Asymptotic invariant tori of perturbed two-body problems

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Abstract

This paper deals with the asymptotic approximation of invariant tori, periodic and quasi-periodic orbits of perturbations of the planar and spatial two-body problems. For that we make use of different types of normalisations and reduction theory.

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1. Introduction

1.1. Perturbed two-body problems

Let us consider 2D and 3D conserved systems formed as the sum of the Kepler problem and the Coriolis term to which we attach a small perturbation, say $P$. The corresponding Hamilton function in a certain frame (synodic) $\{s_1, s_2, s_3\}$ is given by

$$H(x, X; \varepsilon) = \frac{1}{2} x \cdot x - \frac{\mu}{\|x\|} - \Omega (x \times X) \cdot s_3 + P(x, X; \varepsilon),$$

(1)
where \( x = (x, y, z) \) represents the position vector and \( X = (X, Y, Z) \) stands for the velocity. The constant \( \mu > 0 \) is the gravitational constant with physical dimension [length\(^3\)/time\(^2\)]. The primary body rotates with a uniform angular speed \( \Omega \) with physical dimension [1/time]. Besides, \( |P| \ll \|x\|^{-1} - \Omega (x \times x) \cdot s_3 \).

Therefore, concerning the relative size of the two parts of the unperturbed Hamiltonian three possibilities are in order. Specifically, defining \( H_K \) (the pure Kepler Hamiltonian) and \( H_C \) (the so-called Coriolis term or rotating term, see for instance Marsden and Ratiu (1999)) as:

\[
H_K(x, X) = \frac{1}{2} X \cdot X - \frac{\mu}{\|x\|}, \\
H_C(x, X) = -\Omega (x \times X) \cdot s_3 = -\Omega (x Y - y X),
\]

we rewrite \( H = H_0 + H_P \), and one of the following scalings must be chosen: (i) \( |H_K| \approx |H_C| \): moderate rotations, then \( H_0 = H_K + H_C \) and \( H_\Phi = P \); (ii) \( |H_K| \gg |H_C| \): slow rotations, then \( H_0 = H_K \) and \( H_\Phi = H_C + P \); (iii) \( |H_K| \ll |H_C| \): fast rotations, then \( H_0 = H_C \) and \( H_\Phi = H_K + P \).

This Hamiltonian can be understood as a perturbed Kepler problem and appears in many problems in classical and celestial mechanics (the hydrogen atom in crossed electric and magnetic fields (Cushman and Sadowskii, 2000), some cases of restricted three-body problems (Palacián and Yanguas, 2004) or the motion of an artificial satellite subject to the gravity force of an inhomogeneous planet (Deprit, 1981)). There are certain sets of variables which are especially well-suited to deal with this type of system. As we will make use of them, let us briefly describe their main features.

Polar-nodal variables \((r, \vartheta, v, R, \Theta, N)\) are introduced as the symplectic transformation \( \psi : (r, \vartheta, v, R, \Theta, N) \longrightarrow (x, y, z, X, Y, Z) \) such that

\[
\begin{align*}
x &= x' \cos(v) - y' \cos(I) \sin(v), \\
y &= x' \sin(v) + y' \cos(I) \cos(v), \\
z &= y' \sin(I), \\
x' &= r \cos(\vartheta), \\
y' &= r \sin(\vartheta), \\
X &= X' \cos(v) - Y' \cos(I) \sin(v), \\
Y &= X' \sin(v) + Y' \cos(I) \cos(v), \\
Z &= Y' \sin(I),
\end{align*}
\]

where \( x', y', X' \) and \( Y' \) are given by

\[
\begin{align*}
x' &= r \cos(\vartheta) - \frac{\Theta}{r} \sin(\vartheta), \\
y' &= r \sin(\vartheta), \\
X' &= R \cos(\vartheta) - \frac{\Theta}{r} \sin(\vartheta), \\
Y' &= R \sin(\vartheta) + \frac{\Theta}{r} \cos(\vartheta).
\end{align*}
\]

The variable \( \Theta \) is the modulus of the angular momentum vector \( \mathbf{G} = x \times X \) in the synodic frame. The angle conjugate to \( \vartheta \) is the argument of the latitude \( 0 \leq \vartheta < 2\pi \) through the radial direction. Besides, \( r \) is the modulus of vector \( x \), and its conjugate moment \( R \) denotes the radial velocity in the synodic frame. The angle of the node \( v \) is the co-ordinate conjugate to \( N \). In the region of the phase space where \( \mathbf{G} \) does not vanish, we can decompose it uniquely as the product \( \mathbf{G} = \Theta \mathbf{n} \) with \( \Theta > 0 \) and \( \|\mathbf{n}\| = 1 \). Vector \( \mathbf{n} \) indicates the normal direction and is orthogonal to the plane spanned by \( x \) and \( X \): the instantaneous orbital plane. Its inclination with respect to the equatorial plane is given by the angle \( 0 < I < \pi \) such that \( N = \Theta \cos(I) \) and \( N \) is the third component of \( \mathbf{G} \) in
the synodic frame. The domain of validity of polar-nodal variables is the subset of $\mathbb{R}^6$: 

$$\Delta_{pn} = (0, +\infty) \times [0, 2\pi) \times [0, 2\pi) \times \mathbb{R} \times (0, \infty) \times (−\Theta, \Theta).$$

Thus, some types of trajectories cannot be studied with these variables. Specifically they are not useful for collision ($r = 0$), rectilinear ($\Theta = 0$) and equatorial orbits ($\Theta = |N|$).

