A NEW CONSTRUCTION FOR MOTION AND SPEED CAPTURE WITH CONICAL WAVELETS

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ABSTRACT
Motion analysis and in particular, speed and rotation analysis, has been introduced in the 80s using the continuous wavelet transform (CWT) with Morlet wavelets. The motion-tuned WT appeared to be an efficient framework and an alternative to the optical flow (OF), the block matching (BM) or the phase difference, for the study of motion. In particular it has shown better performances in the case of noise, in the case of occlusions and in long temporal-dependency motions. However, the Morlet wavelet has defects that cause some difficulties when performing an analysis. By construction, Cauchy and conical wavelets alleviate these drawbacks. In particular, they can be oriented without interference on other parameters and characteristics of the filter. We have extended these wavelets to the spatio-temporal domain and to motion parameters, and we have studied the capabilities of this new Gaussian-Conical-Morlet wavelet (GCM). We present here the first analysis results with this new construction.

Index Terms— Continuous wavelet transform (CWT), motion estimation (ME), directional wavelets, Morlet, Cauchy and conical wavelets, speed tuning, scene analysis, multiresolution, extraction of motion parameters, sequence retrieval, scalable video coding.

1. INTRODUCTION
The continuous wavelet transform has proved to be a very efficient tool for signal analysis. In the late 80s and the 90s, developments to adapt the wavelet transform to various motions have been proposed by M. Duval-Destin, R. Murenzi and J-P. Antoine [5, 1]. The group of analysis parameters, i.e., usually position, scale and rotation, has been extended to speed, acceleration and deformation. This has led to various types of time dependent wavelets [1, Chap. 10]. Then a very performant algorithm for missile tracking was set up using such wavelets [1] [7, 1] The advantage of velocity detection with the motion-tuned spatio-temporal CWT over other known methods, like optical flow (OF) [2], and even block-matching (BM), has been already discussed in [3]. OF and BM work on the motion of pixels or of blocks, but not on regions or objects. They are not inherently scalable either. Like BM, OF has a short time dependence which is not very accurate for slow motion or trajectory estimation. These three characteristics are the strength of wavelet analysis, and we plan to show it further for pertinent extraction and recognition parameters in sequence analysis, compression and video data mining. Because of its compactness both in the direct space of positions and the dual space of frequencies, the Morlet wavelet (or DDM for Duval-Destin Murenzi [5]) was chosen to be tuned to speed. In Fourier space, this wavelet reads as

$$\hat{\psi}_M(\vec{k}) = \sqrt{\epsilon} \exp(-\frac{1}{2} |A^{-1}(\vec{k} - \vec{k}_0)|^2) + \text{corr}.,$$

where $A = \text{diag}[1, \epsilon^{-1/2}], \epsilon \geq 1$, and the correction term is usually dropped (see [1, Eq(3.18)]). Nevertheless numerous difficulties remain when using this wavelet in directional analysis. Although it has a good capability for directional filtering, its angular selectivity is poor. It is directional in the sense defined in [1, Sec.3.3], namely, “A wavelet $\psi$ is said to be directional if the effective support of its Fourier transform $\hat{\psi}$ is contained in a convex cone in spatial frequency space”, but the anisotropy parameter $\epsilon > 1$ is needed in order to get a decent angular selectivity. In addition, the Morlet wavelet has a major drawback: its angular selectivity increases with the length of the wave vector $k_0$, since the support cone gets narrower, but at the same time the amplitude decreases as $\exp(-|k_0|^2)$. On the other hand, conical and Cauchy wavelets are genuine directional wavelets and they don’t suffer from that defect. We have used these wavelets as a basis for a new construction of motion-tuned, and in particular speed-tuned wavelets. The development of these wavelets and their use in motion analysis is the aim of the present paper.

2. THE NEW MOTION-TUNED CONICAL WAVELET

2.1. The 2D Cauchy wavelet
The 2D Cauchy wavelet is a prototype of conical wavelet, which satisfies the definition of directional wavelet in a strict
sense, since its support is a convex cone $\mathcal{C}$ with apex at the origin [1], namely

$$\{ \vec{k} \in \mathbb{R}^2 | -\alpha \leq \arg \vec{k} \leq \alpha, \alpha < \pi/2 \}.$$ 

