Abstract—This paper proposes an efficient sub-optimal MIMO linear precoder based on the maximization of minimum distance for three virtual subchannels. A new virtual MIMO channel representation with two channel angles allows the parameterization of the linear precoder and the optimization of the distance between signal points at the received constellation. To illustrate the optimization process, a precoder is derived for BPSK and QPSK modulation following the max-SNR approach, which consists in pouring power only on the most favored virtual sub-channel. Simulation results over a Rayleigh channel confirm the interest of this new precoder in terms of bit-error-rate.

I. INTRODUCTION

Over the past few years, the Multiple Input Multiple Output systems using the Orthogonal Frequency Division Multiplex (OFDM) techniques have been widely used in the mobile broadband wireless links because of the significant improvement in spectral efficiency, capacity and data-rate of transmissions in comparison with single antenna or single carrier systems. Through a feedback link, the channel knowledge can be available at the transmitter and precoding techniques can be used to adapt the information signal to the channel and significantly improve the performance of MIMO communications.

If full channel state information is considered at the transmitters (CSI), linear precoders can be designed according to various criteria [1], such as ergodic capacity, received signal-to-noise ratio (SNR) or error probability. By decoupling the MIMO channels into independent and parallel data-streams, the precoder based on maximization of the minimum Euclidean distance (max-$d_{\text{min}}$) gains the diversity orders and performs a significant enhancement in term of bit-error-rate (BER). The optimal precoder was derived for two virtual subchannels using BPSK and QPSK modulations in [2], and a cross-form matrix was derived in [3] to get the max-$d_{\text{min}}$ precoder for any even number of data-stream. An extension to 16-QAM modulation can also be found in [4] that confirm the optimality of the max-$d_{\text{min}}$ criterion. Authors in [5] proposed another approach with lower complexity to reach similar results for BPSK and QPSK modulations.

Introducing a new representation of the virtual diagonal channel with two channel angles, the present paper proposes to adapt the optimization of the minimum distance to three virtual subchannels. As the number of parameters to optimize $d_{\text{min}}$ is too large, a simplified version of the precoder, inspired from the maximization of the Signal-to-Noise Ratio (max-SNR) is designed. This solution consists in retaining the most favored sub-channel and is denoted as SNR-like precoder in the rest of this paper. In fact, the precoder transforms the M-QAM signals on three sub-channels into an M$^3$-QAM modulation on the strongest virtual sub-channel.

Simulation results for BPSK and QPSK modulations over a Rayleigh channel prove the efficiency of the precoder in terms of BER compared to the max-SNR precoder and diagonal precoders in the literature, such as max-$\lambda_{\text{min}}$ [6], water-filling [7], MMSE [8] precoders.

The rest of the paper is organized as follows. The next Section is devoted to the virtual MIMO channel representation and to the minimum distance criterion. The new parameterized form of the precoder for three virtual subchannels is introduced in Section III and the optimization of the $d_{\text{min}}$ criterion with the SNR-like approach is detailed in Section IV. Section V proposes BER simulation results over a Rayleigh channel and comparisons to existing precoders, while conclusions are given in Section VI.

II. OPTIMIZATION OF MINIMUM EUCLIDEAN DISTANCE

A. MIMO channel representation

For a MIMO system with $n_R$ receive, $n_T$ transmit antennas and $b$ independent data stream, the basic system model can be expressed as:

$$\mathbf{y} = \mathbf{GHF}s + \mathbf{G}\nu$$  \hspace{1cm} (1)

where $\mathbf{H}$ is the $n_R \times n_T$ channel matrix, $\mathbf{F}$ is the $n_T \times b$ precoder matrix, $\mathbf{G}$ is the $b \times n_R$ decoder matrix, $s$ is the $b \times 1$ transmitted vector symbol, and $\nu$ is the $n_R \times 1$ additive noise vector.