**Delaunay variables** $(\ell, g, h, L, G, H)$ are a set of action-angle variables defined through polar-nodal variables by means of a generating function built with the “mixed” set of variables $(r, \vartheta, \nu, L, G, H)$. We do not detail the construction of these variables, but we refer to Deprit (1981, 1982).

If $H_K$ stands for the Hamiltonian of the two-body problem, the action $L$ is related to the two-body energy by the identity $H_K = −\mu^2/(2L^2)$, where $\mu$ represents the gravitational constant. The action $G$ is the modulus of the angular momentum, thus $G = \Theta$. The third component of $G$ is $H$, so $H = N$. Moreover $H_C = −\Omega H$ or $H_C = −\Omega G$ in planar systems.

The angle $\ell$ is named as the mean anomaly and is related to the eccentric anomaly $E$ by means of the Kepler equation $\ell = E − e \sin(E)$, where $e$ designates the eccentricity of the orbit, which in terms of Delaunay actions reads $e = \sqrt{1 − G^2/L^2}$; as $e \in [0, 1)$, then $G \in (0, L]$. The angle $E$ is also expressed in terms of the true anomaly $f$.

The angle $g$ is the argument of the pericentre. It is reckoned from the pericentre of the orbit in the instantaneous orbital plane. Then, $g = \vartheta − f$. The angle $h$ is the argument of the node, that is, $h = v$.

Delaunay variables are not valid for circular orbits because $G = L$ or $e = 0$ and the argument of the pericentre is not defined. Besides, collision, rectilinear and equatorial orbits were discarded before, since they were not defined in polar-nodal variables. Thus, the domain of validity of Delaunay variables is given by the subset of $\mathbb{R}^6$: $\Delta_D = [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \times (0, +\infty) \times (0, L) \times (−G, G)$.

### 1.2. Scope and organisation

Our purpose in this paper is to give the guidelines for the construction of periodic, quasi-periodic orbits and 2D invariant tori of a general Hamilton function like (1) by means of a theory which combines perturbation methods based on Lie transformations (Deprit, 1969) with reduction techniques. Moreover we emphasise the algorithmic aspects of our methodology or give the necessary references useful to implement all steps with a symbolic manipulator. Many parts of the approach we present are not new; however, our aim is to put all pieces together in such a way that one can analyse a perturbed Kepler problem following the steps we give below.

Section 2 recalls how to construct formal integrals through Lie transformations, as well as the two techniques for normalising Hamiltonians we use in the paper in a rather generic context. In particular, the algorithm of Section 2.3 is new and can be used in a more general setting. In Section 3 we deal with the reduction process for perturbed Keplerian systems. We explain how to approximate the invariant tori of a given Hamiltonian from the non-degenerate critical points (e.g. equilibria whose linearisations have non-zero eigenvalues) related to their reduced system in Section 4. The conclusions of the paper are addressed in Section 5.
2. Averagings and formal integrals

2.1. Formal integrals through Lie transformations

The central idea is to reduce a Hamiltonian system of the form (1) so that the transformed Hamiltonian $K$ be a system with one degree of freedom less than $H$. Since this approach can be stated for Hamilton functions not necessarily of polynomial nature, we start by focusing on the required steps for any type of analytic Hamiltonian.

Let $J$ denote the skew-symmetric matrix of dimension $2n \times 2n$ and let $s = (s_1, \ldots, s_n, S_1, \ldots, S_n)$ and $t = (t_1, \ldots, t_n, T_1, \ldots, T_n)$ be two symplectic sets of coordinates in $\mathbb{R}^{2n}$. Finally, let $L_T$ denote the Lie operator $L_T(S) = \{S, T\}$. One has the following result.

Theorem 1. Let $M \geq 1$ be given, let $\{P_i\}_{i=-1}^M, \{Q_i\}_{i=1}^M$ and $\{R_i\}_{i=1}^M$ be sequences of linear spaces of smooth functions defined on a common domain $\Omega$ in $\mathbb{R}^{2n}$ and let $G$ be a function in $P_j$, for some $j \geq -1$, with the following properties: (i) $Q_i \subseteq P_i, i = 1, \ldots, M$; (ii) $H_i \in P_i, i = 0, 1, \ldots, M$; (iii) $\{P_i, R_j\} \subseteq P_{i+j}, i + j = 1, \ldots, M$; (iv) for any $D \in P_i, i = 1, \ldots, M$, one can find $E \in Q_i$ and $F \in R_i$ such that

$$E = D + \{H_0, F\} \quad \text{and} \quad \{G, E\} = 0.$$

Then, there exists an analytic function $W$,

$$W(s; \varepsilon) = \sum_{i=0}^{M-1} \frac{\varepsilon^i}{i!} W_{i+1}(s),$$

with $W_i \in R_i, i = 1, \ldots, M$, such that the direct change of variables $s = S(t; \varepsilon)$ is the general solution of

$$\frac{ds}{d\varepsilon} = J \frac{\partial W}{\partial s}(s; \varepsilon), \quad s(0) = t,$$

and transforms the convergent Hamiltonian

$$H(s; \varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} H_i(s),$$

to the convergent Hamiltonian

$$K(t; \varepsilon) = \sum_{i=0}^{M} \frac{\varepsilon^i}{i!} K_i(t) + O(\varepsilon^{M+1}),$$

with $K_i \in Q_i$ and $\{K_i, G\} = 0, i = 1, \ldots, M$. Besides, if $\{H_0, G\} = 0$, then $G$ is a formal integral of $K$.