The Cauchy wavelet is given in Fourier space by

$$\hat{\psi}_{l,m}^{\mathcal{C}(-\alpha,\alpha)}(\vec{k}) = \begin{cases} (\vec{k} \cdot \vec{e}_\alpha)^l (\vec{k} \cdot \vec{e}_\alpha)^m e^{-\frac{\pi}{2} (\vec{k} \cdot \vec{e}_\alpha - \chi(\sigma))^2}, & \text{for } \vec{k} \in \mathcal{C}, \\ 0, & \text{otherwise} \end{cases} \quad (2.1)$$

with $\alpha$ the cone aperture, $\tilde{\alpha}$ the aperture of the dual cone, $\vec{e}_\alpha \equiv \vec{e}(\cos \tilde{\alpha}, \sin \tilde{\alpha})$, $\vec{k} \in \mathcal{C}(-\alpha,\alpha)$, $l$, $m$ the moments and $\tilde{\eta} \in \mathcal{C}(-\alpha,\alpha)$ the cone axis.

### 2.2. The 2D Gaussian-conical wavelet (GC)

By definition, the 2D Cauchy wavelet has the property that its opening angle $\alpha$ is *totally controllable* [1], independently of the amplitude, thus it avoids the drawback of Morlet mentioned above. Nevertheless, although it has a good *angular selectivity*, its *radial selectivity* is poor because the exponential term decays slowly as $|\vec{k}| \to \infty$. This is why this exponential is often replaced by a Gaussian along $k_x$, which concentrates the wavelet on its central frequency $(\sqrt{l+m},0)$ [6]. Then $\sigma > 0$ controls the scale localization of the Gaussian. A *center correction term*, $\chi(\sigma) = \sqrt{l+m} \frac{\sigma-1}{2}$, controls the radial support of $\psi$. This gives the expression of the GC wavelet, in frequency space [1, Eq.(3.37)]:

$$\hat{\psi}_{l,m}^{\text{GC}}(\vec{k}) = \begin{cases} (\vec{k} \cdot \vec{e}_\alpha)^l (\vec{k} \cdot \vec{e}_\alpha)^m e^{-\frac{\pi}{2} (\vec{k} \cdot \vec{e}_\alpha - \chi(\sigma))^2}, & \text{for } \vec{k} \in \mathcal{C}, \\ 0, & \text{otherwise}. \end{cases} \quad (2.2)$$

![Fig. 1. The 2D Gaussian Conical filter in the $(k_x, k_y)$ plane: the "shark" wavelet.](image-url)

### 2.3. Construction of the (new) spatio-temporal GCM

Building a wavelet for directional velocity analysis, in a sequence of images, starts by building a spatio-temporal 2D+T wavelet.

Because the 2D Gaussian conical filter has good capacities in spatial resolution and orientation, as well as radial selectivity, it has been chosen as the "support" for a velocity-tuned wavelet construction. This is why we keep this 2D conical term for the spatial 2D term of our 2D+T wavelet. For the time dimension we made the choice of a Morlet filter. The resulting 2D+T wavelet is constructed in a separable way: the velocity-tuned 2DT wavelet is simply the product of the 2D Gaussian conical with the 1D Morlet. We call the resulting wavelet "2D+T Gaussian Conical Morlet" (GCM). It is given in Fourier space as:

$$\hat{\psi}^{\text{GCM}}(\vec{k}) = \begin{cases} \hat{\psi}^{\text{GC}}(k_x,k_y) \cdot \hat{\psi}^{\text{M}}(\omega), & \text{2D Gaussian conical 1D Morlet}, \\ 0, & \text{otherwise}, \end{cases} \quad (2.3)$$

with $\hat{\psi}^{\text{M}}(\omega) = e^{-\frac{\omega}{2}(\omega-\omega_0)^2}$

### 2.4. Tuning the GCM wavelet to motions

Several motion operators can be applied to the “mother” wavelet, like scaling $D$, translation $T$, rotation $R$ and speed-tuning $\Lambda$:

$$\begin{align*}
\hat{D}_{a,s}^{x,a} \hat{\psi}(\vec{k},\omega) &= a_s a_t^{1/2} \hat{\psi}(a_s k_x, a_t \omega), \\
\hat{T}_{\vec{b},\tau} \hat{\psi}(\vec{k},\omega) &= e^{-i(\vec{k} \cdot \vec{b} + \omega \tau)} \hat{\psi}(\vec{k},\omega), \\
\hat{R}_q \hat{\psi}(\vec{k},\omega) &= \hat{\psi}(e^{i q \vec{k}}, e^{-q \omega}) \\
\hat{\Lambda}_{\theta,c} \hat{\psi}(\vec{k},\omega) &= \hat{\psi}(e^{i \frac{\omega}{2} \theta}, e^{-\frac{\omega}{2} c \omega}).
\end{align*} \quad (2.4)$$