When the transmitter and receiver have the perfect channel state information (CSI), it was shown that the diagonalized channel matrix and whitened noise can be obtained by using a virtual transformation [2]. The approach is based on the following decompositions $\mathbf{F} = \mathbf{F}_v\mathbf{F}_d$ and $\mathbf{G} = \mathbf{G}_d\mathbf{G}_v$, and the received signal in (1) can be re-expressed as

$$\mathbf{y} = \mathbf{G}_d\mathbf{H}_v\mathbf{F}_d s + \mathbf{G}_d\nu_v$$  \hspace{1cm} (2)

with $\mathbf{H}_v = \mathbf{G}_d\mathbf{H}\mathbf{F}_v$ is the $b \times b$ eigen-channel matrix, $\nu_v = \mathbf{G}_d\nu$ is the $b \times 1$ transformed additive noise vector. Thanks to the singular value decomposition, the eigen-channel matrix is diagonal and denoted as

$$\mathbf{H}_v = \text{diag}(\sigma_1, \ldots, \sigma_b)$$  \hspace{1cm} (3)
where $\sigma_i$ stands for every subchannel gain (sorted by decreasing order).

The virtual precoder and decoder matrices $F_d$ and $G_d$ are used to solve the optimization criteria such as maximizing the post-processing SNR [9], maximizing the channel capacity [7], minimizing the mean square error [8], or minimizing the BER [10]. These precoders belong to the diagonal group and lead to power allocation on $b$ parallel independent datastreams.

In this paper, we concentrate to the non-diagonal precoder proposed in [2] by optimizing the minimum Euclidean distance. The MIMO system of a non-diagonal precoder is shown by the block diagram in Fig. 1. At the receive side, an ML detection is considered so the decoder matrix $G$ has no impact on the performance and is consequently assumed to be $G_d = I_b$, with $I_b$ is the identity matrix of size $b \times b$.

### B. Minimum Euclidean distance

The minimum Euclidean distance of the received constellation $d_{\min}$ is defined by

$$d_{\min}^2 = \min_{s_k, s_l \in S, s_k \neq s_l} \|H_v F_d (s_k - s_l)\|^2 \quad (4)$$

with $s_k$ and $s_l$ are two symbol vectors belonging to the set of all possible transmitted vectors $S$. Then, the minimum Euclidean distance precoder $F_{d_{\min}}$ is the solution of

$$F_{d_{\min}}^* = \arg \max_{F_d} d_{\min} (F_d) \quad (5)$$

under the power constraint

$$\text{trace}\{F_d F_d^*\} = \text{trace}\{FF^*\} = p_0 \quad (6)$$

where $p_0$ is the mean available transmit power. Determining the precoder matrix $F_{d_{\min}}$ in (5) is difficult because the solution depends on the constellation size and the space of solution is large. For these reasons, the proposed technique is only available for small $b$ virtual channels. Let us remark that $b \leq \text{rank}(H) \leq \min(n_T, n_R)$ so $n_T$ and $n_R$ can still be larger than $b$.

### C. Two-dimensional $d_{\min}$ precoder

The virtual channel matrix of two-dimensional virtual system can be express as:

$$H_v = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \rho \begin{pmatrix} \cos\gamma & 0 \\ 0 & \sin\gamma \end{pmatrix} \quad (7)$$

where $\rho = \sqrt{\sigma_1^2 + \sigma_2^2}$ and $\gamma = \arctan(\frac{\sigma_2}{\sigma_1})$ stand respectively for the channel gain and channel angle. Since the diagonal elements of $H_v$ are in decreasing order, we have the channel angles $\gamma$ such that $0 < \gamma \leq \pi/4$.

It is shown in [2] that the parameterized form of the precoder using trigonometric functions can be expressed as:

$$F_d = \sqrt{p_0} \begin{pmatrix} \cos\psi & 0 \\ 0 & \sin\psi \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp(\jmath\varphi) \end{pmatrix} \quad (8)$$

Considering all symmetries in usual constellation, the influence of the angles on the Euclidean distance has to be studied for $0 < \psi, \theta < \pi/4$ and $0 \leq \varphi < \pi/2$. The parameter $\psi$ is linked to the power allocation on the eigen-subchannels. $\theta$ and $\varphi$ correspond to scaling and rotation of the received constellation, respectively.