A crucial remark is that whenever $G$ is an integral of $H_0$, the effect of constructing $K_i \in \ker(L_Q)$ for $i = 1, \ldots, M$, is to extend (formally) the integral of the unperturbed system to the whole transformed Hamiltonian $K$. This means that the choice
of $G$ can be done adequately if one knows previously the integrals of $H_0$. In these cases we obtain an integral of the truncated system $K$ independent of $K$. The function

$$I(s; \varepsilon) = G(s) + \sum_{i=1}^{M} \varepsilon^i \frac{d^i}{dt^i} L_{-W}(G(s)),$$

where $L_{-W}(A) = [W, A]$ and $L'_{-W}(A) = L_{-W}(L_{-W}(A))$, becomes a formal integral of $H$ (within an approximation of $O(\varepsilon^{M+1})$) functionally independent of it. Using $W$ one constructs the inverse change $t = T(s; \varepsilon)$. Note that $G(s) \equiv G(t)$ and $I(S(t); \varepsilon) = G(t) + O(\varepsilon^{M+1})$.

We remark that the determination of “generalised normal forms” allows us to obtain information about the original system, which is not provided by the classical normal form. Indeed, the analysis of the transformed system (equilibria, periodic orbits and centre manifolds) can be used to calculate families of periodic orbits and invariant manifolds of the system defined by $H$. This is done up to an approximation $O(\varepsilon^{M+1})$ and under classical conditions of non-degeneracy (see the reference by Moser (1970)).

As in the calculation of normalised Hamiltonians, the symplectic change of variables $s = S(t; \varepsilon)$ is made through a Lie transformation, in practice. The construction of $K$ must be done order by order from $i = 1$ to $i = M$. For that, the partial differential identity (2), that is, the homology equation

$$L_{H_i}(W_i) + K_i = \tilde{H}_i,$$  \hspace{1cm} (2)

has to be solved with the extra condition $\left[K_i, G\right] = 0$ for $i = 1, \ldots, M$. Note that the terms $H_i$ are known and the solution of (2) is the pair $(W_i, K_i)$. We have to split $\tilde{H}_i$ as $\tilde{H}_i = H_i^1 + H_i^2$, where $H_i^1 \in \text{ker}(L_G)$ and $H_i^2 = \tilde{H}_i - H_i^1$, for each $i = 1, \ldots, M$. In this way, we choose $K_i = H_i^1$ and $W_i$ as a solution of $L_{H_0}(W_i) = \tilde{H}_i^1$.

Our task now is the application of the above result in two different situations. To simplify our presentation we start by considering a two-degree-of-freedom Hamiltonian in action-angle variables $(\vartheta_1, \vartheta_2, I_1, I_2)$ that can be split as $H = H_0 + \frac{p_i^a}{p_i}H_p + \frac{q_j}{q_j}H_q$ with $q > p > 1$, thus $|H_0| \gg |\frac{p_i^a}{p_i}H_p| \gg |\frac{q_j}{q_j}H_q|$. Moreover we assume that $H_0 = \omega_1 I_1$, $H_p = \omega_2 I_2$, whereas

$$H_q = \sum_{j,k \in \mathbb{Z}} a_{j,k} \sin(j \vartheta_1 + k \vartheta_2) + b_{j,k} \cos(j \vartheta_1 + k \vartheta_2),$$

with $a_{j,k}$ and $b_{j,k}$ standing for functions depending on the actions $I_1, I_2$ and the constants of a specific model.

Next we plan to build two formal integrals via adequate Lie transformations. Specifically either $I_1$ or $I_2$ will become the new integrals of the approach after averaging over $\vartheta_1$ or over $\vartheta_2$.

2.2. Averaging over $\vartheta_1$

In the first case we start by making $K_0 = H_0$, $K_p \equiv H_p$ and $K_1 \equiv \cdots \equiv K_{p-1} \equiv K_{p+1} \equiv \cdots \equiv K_{q-1} = 0$. Besides we put $W_1 \equiv \cdots \equiv W_{q-1} = 0$. Next, for each
\( i \in \{q, \ldots, M\} \), we solve the homology equation
\[
\omega_1 \frac{\partial W_i}{\partial \vartheta_1} + K_i = \tilde{H}_i.
\]

Now, as we can write
\[
\tilde{H}_i = \sum_{j,k \in \mathbb{Z}} c_{j,k} \sin (j \vartheta_1 + k \vartheta_2) + d_{j,k} \cos (j \vartheta_1 + k \vartheta_2),
\]
for some functions \( c_{j,k} \) and \( d_{j,k} \) depending on \( I_1 \) and \( I_2 \) we make
\[
K_i = \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{H}_i \, d\vartheta_1, \quad W_i = \frac{1}{\omega_1} \left( \int \tilde{H}_i \, d\vartheta_1 - K_i \vartheta_1 \right).
\]
Each \( \tilde{W}_i \) is a Fourier sum in \( \vartheta_1 \) and \( \vartheta_2 \). This algorithm has been implemented with MATHEMATICA.

