Multiple solutions for a discrete boundary value problem involving the \( p \)-Laplacian

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Abstract

Multiple solutions for a discrete boundary value problem involving the \( p \)-Laplacian are established. Our approach is based on critical point theory.

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1. Introduction

The aim of this paper is to look for the existence of multiple solutions to the following problem

\[ (P_\lambda) \begin{cases} -\Delta (\phi_p (\Delta u(k-1))) = \lambda f(k, u(k)), & k \in [1, T], \\ u(0) = u(T + 1) = 0, \end{cases} \]

where, \( T \) is a fixed positive integer, \( [1, T] \) is the discrete interval \( \{1, \ldots, T\} \), \( \lambda \) is a positive real parameter, \( \Delta u(k) = u(k+1) - u(k) \) is the forward difference operator, \( \phi_p(s) = |s|^{p-2}s \), \( 1 < p < +\infty \) and \( f : [1, T] \times \mathbb{R} \to \mathbb{R} \) is a continuous function.

Our approach is based on the variational framework developed, in [1], by R. P. Agarval, K. Perera and D. O’Regan to study problem \((P_1)\); for \( p = 2 \), see also [2]. While the authors of the papers mentioned focus on the nonlinearity \( f \), here we work with its primitive \( F(\cdot, t) = \int_0^t f(\cdot, \xi)\,d\xi \). More precisely, in order to establish the existence of at least three solutions to \((P_\lambda)\) \((\text{Theorems 3.1 and 3.3})\), we point out a suitable relationship between the behavior of \( F \) with a precise bounded interval of parameters \( \lambda \). The existence of nonnegative solutions \((\text{Theorem 3.2})\) is chiefly obtained by using a useful consequence of the strong comparison principle given in [1] \((\text{Lemma 2.1})\). However, we also obtain at least two positive solutions \((\text{Theorem 3.4})\) even if \( \lambda \) belongs to a precise half-line and \( f \) chances sign. To achieve our goal, we give a finite dimensional version of Theorem 2.1 of [3], which is our main tool for investigating \((P_\lambda)\).

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(Theorem 2.1). For more details on the subject treated here we refer the reader to [4,5] and the references given therein. For completeness, we also mention closely related results given in [6] and in [4] (for $p = 2$) where the existence of at least one solution to $(P_\lambda)$, for each $\lambda$ lying in a suitable interval, has been proved by fixed point theory. Section 2 is devoted to auxiliary results and variational framework. The main results are contained in Section 3.

2. Auxiliary results and variational framework

Let $X$ be a finite dimensional real Banach space and let $J_\lambda : X \to \mathbb{R}$ be a functional satisfying the following structural hypothesis:

(A) $J_\lambda(u) := \Phi(u) + \lambda \Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functionals of class $C^1$ on $X$ with $\Phi$ coercive, i.e. $\lim_{\|u\| \to \infty} \Phi(u) = \infty$, and $\lambda$ is a positive parameter.

Further, for each $r > \inf_X \Phi$, put

$$\varphi_1(r) := \inf_{u \in \Phi^{-1}([-\infty, r])} \frac{\Psi(u) - \inf \Phi^{-1}([-\infty, r])}{r - \Phi(u)} ,$$

$$\varphi_2(r) := \inf_{u \in \Phi^{-1}([-\infty, r])} \sup_{v \in \Phi^{-1}(r, +\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)} ,$$

an immediately consequence of Theorem 2.1 of [3] is as follows.

**Theorem 2.1.** Assume that:

(a1) there exist $r > \inf_X \Phi$ such that $\varphi_1(r) < \varphi_2(r)$;

(a2) for each $\lambda \in [1/\varphi_2(r), 1/\varphi_1(r)]$ one has $\lim_{\|u\| \to \infty} J_\lambda(u) = +\infty$.

Then, for each $\lambda \in [1/\varphi_2(r), 1/\varphi_1(r)]$, $J_\lambda$ has at least three critical points.

For the reader’s convenience we recall a consequence of strong comparison principle [2, Lemma 2.3] which we will use in the sequel in obtaining nonnegative as well as positive solutions to $(P_\lambda)$, i.e. $u(k) > 0$ for each $k \in [1, T]$.

**Lemma 2.1.** If

$$-\Delta(\phi_p(\Delta u(k-1))) \geq 0 , \quad k \in [1, T] ,$$

$$u(0) \geq 0 , \quad u(T+1) \geq 0 ,$$

then either $u$ is positive or $u \equiv 0$.

