A Combinatorial Approach to the Fusion Process for the Symmetric Group

VALENTINA GUIZZI AND PAOLO PAPI†

We give a detailed account of Cherednik’s fusion process for the symmetric group using as a key tool the combinatorics of compatible orders on the set of inversions of permutations.

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INTRODUCTION

In this survey paper we give a detailed account of a construction strictly related to the so-called fusion process for the symmetric group $S_n$.

This procedure gives a way of building up the primitive idempotents in $\mathbb{C}[S_n]$—indeed the Young idempotents are obtained—affording the irreducible representations of $S_n$ as a product of factors $1 + c_{i,j}(i,j) \in \mathbb{C}[S_n]$; these factors should be regarded as values of special rational functions on the hyperplane $\sum_{i=1}^{n} z_i = 0$ in $\mathbb{C}^n$. This was first proved by Jucys [4], then by Cherednik [1, 2] and finally generalized by Nazarov [7] to describe the projective representations of $S_n$.

The fusion process associates to each partition $\lambda \vdash n$, more precisely to the standard column tableau $3$ of shape $\lambda$, the following data:

1. the subvariety $Z_3 = \{(z_1, \ldots, z_n) \in \mathbb{C}^n | z_i = z_j \text{ iff } i \text{ and } j \text{ appear in the same row of } \lambda\}$;

2. a $\mathbb{C}[S_n]$-valued function $\Phi_3(z_1, \ldots, z_n)$, depending only on the differences of coordinates, which is a product of factors of type $\phi_{ij} = 1 - \frac{z_i}{z_j}$, This product does not depend on the order of the pairs $(i, j)$ within a precise class of orders, as discovered by Yang [10].

Then it turns out that the restriction of $\Phi_3$ to $Z_3$ is actually regular at $z_1 = \cdots = z_n$ and that its value is a quasi-idempotent generating the irreducible module associated to $\lambda$.

Our point of view in describing the fusion process is similar to Nazarov’s one [6]; however, our presentation relies mainly on combinatorial methods. In particular, we emphasize the relationships with the combinatorics of compatible orders on the set of inversions, which turns out to be a crucial tool both in proof of the regularity of $\Phi_3$ and in the construction of the idempotent. From this point of view our approach seems to be new.

The paper is organized as follows: in Section 1 we collect the results on compatible orders in root systems which will be used in the following. Moreover, we provide combinatorial recipes to obtain explicitly particular reduced expressions of permutations and the associated compatible orders on inversions which are used in the fusion process. In Section 2 we perform the construction of the idempotents.

† Author to whom correspondence should be addressed.
1. Compatible Orders and Symmetric Group

The symmetric group on \( n \) letters \( S_n \) is generated by the \( n - 1 \) transpositions \( s_1 = (1 \ 2), s_2 = (2 \ 3), \ldots, s_{n-1} = (n-1 \ n) \); therefore, endowed with this distinguished set of generators, \( S_n \) can be regarded as a Coxeter group and more precisely as the Weyl group of a root system \( R \) of type \( A_{n-1} \). Recall that \( R \) can be explicitly realized as follows; let \( \varepsilon_1, \ldots, \varepsilon_n \) denote the standard basis in \( \mathbb{R}^n \) and let \( V \) be the hyperplane in \( \mathbb{R}^n \) of equation \( \sum_{i=1}^{n} z_i = 0 \) (the \( z_i \)'s being coordinates w.r.t. the standard basis). Then

\[
R = \{ \varepsilon_i - \varepsilon_j \mid i \neq j \},
\]

\[
R^+ = \{ \varepsilon_i - \varepsilon_j \mid i < j \}.
\]

\( \Pi = \{ \alpha_1, \ldots, \alpha_{n-1} \} \), \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \), is a basis of simple roots. \( S_n \) acts on \( R \) in the natural way:

\[
\sigma(\varepsilon_i - \varepsilon_j) = \varepsilon_{\sigma(i)} - \varepsilon_{\sigma(j)}.
\]

Notice in particular the following.

(I) The map \((i, j) \mapsto \varepsilon_i - \varepsilon_j\) affords a bijection \( \eta \) between pairs \((i, j)\), \( 1 < i < j < n \) and the positive roots.