In the Keplerian context two possibilities are in order (see further details in Palacín (2002a,b)): (i) either \( \vartheta_1 \) refers to the mean anomaly or (ii) it is the argument of the node (or the argument of the pericentre if we are in a 2D setting). In case (i) we have the so-called Delaunay normalisation procedure Deprit (1981, 1982) and \( H_0 = H_K \) and \( H_p = \frac{p^2}{\ell^2} H_C \) whereas in case (ii) we deal with the elimination of the node and \( H_0 = H_C \) and \( H_p = \frac{p^2}{\ell^2} H_K \); see details in Palacín (2002a,b).

In case (i) the expressions which have to be averaged depend on the mean anomaly through the variables \( r, R, \vartheta \) of \( f \) and so they do not admit explicit expressions in terms of \( \ell \). However, several strategies have been devised to deal with closed expressions, without making Fourier expansions in terms of \( \ell \) or Taylor expansions in terms of the eccentricity. Thus, the problem has been circumvented using adequate changes of variables defined through the eccentric and the true anomalies and making use of polylogarithmic functions (Osácar and Palacín, 1994; Palacín, 2002a) for the exact determination of the generating function. Thus, after dropping higher-order terms the new Hamiltonian is going to be independent of \( \ell \) and, subsequently, \( K \) will enjoy the action \( L \) as a new integral. In case (ii) the normalisation of the argument of the node is implemented straightforwardly up to any order \( M \), and after truncation, \( H \) becomes an integral of \( K \).

### 2.3. Averaging over \( \vartheta_2 \): Nonstandard normalisation

In this second situation, as we want that \( I_2 \) becomes the new integral we cannot use the strategy described above. Indeed we use Lie transformations but in such a way that we calculate each portion \( \tilde{W}_i \) of the generating function in two steps. We detail the process below. For an arbitrary order \( i \in \{q, \ldots, M\} \), the homology equation to be solved on this occasion yields
\[
\left( \begin{array}{c}
\ell \\
\omega_1
\end{array} \right) L_{H_p} (W_{i-p}^*) + L_{H_0} (\tilde{W}_i) + K_i = \tilde{H}_i,
\]
assuming that \( W_{i-p}^* \) corresponds to the unknown part of \( W_{i-p} \), whereas \( \tilde{H}_i \) is the known part of the Lie triangle. The Lie operator associated with \( H_p \) is \( L_{H_p} (\cdot) = \omega_2 \partial (\cdot) / \partial \vartheta_2 \). Note also that \( L_{H_0} (\cdot) = \omega_1 \partial (\cdot) / \partial \vartheta_1 \). The algorithm is given below.
Input: $H_i, i \in \{0, \ldots, n\}; p$ and $q$.

Step (1): Make $K_0 \equiv H_0, K_p \equiv H_p$ and $K_1 \equiv \cdots \equiv K_{p-1} \equiv K_{p+1} \equiv \cdots \equiv K_{q-1} = 0$ and $W_1 \equiv \cdots \equiv W_{q-1} = 0$.

Step (2): Decompose each $\hat{H}_i$ of the form given by (3) as $\hat{H}_i = \hat{H}^*_i + \hat{H}^\#_i + \hat{H}^\&_i$, where

$$\hat{H}^*_i(\vartheta_1, \vartheta_2, I_1, I_2), \quad \hat{H}^\#_i(-, \vartheta_2, I_1, I_2), \quad \hat{H}^\&_i = \hat{H}^\&_i(\vartheta_1, \vartheta_2, I_1, I_2).$$

Step (3): Identify $K_i$ with $\hat{H}^*_i$, which is equivalent to average $\hat{H}_i$ over the angle $\vartheta_2$.

Step (4): Solve $L_{H_p}^i(W_i) = \hat{H}^\&_i$, obtaining $W_i$. Note that $W_i$ is determined after calculating an integral with respect to $\vartheta_1$, and is a periodic function in $\vartheta_2$ and also in $\vartheta_1$, since $\hat{H}^\&_i$ does not have terms independent of $\vartheta_1$ and $\vartheta_2$.

Step (5): Calculate $W^\ast_{i-p}$ from

$$\left(\begin{array}{c}
 i \\
 p
\end{array}\right) L_{H_p}^i(W^\ast_{i-p}) = \hat{H}^\#_i$$

through the computation of an integral with respect to $\vartheta_2$. The terms $\hat{H}^\#_i$ are independent of $\vartheta_1$ and periodic in $\vartheta_2$, thus $W^\ast_{i-p}$ is periodic in $\vartheta_2$ and does not depend on $\vartheta_1$.

Step (6): Add $W^\ast_{i-p}$ to $W_{i-p}$, completing the generating function of order $i - p$.

Step (7): While $i \leq n$ make $i = i + 1$ and go to Step (2), applying the usual recursion of the Lie triangle process.

Output: $K_i$ for $i \in \{0, \ldots, n\}$ and $W_i$ for $i \in \{1, \ldots, n\}$.

This algorithm has been implemented in MATHEMATICA following the steps we have specified above. As in Section 2.2 in perturbed spatial two-body problems we usually have two cases, either $\vartheta_1 = \ell$ and $\vartheta_2 = h$ or $\vartheta_2 = \ell$ and $\vartheta_1 = h$. Thus the average over the mean anomaly together with the search of the corresponding piece of the generating function needs more effort than its equivalent for the node, and one has to make use of the changes and algorithms described in Osácar and Palacián (1994), Palacián (2002a,b). We also refer to cf. Palacián and Yanguas (2004) for an application to the 3D circular restricted three-body problem.

