Solving Singular Initial Value Problems of Emden-Fowler and Lane-Emden Type

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Abstract:
In this paper, Singular initial value problems are investigated by using Taylor series method. The solutions are constructed in the form of a convergent series. The method is applied to Emden-Fowler and Lane-Emden equations. Keywords: Taylor series method, Emden-Fowler equation, Lane-Emden equation.

1 Introduction

Recently, a lot of attention has been focused on the study of singular initial value problems (IVPs) in the second order ordinary differential equations (ODEs). Many problems in mathematical physics and astrophysics can be modeled by the so-called initial value problems (IVPs) of Emden-Fowler type equation, cf [1,2]

\[ y'' + \frac{2}{x} y' + a f(x) g(y) = 0, \quad 0 < x \leq 1 \]  \hspace{1cm} (1.1)

subject to the conditions

\[ y(0) = A, \quad y'(0) = B \]  \hspace{1cm} (1.2)

On the other hand, studies have been carried out on another class of singular initial value problems of the form

\[ y'' + \frac{2}{x} y' + a f(x, y) = g(x), \quad 0 < x \leq 1 \]  \hspace{1cm} (1.3)

subject to the conditions

\[ y(0) = A, \quad y'(0) = B \]  \hspace{1cm} (1.4)
where $A$ and $B$ are constants, when $f(x) = 1$ and $a = 1$, Eq. (1.1) reduces to the Lane-Emden type equation, which with specializing $g(y)$ was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents Chandrasekhar [1] and Davis [2]. Due to the significant applications of Lane-Emden type equations in the scientific community various forms of $g(y)$ have been investigated in many research works. A discussion of the formulation of these models and the physical structure of the solutions can be found in Chandrasekhar [1], Davis [2], Shawagfeh [3], Adomian [4] and Wazwaz [5]. Most algorithms currently in use for handling the Lane-Emden type and Emden-Fowler type problems are based on either series solutions of perturbation techniques. Wazwaz [5] has given a general study to construct exact and series solution to Lane-Emden and Emden-Fowler type problems by employing the Adomian decomposition method. Moreover, a generalization was developed in Wazwaz [5] by replacing the coefficient $\frac{2}{x}$ of $y'$ by $\frac{n}{x}$.

Equation (1.3), has attracted many mathematicians and has been studied from various points of view. Russell and Shampine [6] have investigated (1.3) for the linear function. Three-point difference methods of second order have been used in Russell and Shampine [6]. Moreover, three-point difference method of second order have been also used by Chawla and Katti [7], Chawla et al. [8] and Iyenhgar and Jain [9]. However, Jain and Jain [10] derived three-point difference method of four and six order to solve this problem. Recently, El-sayed [11] used a multi-integral method to investigate the nonlinear problem (1.3) with two boundary conditions.

The Eq.(1.3) cannot have a Taylor series expansion directly over the interval in which a solution is desired. For example, if

$$y'' + \frac{2}{x}y' + y = 0,$$

then $y''$ and higher derivatives do not exist at $x = 0$. It is the aim of this study to study the singular problem (1.3), with initial condition and to make further progress beyond the achievements made so far in this regard. Further applying the proposed algorithm in a generalization of this type of problems. A difficult element in the analysis of this type of equations is the singularity behavior that occurs at $x = 0$.

2 Basic idea of The Method

The Method [12], is a novel and effective method which can solve various nonlinear equations. In theory, the infinite Taylor series can be used to evaluate a function, given its derivative function and its value at some point, consider the nonlinear first-order ODE:

$$y'' + \frac{n}{x}y' + a f(x, y) = g(x), \quad n \in \mathbb{R}$$

The Taylor series for $y(x)$ at $x = x_0$ is

$$y(x) = y(x - x_0) + y'(x - x_0) x + \frac{y''(x - x_0)}{2!} x^2 + ...$$

We solve the Eq.(1.2) by the following steps:
From Eq.(1.2), we have

\[ y' = \frac{1}{2} (xg(x)) - \frac{1}{2} (axf(x, y)) - \frac{1}{2} xy'' \]

or \[ y'' = \frac{1}{3} (xg(x))' - \frac{1}{3} (axf(x, y))' - \frac{1}{3} xy''' \]

or \[ y''' = \frac{1}{4} (xg(x))'' - \frac{1}{4} (axf(x, y))'' - \frac{1}{3} xy^{(4)} \]

\[ \vdots = \ldots \]

\[ y^{(k)} = \frac{1}{k + 1} \left[ (xg(x))^{(k-1)} - (axf(x, y))^{(k-1)} - xy^{(k+1)} \right] \quad k = 1, 2, 3, \ldots \]

Hence, by using \( y(0) = A, y'(0) = B \), we can find \( y''(0), y'''(0), \ldots, y^{(k)}(0) \).