(II) For \( \alpha, \beta \in R^+ \), \( \alpha + \beta \) is a root if and only if \( \text{Supp}(\alpha) \cup \text{Supp}(\beta) \) is formed by consecutive integers (if \( \alpha = \sum_{i=1}^{j} a_i \alpha_i \), \( \text{Supp}(\alpha) = \{ i \mid a_i \neq 0 \} \)).

(III) Write a permutation \( \sigma \in S_n \) as a word in \( 1, 2, \ldots, n \): \( \sigma = (\sigma(1), \ldots, \sigma(n)) \); recall then that the set of inversions for \( \sigma \) is the set of subwords of \( \sigma \) of the form \( (\alpha\beta) \), the elements of \( R(\sigma) \) are in bijection with the inversions for \( \sigma \).

We now recall some results which hold in the general framework of Weyl groups. Let \( W \) be a Weyl group and let \( R \) denote its root system. We say that a linearly ordered subset \( L \subseteq R^+ \) is associated to \( w \in W \) if there exists a reduced expression \( s_{i_1} \cdots s_{i_m} \) of \( w \) such that

\[
L = \{ \alpha_{i_1} < s_{i_1}(\alpha_{i_2}) < \cdots < s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m}) \}.
\]

The order on \( L \) (which depends on the reduced expression) will be called compatible.

**Theorem 1.1** ([8]). A linearly ordered subset \( L \subseteq R^+ \) is associated to some \( w \in W \) if and only if \( L \) verifies the following conditions:

(I) If \( \lambda, \mu \in L \), \( \lambda < \mu \), \( \lambda + \mu \in R \), then \( \lambda + \mu \in L \) and \( \lambda < \lambda + \mu < \mu \).

(II) If \( \lambda + \mu \in L \), \( \lambda, \mu \in R^+ \), then \( \lambda \) or \( \mu \) belongs to \( L \) and one of them precedes \( \lambda + \mu \).

**Definition 1.2.** We say that \( L \) is compatible if it satisfies the following conditions:

(1) If \( \lambda, \mu \in L \), \( \lambda + \mu \in R \), then \( \lambda + \mu \in L \).

(2) If \( \lambda + \mu \in L \), \( \lambda, \mu \in R^+ \), then \( \lambda \) or \( \mu \) belongs to \( L \).

This terminology is justified by the following theorem.

**Theorem 1.3** ([8]). A subset \( L \) is compatible if and only if it is of the form \( R(w) = \{ \alpha \in R^+ \mid w^{-1}(\alpha) < 0 \} \), for some \( w \in W \). Moreover, such \( w \) is unique.

In particular, reduced expressions of \( w \) are in bijection with compatible orders on \( R(w) \). We collect some other consequences of the previous theorems.
COROLLARY 1.4.

$(1)$ The following conditions are equivalent for $L \subseteq R^+$:

(i) $L$ is compatible.

(ii) $L' \equiv R^+ \setminus L$ is compatible.

(iii) $L$ and $L'$ are closed (i.e., they verify condition (1) in the definition 1.2).

Moreover, if $L = R(w)$, then $L' = R(w w_0)$, $w_0$ being the longest element in $W$. Finally, the reverse of a compatible order on $R^+$ is still compatible.

$(2)$ Let $L$ be a compatible set. Any compatible order of $L$ starts with a simple root in $L$.

Given $\alpha \in \Pi \cap L$, there exists a compatible order of $L$ which starts with $\alpha$.

$(3)$ A compatible set $L$ admits a compatible order ending in $\gamma$ if and only if $L \setminus \{\gamma\}$ is compatible.

Now we turn to the symmetric group. Consider $\sigma \in S_n$ and write $\sigma$ in the window notation as $(\sigma(1), \ldots, \sigma(n))$. We say that $i$ precedes $j$ w.r.t. $\sigma$ (notation: $i \rightarrow j$) if $\sigma^{-1}(i) < \sigma^{-1}(j)$; otherwise we say that $i$ follows $j$ ($i \leftarrow j$). Finally, denote by $\sigma_0$ the longest element in $S_n$.