Finally, in order to give the variational formulation of problem $(P_\lambda)$, on the $T$-dimensional Banach space

$$W := \{u : [0, T+1] \to \mathbb{R} : u(0) = u(T+1) = 0\},$$

equipped with the norm

$$\|u\| := \left( \sum_{k=1}^{T+1} |\Delta u(k-1)|^p \right)^{1/p} ,$$

we define the functional $J_\lambda : W \to \mathbb{R}$ by putting, for every $u \in W$,

$$J_\lambda(u) := \sum_{k=1}^{T+1} \left[ \frac{1}{p} |\Delta u(k-1)|^p - \lambda F(k,u(k)) \right] ,$$

where, $F(k,t) := \int_0^t f(k, \xi) d\xi$ for every $(k,t) \in [1, T] \times \mathbb{R}$. An easy computation ensures that $J_\lambda$ turns out to be of class $C^1$ on $W$ with

$$J'_\lambda(u)(v) = \sum_{k=1}^{T+1} \left[ \phi_p(\Delta u(k-1)) \Delta v(k-1) - \lambda f(k,u(k)) v(k) \right] , \quad v \in W .$$
Therefore, taking into account that, for every \( u, v \in W \), we have

\[
- \sum_{k=1}^{T+1} \Delta(\phi_p(\Delta u(k - 1)))v(k) = \sum_{k=1}^{T+1} \phi_p(\Delta u(k - 1)) \Delta v(k - 1).
\]

It is clear that the critical points of \( J_k \) are exactly the solutions of problem \( (P_\lambda) \).

### 3. Results

Let \( c \) and \( d \) be two positive constants, we write

\[
\Theta(c) := \frac{\sum_{k=1}^{T} \sup_{|\xi| \leq c} F(k, \xi)}{c^p} \quad \text{and} \quad \Gamma(d) := \frac{\sum_{k=1}^{T} \left[ F(k, d) - \sup_{|\xi| \leq c} F(k, \xi) \right]}{d^p}.
\]

We now give the following theorem.

**Theorem 3.1.** Assume that there exist four positive constants \( a, c, d \) and \( s \) with \( c < d \) and \( s < p \) such that

1. \( \Theta(c) < \left( \frac{2^2}{T + 1} \right)^{p-1} \Gamma(d); \)
2. \( F(k, \xi) \leq a(1 + |\xi|^p) \) for all \( (k, \xi) \in [1, T] \times \mathbb{R} \).

Then, for every \( \lambda \in \left[ \frac{2}{pT + 1}, \frac{2p}{p \Theta(c)(T + 1)^{p-1}} \right] \), problem \( (P_\lambda) \) admits at least three solutions.

**Proof.** In applying Theorem 2.1, choice \( X = W \), for every \( u \in W \), put

\[
\Phi(u) := \sum_{k=1}^{T+1} \left( \frac{1}{p} |\Delta u(k - 1)|^p \right), \quad \Psi(u) := -\sum_{k=1}^{T} F(k, u(k)),
\]

and, for each \( \lambda > 0 \), \( J_k(u) := \Phi(u) + \lambda \Psi(u) \). Clearly, \( J_k \) satisfies condition \( (A) \) and taking,

\[
r = \frac{(2c)^p}{p(T + 1)^{p-1}},
\]

we claim that \( \varphi_1(r) < \varphi_2(r) \). To this end, we observe that for every \( u \in W \), there exists \( j \in [1, T] \) such that \( u(j) = \max_{k \in [1, T]} |u(k)| \). Therefore, taking in account that \( u(0) = u(T + 1) = 0 \), an easy computation ensures that

\[
u(j) \leq \frac{1}{2} \sum_{k=1}^{T+1} |u(k + 1) - u(k)|,
\]

and by using the discrete Hölder inequality, one has

\[
\max_{k \in [1, T]} |u(k)| \leq \frac{(T + 1)^{(p-1)/p}}{2} \|u\|. \tag{2}
\]

From this follows that

\[
\varphi_1(r) = \inf_{\|u\| < (pr)^{1/p}} \frac{\Psi(u) - \inf_{\|u\| \leq (pr)^{1/p}} \Psi}{r - \|u\|^p} \leq \inf_{\|u\| \leq (pr)^{1/p}} \frac{\Psi}{r} \leq \frac{(T + 1)^{p-1}}{2p} \Theta(c).
\]

Further, since one has \( c < \left( \frac{T + 1}{2} \right)^{(p-1)/p} \), it results that \( \|v\| > (pr)^{1/p} \) where,

\[
v(k) = \begin{cases} 
   d & \text{if } k \in [1, T], \\
   0 & \text{otherwise.}
\end{cases}
\]
Hence, bearing in mind that \( I'(d) > 0 \), by (b1) and (2), we get

\[
\varphi_2(r) \geq p \inf_{|u| < (pr)^{1/p}} \frac{\sum_{k=1}^{T} F(k, d) - \sum_{k=1}^{T} \sup_{|\xi| \leq c} F(k, \xi)}{2d^p - \|u\|^p} > \frac{p(T + 1)^{p-1}}{2^p} \Theta(c).
\]