3. Reductions and invariant theory

The averaging techniques explained in Section 2 lead to the introduction of a formal integral. The appearance of this integral allows the initial system to be reduced by one degree of freedom, i.e. the normalised Hamiltonian $K$ defines a dynamical system of two degrees of freedom.

For each normal form transformation, we have to describe the phase space where $K$ is defined. Once the integral is fixed, the reduced phase space has dimension four. It is constructed according to the integral introduced in the Lie transformation. Therefore, two different phase spaces are considered for perturbed Keplerian systems: one is associated with the integral $L$ and the other is associated with $H$. One should realise that the integral $G$ we take (either $G = H_K$ or $G = H_C$) represents a maximally superintegrable system, that is, it possesses five independent integrals of motion. Thence, we can employ, after
fixing the value of one of the five, the other four to parametrise the reduced phase space. This is why the phase space out of the reduction process has dimension four.

Note that $H_K$ and $H_C$ define complete Hamiltonian vector fields, see Abraham and Marsden (1985). Therefore their flows define group actions and hence the reduction theorem can be applied in both cases; see Meyer (1973), Arms et al. (1991) for more details.

3.1. The action $L$ becomes an integral

The invariants associated with $L$ are the functions which are constants on the solutions of the system defined by $H_K$. All these integrals can be expressed as functions of $L$, the components of the angular momentum vector $\mathbf{G} = (G_1, G_2, G_3)$ (note that $G_1^2 + G_2^2 + G_3^2 = L^2$ and $G_3 = H$) and the Laplace vector $\mathbf{A} = (A_1, A_2, A_3)$ defined through

$$\mathbf{A} = \frac{1}{\mu} (\mathbf{X} \times \mathbf{G}) - \frac{\mathbf{x}}{\|\mathbf{x}\|},$$

see cf. Cushman (1983) for more details. Observe that $\|\mathbf{G}\| = G$, $\|\mathbf{A}\| = e$ and $\mathbf{G} \cdot \mathbf{A} = 0$.

Cushman (1983) introduced the mapping $\rho : \mathbb{R}^6 \setminus \{(0) \times \mathbb{R}^3\} \rightarrow \mathbb{R}^5 : (\mathbf{x}, \mathbf{X}) \mapsto (\mathbf{a}, \mathbf{b}) \equiv (\mathbf{G} + L \mathbf{A}, \mathbf{G} - L \mathbf{A})$, with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Explicitly, the functions $a_i$ and $b_i$ can be given in terms of the co-ordinates $\mathbf{x}$ and $\mathbf{X}$.

Now, fixing a value of $-\mu^2/(2 L^2) < 0$, the product of the two-spheres

$$S^2_L \times S^2_L = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1^2 + b_2^2 + b_3^2 = L^2\},$$

is the phase space for Hamiltonian systems of Keplerian type independent of $\ell$, that is, for Hamiltonians for which $L$ is an integral. This result was first reported by Moser (1970) using a regularisation technique based on stereographic projections. Observe that $S^2_L \times S^2_L$ is a smooth space and therefore the reduction is regular (Meyer, 1973; Abraham and Marsden, 1985). In 2D, the corresponding reduced phase space is $S^2_L$.

It is not hard to derive formulae for the functions $G$, $H$, $cos g$, $sin g$, $cos h$ and $sin h$ in terms of $\mathbf{a}$ and $\mathbf{b}$; see Cushman (1983). Now, a Hamiltonian $K$ independent of $\ell$ can be written as a function of the invariants $\mathbf{a}$ and $\mathbf{b}$ and the constant $L > 0$, i.e. $K \equiv K(\mathbf{a}, \mathbf{b}; L)$. Note that the way in which the invariants appear in the Hamiltonian $K$ depends on each specific problem.

The functions $a_i$ and $b_i$ are the invariants associated with $S^2_L \times S^2_L$. These elements together with the constraints $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$ define the reduced phase space, so they are called generators or co-ordinates of the reduced phase space.

Since $2 G = ((a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2)^{1/2}$, one has that $G = 0$ in $S^2_L \times S^2_L$ if and only if $a_1 + b_1 = a_2 + b_2 = a_3 + b_3 \equiv 0$, $a_1^2 + a_2^2 + a_3^2 = L^2$ and $b_1^2 + b_2^2 + b_3^2 = L^2$. These relations define a two-sphere $R^2_L = \{(\mathbf{a}, -\mathbf{a}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2\}$, and rectilinear trajectories could be analysed.

Circular orbits are connected to the condition $G = L$. Thus, in terms of $\mathbf{a}$ and $\mathbf{b}$ they are given by the three-dimensional set $G_L = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1^2 + b_2^2 + b_3^2 = L^2, a_1 b_1 + a_2 b_2 + a_3 b_3 = L^2\}$. 
Similarly, equatorial trajectories (they satisfy $G = |H|$) can be treated with the invariants, as they are described by the two-dimensional set $E_L = \{(a, b) \in \mathbb{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, b_1 = -a_1, b_2 = -a_2, b_3 = a_3\}$.

The above shows how the introduction of the invariants extends the use of Delaunay variables as we can include equatorial, circular and rectilinear orbits. The Poisson brackets involving the $a_i$'s and $b_i$'s can be seen in Cushman (1983).