DEFINITION 1.5. Fix $\sigma \in S_n$; define a total order $\prec$ on the set of pairs $(i, j)$, $i < j$ (hence an order on $R^+$) as follows; set

$$A_\sigma = \{(i, j) \mid i < j, i \rightarrow j \} = \{(i, j) \mid i < j, \sigma^{-1}(i) < \sigma^{-1}(j)\},$$

$$B_\sigma = \{(i, j) \mid i < j, i \leftarrow j \} = \{(i, j) \mid i < j, \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

$(1)$ If $(i, j), (h, k) \in A_\sigma$

$$(i, j) \prec (h, k) \iff j < k \text{ or } j = k \text{ and } i \rightarrow h.$$

$(2)$ If $(i, j), (h, k) \in B_\sigma$

$$(i, j) \prec (h, k) \iff j > k \text{ or } j = k \text{ and } i \leftarrow h.$$

$(3)$ $A_\sigma \prec B_\sigma$.

EXAMPLE 1.6. Consider $\sigma = (152643)$; then the order $\prec$ is

$(12), (13), (23), (14), (24), (15), (16), (56), (26), (46), (36), (25), (45), (35), (34).$

The vertical bar separates $A_\sigma$ from $B_\sigma$. The reduced expression of $\sigma_0$ which induces the previous order is

$$\sigma_0 = s_1 s_2 s_1 s_3 s_2 s_4 s_3 s_2 s_1 s_4 s_3 s_2 s_3.$$

PROPOSITION 1.7.

$(1)$ $\prec$ is indeed a compatible order in $R^+$.

$(2)$ $A_\sigma \leftrightarrow R(\sigma \sigma_0), B_\sigma \leftrightarrow R(\sigma)$.

PROOF. The compatibility of the order can be verified by checking the axioms in 1.2.

By the very definition, $B_\sigma$ is the set of inversions of $\sigma$, therefore, as noticed in (III), it is in bijection with $R(\sigma)$. As $A_\sigma$ is the complement of $B_\sigma$ in $\{(i, j) \mid i < j\}$, we obtain the first statement in (2) from 1.4 (1).
Remark 1.8. The order $\prec$ can be recovered in a combinatorial way using a variation of the Rothe’s algorithm described in [9]; recall that given a permutation matrix, we can associate to it four diagrams obtained in the following way: consider, among the pairs of 1’s in the matrix, those determining a lower-right (resp. upper-right, upper-left, lower-left) corner and colour the entries of the matrix corresponding to the corners (see next paragraph); we can choose the labelling in such a way that the labelled entries describe in a combinatorial way the set of inversions for the given permutation $\sigma$ (resp. $\sigma_0$, $\sigma_0\sigma_0$, $\sigma_0\sigma$); for further information see [5]. We need to use simultaneously two of these diagrams. Proceed as follows: draw a table with $n^2$ square boxes, indexed as matrix entries; among the pairs of boxes $(A, B)$ having ‘coordinates’ $(i, \sigma(i))$ consider the following two types:

- those for which $A$ lies on the left and above with respect to $B$: we call them distinguished pairs of type 1; they are exactly the pairs
  \[ A_\sigma = \{(i, \sigma(i)), (j, \sigma(j)) \mid i < j, \sigma(i) < \sigma(j)\}; \]

- those for which $A$ lies on the left and below with respect to $B$: we call them distinguished pairs of type 2; they are exactly the pairs
  \[ B_\sigma = \{(i, \sigma(i)), (j, \sigma(j)) \mid i > j, \sigma(i) < \sigma(j)\}. \]

Then, for any distinguished pair $(i, \sigma(i)), (j, \sigma(j))$, label with $(\sigma(i), \sigma(j))$ the box having ‘coordinates’ $(i, \sigma(j))$, i.e., the one lying on the same row of $A$ and on the same column of $B$:

\[
\begin{align*}
*_{i,\sigma(i)} &\rightarrow (\sigma(i), \sigma(j))_{i,\sigma(j)} & \star_{j,\sigma(j)} \\
\uparrow & & \downarrow \\
\star_{j,\sigma(j)} & *_{i,\sigma(i)} & (\sigma(i), \sigma(j))_{i,\sigma(j)}
\end{align*}
\]

Finally read, from the left to the right and from the top to the bottom, the labels coming from pairs of type 1 and then read, from the right to the left and from the top to the bottom, the labels coming from pairs of type 2, e.g., for $\sigma = (152643)$:

We claim that the ordered string of labels (thought of as roots) obtained in this way corresponds to the order $\prec$ on $R^+$. 