By Taylor series method with \( x = 0 \), we have

\[ y(x) = y(0) + y'(0) x + \frac{y''(0)}{2!} x^2 + \ldots \]

substituting the initial condition and the values find, we obtain the solution.

3 Generalization of The Method

A generalization of the Lane-Emden type equation has been studied by Wazwaz [5]. We replace the standard coefficient of \( y'(x) \) by \( \frac{n}{x} \), for real \( n \). In other words, a general equation,

\[ y'' + \frac{n}{x} y' + a f(x, y) = g(x), \quad n, a \in \mathbb{R} \]  \hspace{1cm} (3.1)

subject to the conditions

\[ y(0) = A, y'(0) = B \]  \hspace{1cm} (3.2)

can be formulated. From Eq.(3.1),

\[ y' = \frac{1}{2} (xg(x)) - \frac{1}{2} (axf(x, y)) - \frac{1}{2} xy'' \]

\[ y'' = \frac{1}{n + 1} \left[ (xg(x))' - (axf(x, y))' - xy''' \right] \]

\[ y''' = \frac{1}{n + 2} \left[ (xg(x))'' - (axf(x, y))'' - xy^{(4)} \right] \]

\[ \vdots = \ldots \]

\[ y^{(k)} = \frac{1}{n + (k-1)} \left[ (xg(x))^{(k-1)} - (axf(x, y))^{(k-1)} - xy^{(k+1)} \right] \quad k = 1, 2, 3, \ldots \]

Hence, by using \( y(0) = A, y'(0) = B \), we can find \( y''(0), y'''(0), \ldots, y^{(k)}(0) \).

By Taylor series method with \( x = 0 \), we have

\[ y(x) = y(0) + y'(0) x + \frac{y''(0)}{2!} x^2 + \ldots \]

substituting the initial condition and the values find, we obtain the solution.

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4 Application of the Method

In order to assess both the applicability and the accuracy of the method, we apply the algorithm to Emden-Fowler and Lane-Emden type equations as follows.

4.1 Example 1

First we consider following Lane-Emden equation [13],

\[ y'' + \frac{2}{x} y' + y = x^5 + 30 x^3, \quad 0 < x \leq 1 \tag{4.1} \]

subject to the boundary conditions

\[ y(0) = 0, \quad y'(0) = 0 \tag{4.2} \]

From Eq.(4.1), we have

\[
\begin{align*}
    y' &= \frac{1}{2} [x^6 + 30 x^4] - \frac{1}{2} x y - \frac{1}{2} x y'' \\
    y'' &= \frac{1}{3} [6 x^5 + 120 x^3] - \frac{1}{3} [x y' + y] - \frac{1}{3} x y'' \\
    y''' &= \frac{1}{4} [30 x^4 + 360 x^2] - \frac{1}{4} [x y'' + 2 y'] - \frac{1}{4} x y^{(4)} \\
    y^{(4)} &= \frac{1}{5} [120 x^3 + 720 x] - \frac{1}{5} [x y''' + 3 y'] - \frac{1}{5} x y^{(5)} \\
    y^{(5)} &= \frac{1}{6} [360 x^2 + 720] - \frac{1}{6} [x y^{(4)} + 4 y''] - \frac{1}{6} x y^{(6)} \\
    y^{(6)} &= \frac{1}{7} [720 x] - \frac{1}{7} [x y^{(5)} + 5 y^{(4)}] - \frac{1}{7} x y^{(7)} \\
    y^{(7)} &= \frac{1}{8} [720] - \frac{1}{8} [x y^{(6)} + 6 y^{(5)}] - \frac{1}{8} x y^{(8)}
\end{align*}
\]

using (4.2) with \( x = 0 \), we find

\[ y''(0) = 0, \quad y'''(0) = 0, \quad y^{(4)}(0) = 0, \quad y^{(5)}(0) = 120, \quad y^{(6)}(0) = 0, \quad y^{(7)}(0) = 0, \ldots \]

by Taylor series method with \( x = 0 \) we have

\[ y(x) = y(0) + y'(0) x + \frac{y''(0)}{2!} x^2 + \ldots \]

substituting the initial condition and the values find, we obtain the closed form solution,

\[ y(x) = x^5 \]

as obtained by Erturk [13] by DT.
### 4.2 Example 2

Now we consider the following nonlinear, homogeneous Emden-Fowler type equation [14,15]

\[ y'' + \frac{8}{x} y' + 18 a y + 4 a y \log y = 0 \quad (4.3) \]

subject to the boundary conditions

\[ y(0) = 1, \quad y'(0) = 0 \quad (4.4) \]