3.2. The action $H$ becomes an integral

The integrals associated with $H$ are the constant functions on the solutions of the differential system defined by $H_{\mathcal{C}} = -\Omega H$. The invariant polynomials related to $H$ are built in Cartesian co-ordinates. Starting with an arbitrary polynomial of degree one, say $p_1$, we determine the coefficients of $p_1$ so that $\{p_1, x Y - y X\} = 0$. Next we pass to $p_2$, an arbitrary polynomial of degree two, calculating the conditions on its coefficients so that $\{p_2, x Y - y X\} = 0$. We could continue with higher-order polynomials. A set of generators of the invariants is given through vector $c = (c_1 \ldots, c_6)$, where:

\[
\begin{align*}
  c_1 &= x^2 + y^2, & c_2 &= x X + y Y, & c_3 &= z, \\
  c_4 &= x^2 + Y^2, & c_5 &= x Y - y X, & c_6 &= Z,
\end{align*}
\]

since higher-degree invariants can be written in terms of the $c_i$'s. Besides, the invariants $c_i$'s satisfy the relation $c_1 c_4 = c_2^2 + c_3^2$. Moreover, this is the only independent constraint as there must be five functionally-independent generators out of the six invariants of (4).

It is possible to express $c$ as a combination of polar-nodal and Delaunay variables. However, one can identify $c_5$ with $H$. Fixing a value of $H$ (with $|H| \leq G$), this integral $H$ can be understood as an $S^1$-action, or the action of the one-dimensional unitary group $U(1)$ over the space of co-ordinates and moments such that

\[
\rho : S^1 \times (\mathbb{R}^6 \setminus ((0) \times \mathbb{R}^3)) \rightarrow \mathbb{R}^3 \times \mathbb{R}^3
\]

\[
(R_h, (x, X)) \mapsto (R_h x, R_h X),
\]

where

\[
R_h = \begin{pmatrix}
\cos h & \sin h & 0 \\
-\sin h & \cos h & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{with } 0 \leq h < 2\pi.
\]

In fact, if $SO(3)$ denotes the special orthogonal group, its subgroup: $O_{S_3} = \{O \in SO(3) \mid O s_3 = s_3\} = \{R_h \mid 0 \leq h < 2\pi\}$ is diffeomorphic to $S^1$.

This is a singular (or non-free) action because there are non-trivial isotropy groups. The subspace $\{(0, 0, z) \mid z \in \mathbb{R}\}$ is invariant under all rotations around the $z$ axis. Thus, the reduction due to the axial symmetry is singular, in contrast to the regular reduction obtained by doing $L$ an integral, where all the isotropy groups were trivial. Then we apply a singular reduction treatment (Arms et al., 1991).

The reduced phase space is given as the quotient space $\mathbb{R}^6/\rho = \mathbb{R}^6/(S^1 \times S^1)_H$ for a fixed value of $H$, that is,

\[
\mathbb{R}^6/(S^1 \times S^1)_H = \{c \in \mathbb{R}^6 \mid c_1 c_4 = c_2^2 + c_3^2, c_5 = H, c_1, c_4 \geq 0\}.
\]
It is a four-dimensional space whose generators are the invariants \( \mathbf{c} \) defined by (4) with the constraint \( c_1 c_4 = c_2^2 + c_3^2 \) and \( c_5 = H \). In 2D \( H_0 = -\mu^2/(2 L^2) - \Omega G \) and the reduced phase space after making \(-\Omega G\) an integral out of the normal form turns out to be \( \{ i \in \mathbb{R}^4 \mid i_1 i_2 = i_3^2 + i_4^2, \ i_4 = G, i_1, i_2 \geq 0 \} \) where \( i_1 = x^2 + y^2, i_2 = X^2 + Y^2, i_3 = x X + y Y \) and \( i_4 = x Y - y X \).

This time \( K \) can be written as a function of the invariants \( \mathbf{c} \) and the constant \(|H| \leq G\) as a parameter, e.g. \( K \equiv K(\mathbf{c}; H) \). The Poisson brackets among the components of \( \mathbf{c} \) are needed to analyse a certain normalised Hamiltonian. The list appears in Palacián (2002a).

Rectilinear orbits can be considered in (6). Note that \( G = 0 \) if and only if \( x Y - y X = 0, x Z - z X = 0 \) and \( z Y - y Z = 0 \). Combining this with the constraints given in (6), we have that the space of rectilinear trajectories is a three-dimensional subset of \( \mathbb{R}^4/S^1 \times S^1 \) defined as \( \mathcal{R}_H = \{ \mathbf{c} \in \mathbb{R}^6 \mid c_5 = 0, c_1 c_4 = c_2^2, c_1 c_6 = c_2 c_3, c_2 c_6 = c_2 c_4, c_1, c_4 \geq 0 \} \). So rectilinear orbits, excepting circular and equatorial orbits can be treated without restrictions as \( \mathbf{c} \) is not derived from Delaunay variables. For instance, equatorial trajectories are in the two-dimensional set \( \mathcal{E}_H = \{ \mathbf{c} \in \mathbb{R}^6 \mid c_1 c_4 = c_2^2 + c_3^2, c_3 = c_6 = 0, c_5 = H, c_1, c_4 \geq 0 \} \).

3.3. Further reduction

In some cases it is possible to further reduce the normalised Hamiltonian \( K \) and construct another Hamiltonian of one degree of freedom. This is the so-called integral approximation of \( H \).