By applying a numerical approach to the symmetric constellations, the authors in [2] and [4] showed the optimal minimum distance-based precoder for usual modulations. The 2-D $d_{\min}$ precoder take profit of the spatial diversity better than the diagonal precoders so it achieves a significant SNR gain when $n_T$ and $n_R$ increase but the precoder is limited for $b = 2$. In the following section, we introduce an extension of $max-d_{\min}$ precoder for $b = 3$.

### III. PARAMETERIZED FORM OF THE 3-D $d_{\min}$ PRECODER

The virtual channel matrix of a 3-D virtual system can be parameterized as:

$$H_v = \rho \begin{pmatrix} \cos\gamma_1 & 0 & 0 \\ 0 & \sin\gamma_1 \cos\gamma_2 & 0 \\ 0 & 0 & \sin\gamma_1 \sin\gamma_2 \end{pmatrix} \quad (9)$$

with $\rho$ is the channel gain, $\gamma_1$ and $\gamma_2$ are channel angles respectively. Let us denote that the diagonal entries of the virtual channel matrix $H_v = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$ are sorted in decreasing order such that the channel angles $0 \leq \gamma_2 \leq \pi/4$ and $\cos\gamma_2 \leq \cotan\gamma_1$.

By using a singular value decomposition, the precoding matrix in (5) can be performed as:

$$F_d = A \Sigma B^* \quad (10)$$

where $A$ and $B^*$ are $3 \times 3$ unitary matrices, and $\Sigma$ is $3 \times 3$ diagonal matrix with real positive values in decreasing order. The matrix $\Sigma$ must fulfill the power constraint across all transmit antennas (i.e., $\text{trace}\{\Sigma \Sigma^*\} = p_0$). Hence, $\Sigma$ can be expressed as follow:

$$\Sigma = \sqrt{p_0} \begin{pmatrix} \cos\psi_1 & 0 & 0 \\ 0 & \sin\psi_1 \cos\psi_2 & 0 \\ 0 & 0 & \sin\psi_1 \sin\psi_2 \end{pmatrix} \quad (11)$$

with $0 \leq \psi_2 \leq \pi/4$ and $\cos\psi_2 \leq \cotan\psi_1$.

The authors in [6] showed a lower bound for the minimum Euclidean distance:

$$d_{\min}^2 \geq \lambda_{\min}(SNR(F_d)) \min_{s_k, s_l \in S, s_k \neq s_l} \|(s_k - s_l)\|^2 \quad (12)$$
where $\lambda_{\text{min}}(\text{SNR}(F_d))$ is the minimum eigenvalue of the SNR-like matrix given by $\text{SNR}(F_d) = H_d F_d F_d^* H_d$. One should be reminded that the higher $\lambda_{\text{min}}(\text{SNR}(F_d))$ possibly force $d_{\text{min}}$ to higher values. Therefore, we can reduce the complexity of the precoding matrix $F_d^{d_{\text{min}}}$ by maximizing the smallest singular values of $H_d F_d$. Let us remark that the matrix $B^*$ has no effect on the singular values of $H_d F_d$ so the values can be chosen from $H_d A \Sigma$.

It can be proved that the highest singular values of $H_d F_d$ are obtained when $A$ is an identity matrix ($A = I_3$). Then, we only consider the matrices $\Sigma$ and $B^*$ that maximize $d_{\text{min}}$. The author in [11] shows that a unitary matrix $\Sigma$ we only consider the matrices with angles $\theta$ received constellation scaling and rotation, respectively. Like each virtual subchannel. The angle $\theta$ distance can be studied only for $0 \leq \theta \leq \pi$.