Like for the speed-tuned Morlet wavelet, the application of the above operators to the GCM wavelet is done separately on either the spatial 2D conical or the temporal 1D Morlet, depending on the temporal or spatial nature of the transformation. The values of $p = 2/3$ and $q = 1/3$ align the deformation of the space-time domain with speed [5]. After implementation of the transforms defined in (2.4), we can give the expression of the 2D+T GCM in terms of the group parameters $g = \{a_s, a_t, \theta, c\}$:

$$\hat{\psi}_{l,m}^{\text{GCM}}(k_x, k_y, \omega) = \begin{cases} a_s^{-1} a_t^{1/2} (a_s r^{-\theta} k_{xy} \vec{e}_\alpha)^l (a_t r^{-\theta} k_{xy} \vec{e}_\alpha)^m e^{-\frac{\pi}{2} (a_s k_x - \chi(\sigma))^2} e^{-\frac{\pi}{2} (a_t \omega - \omega_0)^2}, & \text{for } \vec{k} \in \mathcal{C}(-\alpha,\alpha), \\ 0, & \text{otherwise}. \end{cases} \quad (2.5)$$

The 2D+T GCM wavelet tuned to speed is shown in the Fourier space $(k_x, k_y, \omega)$ on Fig. 2.

### 2.5. Central frequency correction

To facilitate the initial speed analysis, the wavelet can be centered in the Fourier plane. This is made possible with GCM. It is less convenient with Morlet whose aperture varies with $\omega_0$. Once “centered” in Fourier, GCM is no longer a wavelet, but a simple low-pass filter. Due to its aspect-ratio, this “GCM filter” can easily capture the correct signal spectrum inclination for a low-frequency band limited by the values of $a_s$ and $a_t$. 
For tuning to a specific scale, GCM must not be centered and will detect speed along the hyperboloid of speed-tunings (see Fig. 3). To center GCM, we first must perform an adaptive correction of the central frequency. The original central wave vector is $\vec{k}_0 = (\sqrt{l+m}, 0)$. With all motion transforms, the central frequency becomes:

$$\vec{k}_0 = \frac{1}{a_s a_t} \frac{\sqrt{l+m}}{c^2} (\cos \theta, \sin \theta)$$  \hspace{1cm} (2.6)

### 2.6. Analysis algorithm

The wavelet transform is computed by taking the product of the sequence and the wavelet FFTs, followed by an inverse FFT. This yields the wavelet transform of the sequence: $W_0(\vec{b}, N_i, \theta; a_s, a_t, c_j)$. We then sum, for a selected group $N_0$ of frames and for each speed tuning, the energy density $E_{tot}(c_j)$.

$$E_{tot}(c_j) = \sum_{i \in N_0} \sum_y \sum_x |W(x, y, N_i, c_j)|^2$$  \hspace{1cm} (2.7)

where $(x, y)$ are the pixel coordinates in a frame. The correct object speed $v_r$ is detected when the maximal energy is reached on the curve $E_{tot}(c_j)$. The algorithm complexity is the one of a 3D-CWT, with the remark that GCM coefficients matrix is sparse which enables additional savings.

### 3. EXPERIMENTATION

We first want to recall the results obtained in speed computation [3] between the Morlet speed-tuned CWT and the OF:

1. MTSTWT with 3 wavelets tuned to 3 speeds (3, 6, 10 pixels/fr) on a 360 $\times$ 240 $\times$ 8 sequence block (Tennis sequence) at the highest resolution of scale $t_{MTSTWT}(1200$ms $) + 3 \times$ IFFT$3D(3 \times 380$ms$) = 2.4$s.