From Eq.(4.3), we have

\[
\begin{align*}
y' &= -\frac{a}{8} [18 y x + 4 x y \log y - \frac{1}{8} x y''] \\
y'' &= -\frac{a}{9} [18 y x + 22 x y' + 4 y \log y + 4 x \log y y''] - \frac{1}{9} x y''' \\
y''' &= -\frac{a}{10} [4 y' + 22 x y'' + 8 y' \log y + 4 x \log y y'' + 4 x y (y')^2] - \frac{1}{10} x y^{(4)}
\end{align*}
\]

using (4.4) with \( x = 0 \), we find

\[
\begin{align*}
y''(0) &= -2a, \quad y'''(0) = 0, \quad y^{(4)}(0) = 12a^2, \quad y^{(5)}(0) = 0, \quad y^{(6)}(0) = -120a^3, \quad y^{(7)}(0) = 0, ...
\end{align*}
\]

by Taylor series method with \( x = 0 \) we have

\[ y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!} x^2 + ... \]

substituting the initial condition and the values find, we obtain the closed-form solution,

\[ y(x) = e^{-a x^2}. \]

as obtained by Chowdhury [14] by HPM and Wazwaz [15] by ADM.

### 4.3 Example 3

Finally we consider the nonlinear, homogeneous Emden-Fowler type equation [14,15]

\[ y'' + \frac{5}{x} y' + 8a (e^y + 2 e^{y/2}) = 0 \quad (4.5) \]

subject to the boundary conditions

\[ y(0) = 0, \quad y'(0) = 0 \quad (4.6) \]

From Eq.(4.5), we have

\[
\begin{align*}
y' &= -\frac{8a}{5} [x (e^y + 2 e^{y/2})] - \frac{1}{5} x y'' \\
y'' &= -\frac{8a}{6} [ (e^y + 2 e^{y/2}) + x (e^y + e^{y/2}) y'] - \frac{1}{6} x y''
\end{align*}
\]
\[ y''' = -\frac{8a}{7} \left[ 2(e^y + e^{y/2})y' + x(e^y + e^{y/2})y'' + x(e^y + \frac{1}{2} e^{y/2}) (y')^2 \right] - \frac{1}{7} x y^{(4)} \]

... = ...

using (4.4) with \( x = 0 \), we find

\[ y''(0) = -4a, \ y'''(0) = 0, \ y^{(4)}(0) = 24a^2, \ y^{(5)}(0) = 0, \ y^{(6)}(0) = -480a^3, \ y^{(7)}(0) = 0, \ldots \]

by Taylor series method with \( x = 0 \) we have

\[ y(x) = y(0) + y'(0) x + \frac{y''(0)}{2!} x^2 + \ldots \]

substituting the initial condition and the values find, we obtain the closed-form solution,

\[ y(x) = -2 \ln(1 + ax^2). \]

as obtained by Chowdhury [14] by HPM and Wazwaz [15] by ADM.

5 Conclusions

In this paper, it was shown that, with the proper use of the taylor series method, it is possible to obtain an analytic solution to a class of both linear and nonlinear singular IVPs. Some examples are given to illustrate the validity and accuracy of this procedure. In example 1 we obtained the exact solution same as Erturk [13] and examples 2 and 3, we found the same closed form solutions by Chowdhury [14] by HPM and Wazwaz [15] by ADM. Based on the cases investigated in this task, the present technique is simple, more efficient and reliable. The difficulty in this type of equations, due to the existence of singular point at \( x = 0 \), is overcome here. Our goal has been achieved by formally deriving analytical solution with a high degree of accuracy. The computational work has been reduced compared to other existing techniques. Finally, we conclude that our method is a promising tool for both linear and nonlinear singular IVPs.

References