The optimal precoder with the $\theta$ criterion can now be simplified as:

$$B^* = B_{\beta} B_{\theta} B_{\varphi} \quad (13)$$

where $B_{\beta} = \begin{pmatrix} e^{i\beta_1} & 0 & 0 \\ 0 & e^{i\beta_2} & 0 \\ 0 & 0 & e^{i\beta_3} \end{pmatrix}$, $B_{\theta} = \begin{pmatrix} c_1 & s_1 & c_2 \\ -c_1 c_2 s_3 & s_1 c_3 & c_2 s_3 \\ s_1 c_2 s_3 & -s_1 c_3 & c_2 s_3 \end{pmatrix}$, $B_{\varphi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix}$.

Let us define $\tilde{x}$ a difference vector as $\tilde{x} = s_k - s_l$ with $s_k \neq s_l$. The difference vector distance is given by

$$d_k = ||H_d \Sigma B^* \tilde{x}|| = ||H_d \Sigma B_{\beta} B_{\theta} B_{\varphi} \tilde{x}|| = ||B_{\beta} H_d \Sigma B_{\theta} B_{\varphi} \tilde{x}|| = ||B_{\beta} H_d \Sigma B_{\theta} B_{\varphi} \tilde{x}|| \quad (14)$$

Thanks to the the symmetric properties of $\tilde{x}$ (e.g., centered square constellation) and the equation (14), we can conclude that the matrix $B_{\beta}$ has no influence on $d_{\text{min}}$ and the unitary matrix $B^*$ can be simplified as:

$$B^* = B_{\theta} B_{\varphi} \quad (15)$$

where the influence of the angles on the minimum Euclidean distance can be studied only for $0 \leq \varphi_1, \varphi_2, \varphi_3 \leq \pi$, $0 \leq \theta_1, \theta_3 \leq \pi/2$ and $0 \leq \theta_2 \leq \pi/4$.

The parameterized form of the 3-D max-$d_{\text{min}}$ precoder in (10) can be now simplified as:

$$F_d = \Sigma B_{\theta} B_{\varphi} \quad (16)$$

To determine the matrix $F_d$ maximizing the minimum Euclidean distance $d_{\text{min}}$, the effects exerted by the different angles $\psi_i$, $\theta_i$ and $\varphi_i$ are considered. $\psi_i$ allocates power on each virtual subchannel. The angle $\theta_i$ and $\varphi_i$ correspond to the received constellation scaling and rotation, respectively. Like the 2-D precoder case, when $\theta_i$ and $\varphi_i$ are all null, the matrix $F_d$ is diagonal and leads to the eigen-mode power allocation strategies.

Thanks to the equation (16), the angles corresponding to the optimal precoder with the $d_{\text{min}}$ criterion can now be determined.

IV. OPTIMIZATION OF 3-D max-$d_{\text{min}}$ PRECODER

To illustrate the method of optimization, let us consider the SNR-like max-$d_{\text{min}}$ precoder. The precoder can be seen as the max-SNR design, which pours power only on the most favored virtual sub-channel. In fact, the precoder transforms the M-QAM signals on three sub-channels into a $M^3$-QAM modulation on the strongest virtual sub-channel. The signal is then entirely transmitted on this sub-channel, but all antennas are used physically at both the transmitter and the receiver.

A. Optimal SNR-like max-$d_{\text{min}}$ precoder for a BPSK modulation

For a BPSK modulation, the symbols on each data stream belong to the set $\{1, -1\}$. Then, the differences vectors defined by the differences between the possible transmitted vectors, i.e., $\tilde{x} = s_k - s_l$ with $s_k \neq s_l$ are combinations of the set $\{0, 2\}$. By eliminating the collinear vectors, we can reduce the set of received different vectors $\tilde{x}$. Of course, the case of $b = 3$ data streams, the set of different vectors is denoted as $\tilde{X}_{BPSK}$ has only 13 elements.

The precoder is obtained for $\Sigma = \text{diag}(1, 0, 0)$, meaning that only the first virtual sub-channel is used. A numerical search over $\theta$ and $\varphi$ to maximize $d_{\text{min}}$ for each channel angles $\gamma_i$ show that $\theta_2 = \pi/4$, $\theta_3 = 0$ and $\varphi_1 = 0$. A received constellation on the strongest sub-channel is represented on Fig.2. In the figure, the points denoted from 1 to 8 correspond to the 8 possible received symbols.