In fact, the map \(((i, \sigma(i)), (j, \sigma(j))) \mapsto (\sigma(i), \sigma(j))\) establishes bijections \(A' \sigma \leftrightarrow A \sigma\), \(B' \sigma \leftrightarrow B \sigma\); moreover, if \((i, k), (j, k)\) belong to \(A \sigma\) (resp. \(B \sigma\)) and \((i, k)\) precedes \((j, k)\) in the ‘reading’ order, this means that \(\sigma^{-1}(i) < \sigma^{-1}(j)\), i.e., \(i \rightarrow j\) (resp. \(\sigma^{-1}(i) > \sigma^{-1}(j)\), i.e., \(i \leftarrow j\)). From these remarks the claim follows. The order \(\prec\) is indeed induced by the following reduced expression of \(\sigma_0\) (cf. [7]):

\[
\prod_{2 \leq k \leq n} (s_{k-1} \cdots s_{k-a_k}) \prod_{2 \leq k \leq n} (s_{n-k+a'_k} \cdots s_{n-k+1}).
\]

Here \(a_k = \#\{i \mid i < k, i \rightarrow k\}\), \(a'_k = \#\{i \mid i < k, i \leftarrow k\}\). This reduced expression can be deduced by the Rothe diagram in the following way. For any distinguished pair \((A, B)\) of type 1 (resp. type 2), mark with a circle (resp. a cross) the box which lies on the same row of \(A\) and on the same column of \(B\). Finally, proceed as follows.

1. As \(k\) varies from 1 to \(n\), label by \(k-1, k-2, \ldots\) from the top to the bottom the ‘circled’ boxes lying on the \(k\)-th column. Write then the string obtained juxtaposing from the left to the right the simple reflections relative to the labels on the columns proceeding from the left to the right and from the top to the bottom: this string is a reduced expression of \(\sigma \sigma_0\).

2. As \(k\) varies from 1 to \(n\), label by \(1, 2, \ldots\) from the bottom to the top the ‘crossed’ boxes lying on the \((n-k+1)\)-th column. Write then the string obtained juxtaposing from the left to the right the simple reflections relative to the labels on the columns proceeding from the right to the left and from the top to the bottom: this string is a reduced expression of \(\sigma_0 \sigma^{-1} \sigma_0\).

The following picture illustrates this procedure in the case of the previous example.

By juxtaposing the reduced expression found in (1), (2), respectively, we obtain the reduced expression of \(\sigma_0\) we were looking for. To prove that this algorithm selects the reduced expression displayed in (1.1), it suffices to observe that

\[
\text{# circled box on the } k\text{-th column} = \text{# crossed box on the } k\text{-th column}
\]

\[
\text{# } \{i \mid \sigma(i) < k, i < \sigma^{-1}(k)\} = \text{# } \{i \mid \sigma(i) < k, i > \sigma^{-1}(k)\} = a_k.
\]

\[
\text{# } \{j \mid j < k, \sigma^{-1}(j) < \sigma^{-1}(k)\} = a'_k.
\]

It remains to verify that the reduced expression in (1.1) induces \(\prec\); this can be shown, mutatis mutandis, through the same inductive argument used to prove the main theorem of [9].

In the next section we will need the following fact.
PROPOSITION 1.9. Let $L \subseteq R^+$ be a compatible set; assume that there exists a compatible order on $L$ which ends with $\alpha + \beta$, $\beta$. Then there is a compatible order which ends with $\alpha$, $\alpha + \beta$, $\beta$.

The statement means that if, in the hypothesis of the proposition, 62 and 64 are inversions for $\sigma$, then $\sigma = \cdots 642 \cdots$ and moreover $\sigma$ has a reduced decomposition $\sigma = s_{i_1} \cdots s_{i_k}$ with last steps $s_{i_1} \cdots s_{i_{k-1}} = \cdots 246 \cdots$, $s_{i_1} \cdots s_{i_{k-2}} = \cdots 426 \cdots$, $s_{i_1} \cdots s_{i_{k-1}} = \cdots 462 \cdots$, $s_{i_1} \cdots s_{i_k} = \cdots 642 \cdots$.

PROOF. Set $M = L \setminus \{\alpha + \beta, \beta\}$; $M$ is compatible and, by compatibility of the order on $L$, $\alpha \in M$. By corollary 1.4 (3), to obtain the assertion it suffices to prove that $N = M \setminus \{\alpha\}$ is a compatible set. We have to verify the axioms of Definition 1.2 for $N$; we are in turn reduced to prove that

(a) $\forall x, y \in N \quad x + y \neq \alpha$.
(b) $\alpha + x \in N \implies x \in N$.