2. Fast Optical Flow with wavelets [2] between two frames and 4 different resolutions $t_{OF} = 10$s.

We now perform a comparison between the 2D+T Morlet and the 2D+T GCM. We use a test sequence of $128 \times 128 \times 32$, that includes the motion, at constant speed, of a 2D non-symmetrical Gaussian. The angles of the Gaussian and of its trajectory are varied (along OX, at 45° and along OY). The first experiments have been done with a symmetrical Gaussian travelling at constant speed in the plane. We checked that speed capture is as good with GCM as with Morlet. But this is not the real interest of this study. We then modified the Gaussian to exhibit a strong anisotropy in the direction OX, OY or at 45° in the spatial plane. For this we gave it, for example, a large $\sigma_y$ (i.e., in the OY direction) and a small $\sigma_x$. This orientation will be the “signature” of the object and will enable to “capture” it in a sequence and to assign it its speed (Fig. 3). We also varied the orientation of the trajectory between $-\pi/2$ and $+\pi/2$, with respect to the OX axis in the spatial domain. For each of those orientations, we remind the rotation by $\pi/2$ of the spectrum in the dual, Fourier, domain. In the 3D Fig. 3, the sequence spectrum is oriented along OX and we show the wavelet family of speed-tuned Morlet and GCM wavelets tuned to speed. For Morlet, the wavelet elongates and moves along a “hyperbolic-like” curve with speed-tuning. We will show further that the poor angular selectivity, together with speed-tuning, will make the directional capture of the spectrum very difficult. GCM is centered in the middle of the Fourier space and exhibits a high angular selectivity. This makes it very performant in capturing the signal speed at any angle within a constant and narrow conical aperture $\alpha = \pi/16$ and not outside it. In Fig. 4 we show the orientations that Morlet and GCM can take for $0 \leq \theta \leq +\pi/2$. This figure speaks by itself. Morlet is shown with 3 orientations. For each one, we increase the couple $(k_0, \epsilon_y)$. We show 4 couples $k_0 = \{6, 12, 22, 35\}$ and $\epsilon_y = \{1, 2, 5, 13\}$. 
We can see that the angular selectivity is very hard to adjust. For weak values of $k_0$ and/or $\epsilon_y$, the aperture is very large. This aperture decreases by increasing $k_0$ and $\epsilon_y$. But as $k_0$ increases, Morlet moves away from the Fourier center $(0,0)$. With GCM the couple orientation/aperture is extremely simple to adjust. We show 5 orientations $\theta = \{0 \text{ to } \pi/2\}$ with GCM tuned to aperture $\alpha = \{\pi/256 \text{ to } \pi/16\}$. This proves the very good aperture selectivity of GCM. In Fig. 5, we plot the curve $v_m$ vs. $\theta_{wav}$ for $-\pi/2 \leq \theta \leq +\pi/2$, with GCM. The correct speed is captured when the wavelet orientation ($\theta = 0$) exactly corresponds to the spectrum orientation (OX). This proves the good angular selectivity of GCM, that could not be reached with Morlet, and its efficiency to detect the correct speed of the sequence in a very narrow angular aperture and not elsewhere.

![Fig. 4](image1.png)

**Fig. 4.** Comparison of angular selectivity between Morlet and GCM, for $0 \leq \theta \leq +\pi/2$: (a) Morlet is shown with 3 orientations and, for each one, we increase the couple $(k_0, \epsilon_y)$. We show 4 couples $k_0 = \{6, 12, 22, 35\}$ and $\epsilon_y = \{1, 2, 5, 13\}$. (b) We show 5 orientations $\theta = \{0 \text{ to } \pi/2\}$ with GCM tuned to apertures $\alpha = \{\pi/256 \text{ to } \pi/16\}$.

![Fig. 5](image2.png)

**Fig. 5.** Result: Performances of GCM in directional speed capture: we plot the curve $v_m$ vs. $\theta_{wav}$ for $-\pi/2 \leq \theta \leq +\pi/2$ (see Fig. 4b) for an aperture $\alpha = \pi/16$. The correct speed is captured when the wavelet orientation ($\theta = 0$) exactly corresponds to the spectrum orientation (OX).

4. CONCLUSION

We have presented a new tool based on the spatio-temporal CWT. The multiresolution ME framework obtained extends a former one based on the Morlet wavelet. The CWT outperforms spatial-only methods like curvelets [4] as well as spatio-temporal “two-frame-at-a-time” methods like optical flow (OF) [2], block matching (BM) and phase difference. The advantage of CWT over two-frame methods, namely robustness to occlusions and to object signature variation and long temporal dependence, has already been proved in [7, 1] even in the case of complex motions. Our contribution essentially lies in the fact that we have replaced the 2D+T Morlet wavelet by genuine directional filters, namely, the Cauchy conical wavelets, building upon former works on wavelet speed tuning and spatial conical waves. Combining a 1D Morlet wavelet for the temporal dependence with a 2D Gaussian conical wavelet for the spatial dependence, we have obtained a new spatio-temporal wavelet, tuned to velocity and other motions (scaling and rotation), which yields a real scalable, directional and kinematical analysis. The outcome is a new sophisticated and efficient tool. The next steps should involve, e.g., accurately comparing the spatial performances of 2D GCM with curvelets, comparing the spatio-temporal performances of 2D+T GCM with OF and other speed quantization tools, and studying the performances in real image sequences. This will be the subject of our future work.

5. REFERENCES