![Fig. 2. Received constellation of the precoder for BPSK](image)

The received symbol $z$ can be defined by:

$$z = \rho \sqrt{P_0} \cos \gamma_1 \left( \cos \theta_1 s_1 + \frac{\sin \theta_1}{\sqrt{2}} e^{i\varphi_2} s_2 + \frac{\sin \theta_1}{\sqrt{2}} e^{i\varphi_3} s_3 \right)$$

where $s = [s_1 s_2 s_3]^T$ is the transmitted symbol.

The equation (17) show that the angles $\varphi_2$ and $\varphi_3$ have the same roles. Without loss of generality, we can assume that $\varphi_3 \leq \varphi_2 \leq \pi/2$. One should be reminded that the smallest distance is maximized such that the nearest neighbors have the same distances. Thanks to the received constellation presented in Fig.2, the optimized solution is obtained if the point 7 is the center of the rectangular created by the points $(2, 4, 6, 8)$, i.e. $d_{\text{min}}$ is obtained with $\varphi_2 = \pi/2$ and $d_{7,4} = d_{7,6} = d_{7,8}$ which
correspond to Euclidean distance of the 3 different vectors $\hat{x}_1 = [2 0 -2]^T$, $\hat{x}_2 = [0 2 -2]^T$ and $\hat{x}_3 = [0 0 2]^T$.

The corresponding normalized distances can be expressed as
\[
\begin{pmatrix}
\begin{array}{c}
\frac{d^2_{x_1}}{d_{x_1}^2} = 2 \cos^2 \gamma_1 (\cos^2 \theta_1 + 1 - \sqrt{2} \sin(2\theta_1)) \cos \varphi_3 \\
\frac{d^2_{x_2}}{d_{x_2}^2} = 2 \cos^2 \gamma_1 \sin^2 \theta_1 (2 - 2 \sin \varphi_3) \\
\frac{d^2_{x_3}}{d_{x_3}^2} = 2 \cos^2 \gamma_1 \sin \theta_1
\end{array}
\end{pmatrix},
\]
by solving the system $d^2_{x_1} = d^2_{x_2} = d^2_{x_3}$, we obtain
\[
\begin{pmatrix}
\varphi_3 = \pi/6 \\
\theta_1 = \arctan \sqrt{\frac{2}{3}}
\end{pmatrix}
\tag{18}
\]

The precoder $\mathbf{F}_{BPSK}$ is then defined by
\[
\mathbf{F}_{BPSK} = \frac{\sqrt{P_0}}{\sqrt{5}} \begin{pmatrix}
\sqrt{3} e^{i\pi/2} & e^{i\pi/6} \\
0 & 0 \\
0 & 0
\end{pmatrix}
\tag{19}
\]

**B. Optimal SNR-like $d_{\text{min}}$ precoder for a QPSK modulation**

Firstly, if a QPSK modulation is considered at the transmitter, the symbols belong to the set:
\[
S = \frac{1}{\sqrt{2}} \{1 + i, -1 + i, -1 - i, -1 - i\}
\tag{20}
\]

Like the BPSK case, the set of different vectors denoted as $\mathbf{X}_{QPSK}$ can eliminate the collinear vectors. The optimized solution of the precoder for QPSK is illustrated in the Fig. 5. The rotation angle $\varphi_1$ and scaling angle $\theta_3$ have no influence on the performance and are consequently assumed to be 0.

$\begin{bmatrix}
\sqrt{2}+i\sqrt{2}, \sqrt{2}, \sqrt{2}\n\end{bmatrix}^T$ have the same received Euclidean distances.

