(x)) \, dx = \bar{U},
\]

where \( \lambda \) is the customer arrival rate, \( F(x) \) is the cumulative distribution of service times, \( T(x) \) is the expected sojourn time of a customer conditioned on service requirement being \( x \), and \( \bar{U} \) is the average unfinished work, independent of the scheduling policy.

The impact of infinite second moment service time distributions on the tail of the unconditional sojourn time distribution for different scheduling policies have recently received much attention. See for example [BBNQZ03] for a survey and [NQ02, NWZ05] for results concerning FB.

The FB scheduling policy consists in giving the full server capacity to the jobs with the least attained service. It is also called the least attained service policy (LAS) or shortest elapsed time SET.

In the following we will consider the M/G/1 queue served according to a FB scheduling policy. Let \( X \) be the random service time of a job and \( F(x) \) be the service time distribution, and \( \lambda \) be the job arrival rate. The expected conditional response time for the M/G/1 queue with FB may be expressed in terms of the moments of the truncated service times [Kle76] and the truncated load: \( E[X^n] = \int_0^x ny^{n-1}(1 - F(y))dy \) and \( \rho_x = \lambda E[X]. \)

The expected conditional response time for the M/G/1 FB queue is given by [Kle76]:

\[
T_{FB}(x) = \frac{\lambda E[X^2]}{2(1 - \rho_x)} + x \frac{1}{1 - \rho_x}.
\]
We would like to characterize the difference between expected response times obtained with FB and PS, which after some calculations may be expressed as:

\[
T_{FB}(x) - T_{PS}(x) = \lambda \int_0^x y(1 - F(y))dy / (1 - \rho_x)^2 - \frac{x \int_x^\infty (1 - F(y))dy}{(1 - \rho_x)(1 - \rho)}.
\]

2. Asymptotic Analysis

We first show the following proposition concerning the asymptotic slowdown for FB which has been previously proved for service time distributions with finite second moments [HBSW02] and which may be extended to any distribution with finite first moment using Equation 3:

**Proposition 2.1.** For an M/G/1 queue under the FB scheduling policy the asymptotic slowdown is:

\[
\lim_{x \to \infty} \frac{T_{FB}(x)}{x} = \frac{1}{1 - \rho}.
\]

This asymptotic limit had previously been shown to hold for the class of service requirement distributions which are of intermediate regular variation in the case of the M/G/1 queue [NQ02] and of the GI/GI/1 queue [NWZ05]. We only suppose a finite first moment.

More precise results may be obtained in the case of service time distributions with finite second moments or in the case of regularly varying distributions which include the Pareto distributions. We recall the definition of regularly varying distributions ([Fel71], chap.VIII.8). A distribution function \( F(x) \) on \([0, \infty)\) is regularly varying with index \(-\nu\) if: \( 1 - F(x) = x^{-\nu}L(x), \nu \geq 0 \), where \( L \) is a slowly varying function, i.e., \( \lim_{x \to \infty} L(tx)/L(x) = 1, t > 0 \). The notation \( f(x) / g(x) \) will be used for \( \lim_{x \to \infty} (f(x)/g(x)) = 1 \).

**Proposition 2.2.** For an M/G/1 FB queue under load \( \rho < 1 \):

1. If \( E[X^2] < \infty \): \( \lim_{x \to \infty} (T_{FB}(x) - T_{PS}(x)) = \frac{\lambda E[X^2]}{(1 - \rho)^2} \geq 0 \).
2. If \( F \) is regularly varying, then: \( T_{FB}(x) - T_{PS}(x) \sim \frac{\lambda x^{2-\nu}}{(1 - \rho)^2} L(x)^{2\nu - 3} / (2 - \nu) \). The difference is negative if \( \nu \leq \frac{3}{2} \) and tends to infinity.

The proof is based on the following lemma derived from Karamata ([Fel71], chap.VIII.9):

**Lemma 2.1.** If \( 1 - F \) varies regularly with index \(-\nu\), then: \( \lim_{x \to \infty} \int_0^x y(1 - F(y))dy / (1 - F(y))dy = \frac{\nu - 1}{2 - \nu} \)

We conclude from proposition 2.2 that for very heavy tailed distributions (i.e. \( \nu < 1.5 \)), large jobs may benefit from a scheduling policy favoring shorts jobs. Another consequence of this proposition is that for such distributions the slowdown, \( T(x)/x \), is asymptotically increasing. Two consequences are that customers with large service requirements do not have smaller slowdown than intermediate size customers and it may thus be possible that FB be fair for certain service distributions with the definition of [WHB03] (see [WHB03] for discussion on unfairness of FB). In comparison the following proposition shows that the shortest remaining processing time (SRPT) scheduling policy is always asymptotically better than PS as soon the service time distribution has an infinite second moment.

**Proposition 2.3.** For an M/G/1 SRPT queue under load \( \rho < 1 \), with a service time distribution with infinite second moment: \( T_{SRPT}(x) - T_{PS}(x) \sim -\frac{\lambda E[X^2] / E[X] - E[X]}{(1 - \rho)^2} \).