(4)

Combining (3) and (4), it is clear that the above claim is proved. Now, owing to (b2) and again by (2), for every \( u \in W \) and \( \lambda > 0 \), we have

\[
J_\lambda(u) \geq \frac{\|u\|^p}{p} - \lambda \sum_{k=1}^{T} a(1 + |u(k)|^s)
\geq \frac{\|u\|^p}{p} - a\lambda T \left[ \frac{(T + 1)^{(p-1)/p}}{2} \right] \|u\|^s - a\lambda T,
\]

which clearly ensures that \( J_\lambda \) turns out to be coercive. So, the assumptions of Theorem 2.1 are satisfied and our conclusion follows. □

Remark 3.1. In many situations it is also important to obtain at least one solutions to \((P_\lambda)\), see for instance Theorem 1.1 of [1]. In this order of ideas, a careful reading of the above proof reveals that condition (b2) ensures the existence of at least one solution to \((P_\lambda)\) for every \( \lambda > 0 \). Whereas, for \( c > 0 \), arguing again as above, but taking into account the proof of Theorem 2.1, we can show that, for every \( \lambda \in \left[ 0, \frac{2^p}{p\Theta(c)(T+1)^{p-1}} \right] \), the same conclusion still holds without any additional assumption.

Let \( h : \mathbb{R} \to \mathbb{R}, q : [1, T] \to \mathbb{R} \) be two nonnegative functions with \( Q := \sum_{k=1}^{T} q(k) > 0 \). Put, for every \( t \in \mathbb{R} \), \( H(t) := \int_{0}^{t} h(\xi)d\xi \), a simple consequence of the previous result is the following

**Theorem 3.2.** Assume that there exist four positive constants \( \rho, c, d \) and \( s \) with \( c < d < s < p \) such that

(b3) \( \frac{H(c)}{c^p} < \frac{2^{p-1}}{(T+1)^{p-1}} \frac{H(d)}{d^p} \),

(b2) \( H(t) \leq \rho (1 + |t|^s) \), for all \( t \in \mathbb{R} \).

Then, for each \( \lambda \in \left[ 0, \frac{2^p}{pQ(T+1)^{p-1}} \right] \), the problem

\[
\begin{align*}
- \Delta(f_{\rho}(\Delta u(k-1))) &= \lambda q(k)h(u(k)), & k \in [1, T], \\
u(0) &= u(T + 1) = 0,
\end{align*}
\]

admits at least three nonnegative solutions.

**Proof.** Taking into account Lemma 2.1 as well as that \((b3)\) implies

\[
\frac{H(c)}{c^p} < \left( \frac{2}{T + 1} \right)^{p-1} \frac{H(d) - H(c)}{d^p}.
\]

We see at once that our conclusion follows from Theorem 3.1 by choosing \( f(k, t) = q(k)h(t) \) for each \((k, t) \in [1, T] \times \mathbb{R} \) and \( a = \rho Q \). □

**Remark 3.2.** We explicitly observe that, if \( h(0) > 0 \), then in Theorem 3.2 we can replace the word nonnegative with positive.

**Example 3.1.** Write, for each \( k \in [1, T] \), \( q(k) = k \) and let \( h : \mathbb{R} \to \mathbb{R} \) be defined by putting

\[
h(t) = \begin{cases} 
  e^t, & \text{if } t \leq 12; \\
  e^{12}, & \text{if } t > 12.
\end{cases}
\]
By choosing for instance $\rho = e^{12}$, $c = 1$, $d = 12$, $s = 1$ and $p = 3$, the assumptions of Theorem 3.2 are satisfied. Therefore, for each $\lambda \in \left[164 \cdot 10^6, 344 \cdot 10^6 \right]$, the problem
\[
\begin{align*}
- \Delta (|\Delta u(k - 1)| \Delta u(k - 1)) &= \lambda k h(u(k)), \quad k \in [1, T], \\
u(0) &= u(10) = 0,
\end{align*}
\]
has at least three positive solutions.

Now we discuss the case $s = p$ inside the growth condition $(b_2)$.

**Theorem 3.3.** Assume that there exist three positive constants $a$, $c$, $d$ with $c < d$ such that $(b_1)$ holds and in addition suppose that

$(b_4)$ \quad $F(k, \xi) \leq a(1 + |\xi|^p)$ for all $(k, \xi) \in [1, T] \times \mathbb{R}$, with $a < \frac{2^{p-1}}{T(T+1)^{p-1}} \Gamma(d)$.