The second reduction can be done whether there are no resonant terms in the system. If \( G \) (either \( L \) or \( H \)) is an integral of \( H_0 \), approximate resonances appear for specific relations between the angles \( \ell \) and \( h \) in \( H_1 \). In these circumstances, a second averaging process could be calculated; see some examples in Cushman (1983), Palacián (2002a) and Cushman and Sadovskii (2000). So, if the first system depends on \( h \), the second normalisation process consists in making \( H \) an integral out of it. In contrast, if the first averaged Hamiltonian depends on \( \ell \), the second normal form is built so that the action \( L \) becomes an integral of it.

From a practical point of view, the second reduction can be performed up to any order for moderate, slow and fast rotations. The reason is that as the first normalised Hamilton function defines a two-degree-of-freedom system, so \( L \) or \( H \) becomes an integral, hence \(-\mu^2/(2 L^2)\) or \(-\Omega H\) can be considered as a constant of motion and the corresponding Lie operator needed to perform the second normalisation reduces to, respectively, \( L_{H_0} = -\Omega \partial (\cdot)/\partial \ell \) or \( L_{H_0} = n \partial (\cdot)/\partial \ell \). So, the process to obtain \( K \) and \( \mathcal{W}_i \) up to any order is almost identical to the one explained in Section 2.2.

Now we have to define the doubly-reduced phase space, \( T_{L,H} \), from \( S_L^2 \times S_L^2 \) although we could do it from the generators of \( \mathbb{R}^6/(S^1 \times S^1)_H \). We define an \( S^1 \)-action \( \varrho \) on \( S_L^2 \times S_L^2 \) as \( \varrho : S^1 \times (S_L^2 \times S_L^2) \to S_L^2 \times S_L^2 : (h, (\mathbf{a}, \mathbf{b})) \mapsto (R_h \mathbf{a}, R_h \mathbf{b}) \), where \( R_h \) is the matrix given already in (5).

The algebra of polynomials on \( S_L^2 \times S_L^2 \) invariant under \( \varrho \) is given through

\[
\begin{align*}
\pi_1 &= a_1^2 + a_2^2, & \pi_2 &= a_1 b_2 - a_2 b_1, & \pi_3 &= a_3, \\
\pi_4 &= b_1^2 + b_2^2, & \pi_5 &= a_1 b_1 + a_2 b_2, & \pi_6 &= b_3,
\end{align*}
\]
together with the constraints
\[ \pi_1 + \pi_2^2 = L^2, \quad \pi_4 + \pi_6^2 = L^2, \quad \pi_2^2 + \pi_5^2 = \pi_1 \pi_4. \]  
\[ \text{(8)} \]
Note the similarity between the \( c_i \)'s and the \( \pi_i \)'s. This owes to the fact that both sets are invariant under the action of \( H \). However, they are not equivalent, as the \( \pi_i \)'s are also invariant under the action of \( L \), whereas the \( c_i \)'s are not.

Taking the mapping \( \pi_H : S_L^2 \times S_L^2 \rightarrow \{ H \} \times \mathbb{R}^3 : (\mathbf{a}, \mathbf{b}) \mapsto (H, \tau_1, \tau_2, \tau_3) \equiv (H, \tau) \), where \( \tau_1 = \frac{1}{2}(\pi_3 - \pi_6) \), \( \tau_2 = \pi_2 \) and \( \tau_3 = \pi_5 \) we define the invariants \( \tau_i \)'s, in terms of \( \mathbf{a} \) and \( \mathbf{b} \), as \( \tau_1 = \frac{1}{2}(a_3 - b_3) \), \( \tau_2 = a_1 b_2 - a_2 b_1 \), \( \tau_3 = a_1 b_1 + a_2 b_2 \). As \( 2 \cos L = a_3 + b_3 \) then \( H = \frac{1}{2}(a_3 + b_3) \), \( H = a_3 - \tau_1 \) and \( H = \tau_1 - b_3 \). Note that \( \tau \) may be expressed in Delaunay variables; see Cushman (1983). The constraints (8) are used to define the corresponding phase space. This space is defined as the image of \( S_L^2 \times S_L^2 \) by \( \pi_H \), that is,
\[ T_{L,H} = \pi_H(S_L^2 \times S_L^2) \]
\[ = \{ \tau \in \mathbb{R}^3 \mid \tau_2^2 + \tau_5^2 = [L^2 - (\tau_1 - H)^2][L^2 - (\tau_1 + H)^2] \}, \]  
\[ \text{(9)} \]
for \( 0 \leq |H| \leq L \) and \( L > 0 \). Note that \( \tau_2 \) and \( \tau_3 \) always lie in the interval \( [H^2 - L^2, L^2 - H^2] \) whereas \( \tau_1 \) belongs to \( |H| - L, L - |H| \).

When \( 0 < |H| < L \), \( T_{L,H} \) is diffeomorphic to a two-sphere \( S^2 \) and therefore the reduction is regular in that region of the phase space. However, when \( H = 0 \) then \( T_{L,0} \) is a topological two-sphere with two singular points: the vertices at \((\pm L, 0, 0)\). The reason for the existence of these two points is that the \( S^1 \)-action \( \varphi \) has two fixed points: \( L (\pm 1, 0, 0, \mp 1, 0, 0) \), and consequently \( \varphi \) is not free. Finally, when \( |H| = L \), \( T_{L,H} \) gets reduced to a point.