If (a) does not hold, then in the fixed compatible order on $L$

$$\cdots < x < \cdots < \alpha < \cdots < y < \cdots < \alpha + \beta < \beta.$$ 

Now, by (II), we have either $y + \beta \in R$ or $x + \beta \in R$. We deal with the first case, the other being similar. We have

$$\cdots < x < \cdots < \alpha < \cdots < y < \cdots < \alpha + \beta < \beta.$$ 

The previous relations imply $\alpha + \beta = x + y + \beta < y + \beta < \alpha + \beta$, a contradiction.

To prove (b), assume $\alpha + x \in N$. We have three subcases

(A) $\alpha + \beta + x \in R$ (i.e., $\text{Supp} (\alpha + \beta + x) = \text{Supp} (\alpha) \cup \text{Supp} (\beta) \cup \text{Supp} (x)$).
(B) $\beta - x \in R^+$ (i.e., $\text{Supp} (x) \subseteq \text{Supp} (\beta)$).
(C) $x - \beta \in R^+$ (i.e., $\text{Supp} (\beta) \subseteq \text{Supp} (x)$).

If $x \notin N$ we obtain in any case a contradiction; in fact

(A) We have $\cdots < \alpha < \cdots < \alpha + x < \cdots < \alpha + \beta + x < \cdots < \alpha + \beta < \beta$
and neither $\alpha + \beta$ nor $x$ precedes $\alpha + \beta + x$.
(B) As $L$ is compatible, from $\beta = (\beta - x) + x$ we obtain $\beta - x \in N$ and therefore $(\beta - x) + (x + \alpha) = \alpha + \beta \in N$.
(C) Again, by compatibility of $L$, $\alpha + x = (\alpha + \beta) + (x - \beta)$ implies $x - \beta \in N$; by convexity $(x - \beta) + \beta = x \in L$, hence $x \in N$.

$\square$

2. THE FUSION PROCESS

Set $A = \mathbb{C}[S_n]$ and consider the algebra

$$B = A \otimes \mathbb{C}(z_1, \ldots, z_n).$$

We will be mainly concerned with the following fundamental objects in $B$.

DEFINITION 2.1. For $\alpha \in R^+$, $z = \sum_{i=1}^n z_1 \xi_i$ we set

$$\varphi_\alpha \equiv \varphi_\alpha (z) = 1 - \frac{s_\alpha}{(\alpha, z)}$$

(, ) being the standard inner product in $\mathbb{R}^n$. If $\alpha = \varepsilon_i - \varepsilon_j \in R^+$ we write also $\varphi_{ij}(z_i, z_j)$ (or simply $\varphi_{ij}$).
Our crucial computational tool will be explained in the next proposition, which can be proved with a straightforward verification.

**Proposition 2.2.** Suppose $\alpha, \beta \in R^+$. Then the Yang–Baxter equation in $\mathbb{C}[S_n]$ holds. Indeed:

1. If $\alpha \perp \beta$, then $\varphi_{\alpha} \varphi_{\beta} = \varphi_{\beta} \varphi_{\alpha}$;
2. If $\alpha + \beta \in R$, then $\varphi_{\alpha} \varphi_{\beta} \varphi_{\alpha + \beta} = \varphi_{\beta} \varphi_{\alpha + \beta} \varphi_{\alpha}$.

**Corollary 2.3 (Yang, Baxter).** Fix a compatible order on $L$. Then the function $\prod_{\alpha \in L} \varphi_{\alpha}$ depends only on $L$ and not on the chosen order.