The five corresponding normalized distances (i.e., $d/\sqrt{2P_0\rho^2\cos \gamma_1^2}$) are defined by
\[
\begin{pmatrix}
\frac{d^2_1}{d_{\text{min}}^2} = \cos \theta_1^2 \\
\frac{d^2_2}{d_{\text{min}}^2} = 1 + \sin(2\theta_1) \times (\sin \varphi_3 - \cos \theta_2 \cos \varphi_2) \\
\frac{d^2_3}{d_{\text{min}}^2} = \sin \theta_1 (\cos(\varphi_3 - \varphi_2)) \\
\frac{d^2_4}{d_{\text{min}}^2} = 1 + \cos^2 \theta_1 - \sin \theta_1 \cos \varphi_2 \sin \varphi_2 \\
\frac{d^2_5}{d_{\text{min}}^2} = \sin(2\theta_1) \sin \theta_2 (\cos \varphi_3 + \sin \varphi_3)
\end{pmatrix}
\tag{21}
\]

By considering $d^2_1 = d^2_2 = d^2_3 = d^2_4 = d^2_5$, it is possible to get the four other angles
\[
\begin{pmatrix}
\varphi_2 = \varphi_3 = \pi/12 \\
\theta_1 = \arctan \sqrt{3(\sqrt{3}+1)} \\
\theta_2 = \arctan \frac{1}{2}
\end{pmatrix}
\tag{22}
\]

The minimum Euclidean distance obtained by $\mathbf{F}_{QPSK}$ is then
\[
d^2_{QPSK} = 2P_0\rho^2\cos \gamma_1 \cdot \frac{1}{5\sqrt{3} + 11}
\tag{23}
\]

**V. Simulation results**

Due to an improvement of minimum Euclidean distance, a significant increase of BER performance of SNR-like $d_{\text{min}}$ precoder is expected compared to diagonal precoders. We consider a MIMO-OFDM system using $n_T = 3$ transmit antennas, $n_R = 3$ receive antennas, 128 subcarriers.

The transmission channel is considered to be a Rayleigh channel and noise elements are additive white Gaussian.

Fig. 4 and Fig. 5 illustrate the BER performance with respect to the SNR in a case of BPSK and QPSK modulation, respectively. Four diagonal precoders: beamforming, water-filling [7], MMSE [8], $\lambda_{\text{min}}$ [6] are compared to the SNR-like $d_{\text{min}}$ precoder. In the Fig. 4, we observe a large performance improvement in terms of BER in comparison with the diagonal precoders. As discussed in Section IV, the SNR-like $d_{\text{min}}$ precoder for BPSK is equivalent to the max-SNR design for 8-QAM (odd-k) modulation, therefore, its performance is not plotted in this figure.

In the case of QPSK modulation, the received constellation like the rotation of the 64-QAM modulation of $\mathbf{F}_{QPSK}$ (appearing in Fig. 3) improved the minimum Euclidean distance. One should note that the difference of $d_{\text{min}}$ between max-SNR and our precoder remains constant for every channel angle. For this reason, we observe in Fig. 5 a slight superiority of BER in comparison with max-SNR precoder. Furthermore, when comparing to other diagonal precoders, the improvement of BER performance is significant. This result clearly demonstrates that the $d_{\text{min}}$ criterion is suited for BER minimization when an ML detection is used at the receive side.

![Fig. 3. Received constellation of the precoder for QPSK](image-url)
In addition, the BER improvement of the SNR-like max-d_{min} precoder depends on the channel characteristics. If there is a small dispersion of the sub-channels SNR $\sigma_1$ and $\sigma_3$, the BER performance of SNR-like max-d_{min} precoder nearly do not change, although the other diagonal precoders have large performance improvements in terms of BER. This is confirmed by the Fig. 6 representing the BER curves in the case of $\sigma_1/\sigma_3 \leq 10$.

VI. CONCLUSION

A new precoder for MIMO transmission, which is based on the maximization of the minimum Euclidean distance between received symbols has been introduced for the case of 3-D sub-channels. Thanks to the virtual channel diagonalization, a generalized form of the 3-D max-d_{min} precoder was proposed.

We introduced also 3-D SNR-like max-d_{min} precoder, the sub-optimal solution of max-d_{min} criterion, which concentrate power on the the most favored virtual sub-channel. For a

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