For a regularly varying service time distribution: \( T_{SRPT}(x) - T_{PS}(x) \sim -\frac{\lambda x^{2-\nu}L(x)}{(\nu - 1)^2(2 - \nu)} \)

The difference is asymptotically always negative and has arbitrarily large amplitude.

We now study the performance of FB with respect to PS for small and medium size jobs.
3. The sign of $T_{FB}(x) - T_{PS}(x)$

We now proceed to study the sign of $T_{FB}(x) - T_{PS}(x)$ as a function of load, $\rho$, and service requirement, $x$, for given service time distribution. In the following we impose no restriction on the distribution except that it have a finite mean.

We first study the sign of $T_{FB}(x) - T_{PS}(x)$ for fixed $x$ and varying $\rho$, then for $\rho = 0$ and varying $x$. The sign of $T_{FB}(x) - T_{PS}(x)$ is given by that of $\Delta(x, \rho)$ (or $\Delta(x)$ if there is no ambiguity) defined as:

$$\Delta(x, \rho) = \frac{(1 - \rho)(1 - \rho_x)}{\lambda} (T_{FB}(x) - T_{PS}(x)) = \int_0^x y(1 - F(y))dy \frac{1}{1 + \rho \int_0^y (1 - F(y))dy} - x \int_x^\infty (1 - F(y))dy$$

**Proposition 3.1.** For a given size $x$ which is not the maximum service time, $T_{FB}(x) - T_{PS}(x)$ is either always strictly negative for $0 < \rho < 1$, or there exists a load $\rho^*$ such that:

- $T_{FB}(x) > T_{PS}(x)$ for $0 < \rho < \rho^*$
- $T_{FB}(x) < T_{PS}(x)$ for $\rho^* < \rho < 1$

Such a load, $\rho^*$, exists and $T_{FB}(x) - T_{PS}(x)$ may be positive if and only if

$$\int_0^x y(1 - F(y))dy > x \int_x^\infty (1 - F(y))dy.$$ 

*If there exist a maximum service time, $x_{max}$, then $T_{FB}(x_{max}) > T_{PS}(x_{max})$ for all loads.*

From the previous proposition we conclude that, when the load increases, more service sizes obtain better average performance with FB than with PS. Furthermore:

**Proposition 3.2.** For service sizes, $x$, and for loads $\rho > \rho^*$ for which $T_{FB}(x) < T_{PS}(x)$: $T_{FB}(x) - T_{PS}(x)$ and $(T_{FB}(x) - T_{PS}(x))/T_{PS}(x)$ decrease as the load increases. If $1 - F(x) > 0$:

$$\lim_{\rho \to 1} T_{FB}(x) = \frac{E[X^2]/E[X]}{2(1 - E[X]/E[X])^2} + \frac{x}{1 - E[X]/E[X]}$$

We next study the evolution of the difference for a given load and varying $x$ and we will see that its qualitative behavior is less binary. For this we define $\delta(x) = \Delta(x, 0)$. From Proposition 3.1 service times for which $\delta(x) < 0$ will always have better average response times with FB than with PS, no matter what the system load is. The following proposition provides a criteria for a job size, $x$, to fall in this category.

**Proposition 3.3.** For any given service time distribution:

$$E[X_x] < \frac{2}{3} E[X] \quad \text{(i.e. } \rho_x < \frac{2}{3} \rho \text{)} \implies \forall \rho < 1: \quad T_{FB}(x) < T_{PS}(x).$$

Note that for Internet traffic most of the flows are short and more than half of the load is due to a very small proportion of large flows. Thus the condition $\rho_x < \frac{2}{3} \rho$ covers most of the flows on an Internet link.

The following lemma confirms the intuition that FB favors short service times.

**Lemma 3.1.** For every service distribution, there exists an interval, $(0, \chi_0)$, such that

$$\forall x \in (0, \chi_0) : \quad T_{FB}(x) < T_{PS}(x)$$

*Proof. $\delta(x)$ is a continuous and derivable function, null and with a negative derivative at the origin.*

The following proposition illustrates the range of possible behaviors.

**Proposition 3.4.** For exponentially and Pareto distributed service times with index $-\nu \in (-2, -3/2)$, $T_{FB}(x) < T_{PS}(x)$ for all loads on a unique interval $(0, \chi_0)$.

*For Pareto distributed service times with index $-\nu \in (-3/2, -1)$, $T_{FB}(x) < T_{PS}(x)$ for all loads and all service times $x$.***
For discrete distributions composed of two points of positive measure, \( x_1 \) and \( x_2 \), there may exist two distinct intervals, \((0, \chi_0)\) and \((\chi_1, \chi_2)\), where \( T^{FB}(x) < T^{PS}(x) \) for all loads. This is for example the case if \( x_1 = 1 \) with probability \( 1 - p \), \( x_2 = 4 \) with probability \( p \), and \( p = 1/12 \).

**Proof.** It is sufficient to find the intervals for which \( \delta(x) < 0 \) for each distribution. Carrying through the calculations for each case we find the results of the proposition.

It is thus possible for a work-conserving non anticipating scheduling policy to deliver smaller expected sojourn times over all job sizes.

**References**