**Proof.** Suppose $L = R(w)$. We have already observed that compatible orders of $L$ correspond to reduced expressions of $w$; by a well-known theorem of Iwahori and Matsumoto, it is possible to change any reduced expression into another one by means of braid relations (which involve a rank 2 root system). By the previous proposition, the same happens with the $\varphi_{\alpha}$'s. \qed

Let $\lambda \vdash n$, and let $\Lambda$ be the column tableau of shape $\lambda$, obtained by numbering the boxes in the columns by $1, \ldots, n$ starting from the top to bottom and then from left to right. Let $C_{\Lambda}, R_{\Lambda}$ denote the subgroups of $S_n$ stabilizing the columns and the rows of $\Lambda$, respectively. Introduce the following elements in $A$:

$$P_{\Lambda} = \sum_{\sigma \in R_{\Lambda}} \sigma, \quad Q_{\Lambda} = \sum_{\sigma \in C_{\Lambda}} (-1)^{\sigma} \sigma, \quad E_{\Lambda} = P_{\Lambda} Q_{\Lambda}.$$  

It is a classical theorem, from Young, that $AE_{\Lambda}$ is the irreducible $S_n$-module $M_{\lambda}$ associated to the partition $\lambda$.

Consider an arbitrary diagram of shape $\lambda$ and the associated column tableau $\Lambda$. For $i = 1, \ldots, n$ denote by $c_i$ the content (difference between column and row positions) of the box containing $i$. Define then

$$\Phi_{\Lambda}(z_1, \ldots, z_n) = \prod_{1 \leq i < j \leq n} \varphi_{ij}(z_i + c_i, z_j + c_j). \quad (2.1)$$

where by Corollary 2.3 we can take the product w.r.t. any compatible order on $R^+$.

Consider now the subvariety of $\mathbb{C}^n$ defined by the following equations

$$Z_{\Lambda} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i = z_j \text{ iff } i \text{ and } j \text{ appear in the same row of } \Lambda \}.$$  

**Theorem 2.4.**

1. The restriction of $\Phi_{\Lambda}(z_1, \ldots, z_n)$ to $Z_{\Lambda}$ is regular at $z_1 = \cdots = z_n$.
2. Let $\Phi$ be the value of this restriction at $z_1 = \cdots = z_n$; then $\Phi$ generates under left multiplication by $\mathbb{C}[S_n]$ the irreducible $S_n$-module $M_{\lambda}$. More precisely we have:

$$\Phi = \frac{Q_{\Lambda} P_{\Lambda} Q_{\Lambda}}{\lambda_1! \lambda_2! \cdots} \quad (2.2)$$

($\lambda_i'$ denotes the length of the $i$-th column of $\lambda$).
The proof of the theorem consists in proving that the Yang–Baxter product to which the function $\Phi_\Lambda(z_1, \ldots, z_n)$ reduces after specialization and evaluation at $z_1 = \cdots = z_n$ verifies the properties characterizing the Young idempotents, once one has checked the validity of such a specialization. As an example, consider the case in which $\lambda$ is a hook: then the restriction of $\Phi_\Lambda(z_1, \ldots, z_n)$ to $Z_\Lambda$ is manifestly regular; if, e.g., $\lambda = (2, 1, 1)$ we obtain

$$
\Phi_\Lambda = \frac{1}{2} (1 - (12))(1 + (13)) \left( 1 + \left( \frac{14}{2} \right) \right) \left( 1 + \left( \frac{24}{3} \right) \right) \left( 1 + \left( \frac{23}{2} \right) \right)
$$

which is the quasi-idempotent corresponding to the standard irreducible 3-dimensional representation of $S_3$.

Before proving the theorem we need the following:

**Lemma 2.5 ([6]).** The restriction of the function

$$
\psi_{ih}(z_i, z_h) \psi_{ij}(z_i, z_j) \psi_{hj}(z_h, z_j)
$$

to $z_h = z_i + 1$ is regular at $z_i = z_j$.

**Proof.** Just compute:

$$
\psi_{ih}(z_i, z_h) \psi_{ij}(z_i, z_j) \psi_{hj}(z_h, z_j)|_{z_h = z_i + 1} = (1 - ih) \left( 1 - \frac{(ij) + (hj)}{z_h - z_j} \right),
$$

and this function is clearly regular at $z_i = z_j$. \hfill $\Box$

**Proof of 2.4(1).** Observe that the function $\psi_{ij}(z_i + c_i, z_j + c_j)$ is singular at $z_i = z_j$ if and only if $i$ and $j$ stand on the same diagonal of the tableau $\Lambda$, so that $c_i = c_j$; in this case we will call the pair $(i, j)$ singular. As we are interested to prove the regularity of the restriction of $\Phi_\Lambda(z_1, \ldots, z_n)$ to $Z_\Lambda$, the only singularities we have to take care of are the ones coming from singular pairs.