The variables \( g \) and \( G \) can be expressed in terms of \( \tau \); see Cushman (1983). It is also possible to express the quantities \( \sin L, \cos L, \sin g, \cos g \) and \( G \) in terms of \( \tau, L \) and \( H \). Besides, \( e \) can be put in terms of the invariants, \( L \) and \( H \), through the variable \( G \). Rectilinear orbits satisfy \( G = H = 0 \). Taking also into account the constraint appearing in (9), we know that they are defined on the one-dimensional set: \( \mathbb{R}_{L,0} = \{ \tau \in \mathbb{R}^3 \mid \tau_2 = 0, \tau_3 = \tau_1^2 - L^2 \} \). Thus, excepting orbits with \( ||\mathbf{x}|| = 0 \), we could analyse rectilinear trajectories. Circular orbits are concentrated in a unique point of \( T_{L,H} \) with co-ordinates \((0, 0, L^2 - H^2)\) whereas equatorial trajectories in this doubly-reduced phase space are represented in the negative extreme point of \( T_{L,H} \) with co-ordinates \((0, 0, H^2 - L^2)\).

The twice normalised Hamiltonian is represented by a Hamiltonian expressed in terms of the \( \tau_i \)'s. It defines a one-degree-of-freedom system with \( L \) and \( H \) their integrals and is expressed as \( K(\tau; L, H) \). The Poisson brackets of the \( \tau_i \)'s appear in Cushman (1983) and Palacián (2002a).

### 4. Critical points of the reduced spaces

Once the different normalisations have been achieved and their corresponding Hamilton functions are written in terms of their invariants (either \( K_L \equiv K(\mathbf{a}, \mathbf{b}; L) \), or \( K_H \equiv K(\mathbf{c}; H) \) or \( K_{L,H} \equiv K(\tau; L, H) \)) we need to determine the dynamical systems making use of Liouville identity and the tables of Poisson brackets. Thus we arrive at the three differential systems, see Palacián (2002b):
\[
\begin{align*}
\dot{a}_i &= \sum_{1 \leq j \leq 3} \{a_i, a_j\} \frac{\partial K_L}{\partial a_j} + \sum_{1 \leq j \leq 3} \{a_i, b_j\} \frac{\partial K_L}{\partial b_j}, \\
\dot{b}_i &= \sum_{1 \leq j \leq 3} \{b_i, a_j\} \frac{\partial K_L}{\partial a_j} + \sum_{1 \leq j \leq 3} \{b_i, b_j\} \frac{\partial K_L}{\partial b_j}, \\
\dot{c}_i &= \sum_{1 \leq j \leq 6} \{c_i, c_j\} \frac{\partial K_H}{\partial c_j}, \\
\dot{\tau}_i &= \sum_{1 \leq j \leq 3} \{\tau_i, \tau_j\} \frac{\partial K_{LH}}{\partial \tau_j},
\end{align*}
\]

Equilibrium points of Eqs. (10)–(12) are obtained as the roots of the right members of the corresponding equations equated to zero. Obviously the number of roots varies in each case according to the relative values of the constants and other parameters present in each problem. Moreover, the changes in the number of equilibria lead to various types of bifurcations, namely: saddle-centre, flip, pitchfork and Hamiltonian–Hopf. Examples of this appear in Palacián and Yanguas (2004). The bifurcation sets can be points, lines, surfaces or hypersurfaces, depending on the number of parameters of a specific problem and can be determined using several techniques, such as the calculation of resultants between two polynomials. In addition to that, it is also possible to analyse the nonlinear stability of the equilibria using linearisation, Morse Lemma or index theory.

We must connect the critical points with the invariants of the original Hamiltonian. The normalisation transformations must be carried out up to an order \( M \) which takes into account all terms in \( P \) and such that the corresponding equilibria are isolated, because by Morse theory, higher orders in the normalisation process do not alter the behaviour of the reduced system. We stress that for systems (10) and (11), their isolated and non-degenerate critical points correspond to families of periodic orbits (1D tori) parametrised by \( L \) in Eq. (10) or by \( H \) in Eq. (11), having the same stability character. The isolated and non-degenerate equilibria of the flow defined by \( K_{LH} \) are in correspondence with families of 2D tori parametrised by \( L \) and \( H \) from where we obtain quasi-periodic orbits whenever we fix \( L \) or \( H \). Moreover, bifurcations of critical points are translated into bifurcations of periodic orbits or 2D tori.

The existence and stability of these invariant sets is guaranteed by constructing an adequate Poincaré map and using the Implicit Function Theorem; see the general case (Palacián, 2003). Thus, one can speak of asymptotic approximation of the 1D and 2D invariant tori of the flow defined by Hamiltonian (1). Now, the co-ordinates of the asymptotic 1D invariant tori are explicitly determined through the direct change of variables appearing in Section 2.1 (two changes of co-ordinates for the asymptotic 2D tori). Furthermore in order to get very accurate approximations of the actual tori one must push the Lie transformation to order \( M'>M \) provided that we are inside the domain of convergence.

5. Conclusions

In the context of perturbed two-body problems, we present the steps towards the analysis of existence and approximate calculation of invariant tori and quasi-periodic orbits. The
analysis is carried out combining techniques of averaging and normalisation of Keplerian systems with invariant and reduction theory. In particular we present, in a general setting, a novel algorithm useful to construct a formal integral of a certain perturbed Hamiltonian, which is not related to the unperturbed part of the Hamiltonian. This feature permits us to construct other associated reduced phase spaces and reduced Hamiltonians, therefore extending the qualitative analysis of the original Hamiltonian.

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