Then, for every $\lambda \in \left[\frac{T^2}{pT(aT)} \cdot \frac{2^p}{p(T+1)^{p-1}} \min \left\{ \frac{1}{c}, \frac{1}{aT} \right\} \right]$, problem $(P_\lambda)$ admits at least three solutions.

**Proof.** Arguing as in the proof of Theorem 3.1, it is clear that our conclusion follows again by Theorem 2.1 if we show that $J_\lambda$ turns out to be coercive. Indeed, keeping $\lambda$ fixed as above, since for every $u \in W$, by (2), we get
\[
|u(k)|^p \leq \frac{(T + 1)^{(p-1)}}{2} \|u\|^p, \quad \forall k \in [1, T],
\]

$(b_4)$ ensures that
\[
J_\lambda(u) \geq \left( \frac{1}{p} - \frac{aT(T + 1)^{p-1}}{2^p} \right) \|u\|^p - \lambda a T.
\]
Thus, begin $\lambda < \frac{2^p}{aT(T+1)^{p-1}T}$, our claim holds and the proof is completed. \hfill $\square$

**Remark 3.3.** Evidently, arguing as in the above proof, we get at least one solution to $(P_1)$ if $(b_4)$ holds with $a < \frac{2^p}{pT(T+1)^{p-1}T}$.

**Theorem 3.4.** Assume that there exist four positive constants $a$, $c$, $d$ and $s$ with $c < d$ and $s < p$ such that $(b_2)$ holds and moreover, suppose that

$(b_5)$ \quad $\max_{|\xi| \leq c} F(k, \xi) \leq 0$ for all $k \in [1, T]$;
$(b_6)$ \quad there exists $\bar{k} \in [1, T]$ such that $\int_0^d f(\bar{k}, \xi) d\xi > 0$.

Then, for every $\lambda \in \left[\frac{2d^p}{p \int_0^d f(\bar{k}, \xi) d\xi}, +\infty \right]$, problem $(P_\lambda)$ admits at least two positive solutions.

**Proof.** In applying Theorem 2.1, let $J_\lambda$ be as above and $r = \frac{(2c)^p}{p(T+1)^{p-1}}$. For every $u \in W$ with $\|u\| \leq (pr)^{1/p}$, by $(b_5)$ one has that $\psi(u) \geq 0$. From this, it follows that
\[
\inf_{\|u\| \leq (pr)^{1/p}} \psi(u) = \psi(0) = 0,
\]
which furnishes $\varphi_1(r) = 0$. On the other hand it is easy to see that by setting
\[
w(k) = \begin{cases} d & \text{if } k = \bar{k}; \\ 0 & \text{otherwise}, \end{cases}
\]
one has $\|w\| \geq (pr)^{1/p}$. Hence, by $(b_6)$, we get
\[
\varphi_2(r) \geq p \inf_{\|u\| < (pr)^{1/p}} \frac{\int_0^d f(\bar{k}, \xi) d\xi - \sum_{k=1}^T \sup_{|\xi| \leq c} F(k, \xi)}{2d^p} \geq p \frac{\int_0^d f(\bar{k}, \xi) d\xi}{2d^p} > 0.
\]
So, since (b5) implies $f(k, 0) = 0$, for every $k \in [1, T]$, Theorem 2.1 furnishes at least two nontrivial solutions, say $u_1$ and $u_2$, to $(P_\lambda)$. Moreover, a simple computation shows that $u_1$ and $u_2$ turn out to be also two solutions of the following problem

\[
\begin{cases}
-\Delta(\phi_p(\Delta u(k - 1))) = \lambda \hat{f}(k, u(k)), & k \in [1, T], \\
u(0) = u(T + 1) = 0,
\end{cases}
\]

where, $\hat{f} : [1, T] \times \mathbb{R} \to \mathbb{R}$ is defined by putting

\[
\hat{f}(k, t) = \begin{cases}
f(k, t) & \text{if } t \geq 0; \\
0 & \text{otherwise}.
\end{cases}
\]

We claim that $u_1$ and $u_2$ turn out to be positive. By contradiction, suppose that at least one, said $u_1$, is non-positive. Thus, we have that $u_1$ fulfills the following conditions

\[
-\Delta(\phi_p(\Delta u_1(k - 1))) = 0, \quad k \in [1, T], \quad u_1(0) = u_1(T + 1) = 0.
\]

On the other hand, bearing in mind that $u_1$ is nontrivial and also Lemma 2.1, one has that $u_1$ is positive, which clearly is a contradiction. □

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References