We proceed in two steps. First arrange the factors in (2.1) w.r.t. a suitable compatible order canonically associated to $\Lambda$, in order to satisfy the hypothesis of Proposition 1.9 for any singular pair; then we modify $\Phi_\Lambda(z_1, \ldots, z_n)$ without affecting the value of its restriction to $Z_\Lambda$ at $z_1 = \cdots = z_n$: this allows to conclude using Lemma 2.5.

Let $\sigma_\Lambda \in S_n$ be the row-word associated to the tableau $\Lambda$ (i.e., the permutation $(\sigma_\Lambda(1), \ldots, \sigma_\Lambda(n))$ obtained by reading the column tableau $\Lambda$ by rows, from the top to the bottom and from the left to the right). Let $< \in \sigma_\Lambda$ the total order defined on $R^+$ by $\sigma_\Lambda$ as in Definition 1.5, which is compatible by Proposition 1.7. Note that if $(i, j) \in B_{\sigma_\Lambda}$ the pair $(i, j)$ is not singular because $i$ stands below and to the left of $j$ and so $c_i \neq c_j$.

Now consider a singular pair $(i, j)$; remark that $j \neq i + 1$, so $(i, j)$ does not correspond through $\eta$ to a simple root. Let $h$ be the integer next to $i$ on the same row of $\Lambda$: in particular, $c_h = c_i + 1$. It is easy to see that, in the order we have fixed, the pair following $(i, j)$ is $(h, j)$ because $h < j$ and $h$ appear next after $i$ in the sequence $(\sigma_\Lambda(1), \ldots, \sigma_\Lambda(n))$ defined above. Therefore

$$
\Phi_\Lambda(z_1, \ldots, z_n) = \left( \prod_{(k,l) \prec (i,j)} \varphi_{kl} \right) \cdot \varphi_{ij} \cdot \varphi_{hj} \cdot \left( \prod_{(k,l) \prec (h,j)} \varphi_{kl} \right).
$$
Now using Corollary 2.3 and Proposition 1.9 we can arrange the order of the product in previous formula in such a way to have \( \varphi_{ih}, \varphi_{ij}, \varphi_{kj} \) as adjacent factors:

\[
\Phi_A(z_1, \ldots, z_n) = \left( \prod_{i} \varphi_{kl} \right) \cdot \varphi_{ih} \varphi_{ij} \varphi_{kj} \cdot \left( \prod_{(k,j) \succ (h,j)} \varphi_{kl} \right)
\]

where the arguments of the \( \varphi_{ij} \)'s are omitted for shortness and the first product is taken with respect to a suitable compatible order \( \prec \) on the set \( \{(k, l) \mid (k, l) \prec (i, j), (k, l) \neq (i, h)\} \).

Note now that the restriction of \( \varphi_{ih}(z_i + c_i, z_h + c_h) \) to \( z_j = z_h \) is 1 + \((ih)\)/2 is an idempotent in \( A \), so we can replace \( \varphi_{ih}(z_i + c_i, z_h + c_h) \) by its square without affecting the value of the restriction of \( \Phi_A \) to \( Z_A \); so we obtain

\[
\Phi_A(z_1, \ldots, z_n)_{|_{z_i = z_h}} = \left( \prod_{(k,j) \neq (i,j)} \varphi_{kl} \right)_{|_{z_i = z_h}} \cdot \varphi_{ih} \varphi_{ij} \varphi_{kj} \cdot \left( \prod_{(k,j) \succ (h,j)} \varphi_{kl} \right)_{|_{z_i = z_h}}.
\]

The middle factor is the restriction of \( \varphi_{ih}(z_i, z_h + 1) \varphi_{ij}(z_i, z_j) \varphi_{kj}(z_h + 1, z_j) \) at \( z_i = z_h \), which is regular at \( z_j = z_j \) by Lemma 2.5; iterating this procedure for any singular pair we obtain the first statement of Theorem 2.4.

Let us now define

\[
\Psi_A(z_1, \ldots, z_n) = \prod_{(i,j) \in A_{\sigma_A}} \varphi_{ij}(z_i + c_i, z_j + c_j),
\]

\[
\Theta_A(z_1, \ldots, z_n) = \prod_{(i,j) \in B_{\sigma_A}} \varphi_{ij}(z_i + c_i, z_j + c_j),
\]

so that

\[
\Phi_A(z_1, \ldots, z_n) = \Psi_A(z_1, \ldots, z_n) \Theta_A(z_1, \ldots, z_n).
\]

From what we have already observed about \((i, j) \in B_{\sigma_A}\), it follows that \( \Theta_A \) is regular at \( z_1 = \cdots = z_n \) and its value \( \Theta \) at this point is invertible in \( A \) because each of its factors is of the form \( 1 + a(i, j) \) with a \( \neq \pm 1 \). We denote by \( \Psi \) the value at \( z_1 = \cdots = z_n \) of the restriction of \( \Psi_A(z_1, \ldots, z_n) \) to \( Z_A \). In order to prove part (2) of Theorem 2.4 we need two more preliminary results.

**Proposition 2.6.** \( \Psi_A(z_1, \ldots, z_n) \) is divisible in \( \mathcal{B} \) on the right by \( \varphi_{kl}(z_k + c_k, z_l + c_l) \) where \( k \) and \( l \) stand next to each other on the same row of \( \Lambda \). Moreover, \( \Psi \) is divisible in \( \mathbb{C}[S_n] \) on the right by \( 1 + (kl) \).

**Proof.** To prove the first assertion it suffices, by 1.4 (3) and 2.3, to show that \( A_{\sigma_A} \setminus \{(kl)\} \) is compatible: this is easily done by checking the axioms of Definition 1.2. To prove the other claim note that the restriction of \( \varphi_{kl}(z_k + c_k, z_l + c_l) \) to \( z_k = z_l \) is exactly 1 + \((kl)\).

**Proposition 2.7.** If \( k, k + 1 \) stand in the same column of \( \Lambda \), then \( \Psi_A(z_1, \ldots, z_n) \) is divisible in \( \mathcal{B} \) on the left by \( \varphi_{kl}(z_k + c_k, z_{k+1} + c_{k+1}) \). Moreover, \( \Psi \) is divisible in \( \mathbb{C}[S_n] \) on the left by \( 1 - (kk + 1) \).

**Proof.** We argue as in the proof of 2.6, observing that, by the definition of the order \( \prec \), if \( k \) and \( k + 1 \) stand in the same column of \( \Lambda \), then \( (k, k + 1) \in A_{\sigma_A} \). Then we obtain the first assertion using 1.4 (2) and 2.3. Finally, note that the value of \( \varphi_{kl}(z_k + c_k, z_{k+1} + c_{k+1}) \) at \( z_k = z_{k+1} \) is just \( 1 - (kk + 1) \), so the proposition is proved.
Proof of 2.4(2). From Propositions 2.6 and 2.7 it follows that \( \Psi(kl) = \Psi \) and \( (k + 1) \Psi = -\Psi \). So we have \( q \Psi p = sgn q \Psi \) for any \( p \in R_\lambda \) and \( q \in C_\lambda \); therefore \( \Psi \) equals \( Q_\lambda P_\lambda \) up to a constant factor (see [3, 4.21]).

Observe that the function \( \Phi_\lambda \) is invariant w.r.t. the antiautomorphism of \( B \) mapping \( \sigma \in S_n \) to \( \sigma^{-1} \), for the image of \( \Phi_\lambda \) under this map is just the product of the factors in (2.1) taken in the reverse order (cf. 1.4 (1)). So, recalling that \( \Phi = \Psi \Theta \), it follows from Proposition 2.7 that \( \Phi \) is divisible on the right by \( 1 - (k + 1) \) and therefore \( \Phi q = sgn q \Phi \) for \( q \in C_\lambda \). This implies that, up to a constant \( \alpha \in \mathbb{C} \), \( \Phi \) equals \( Q_\lambda P_\lambda \) and so, using the same argument as above it follows that:

\[
\Phi = \alpha Q_\lambda P_\lambda Q_\lambda.
\]

To complete the proof of (2) we have to show that \( \alpha = \frac{1}{s_1 s_2 \cdots s_n} \). Note that the coefficient at the identity of the right hand side of (2.2) is 1; it remains to prove that the same happens for \( \Phi \).

We will show that the coefficient of the longest element \( \sigma_0 \in S_n \) in \( \Phi \sigma_0 \) is 1. We prove that

\[
\Phi_\lambda(z_1, \ldots, z_n) \sigma_0 = \prod_{i<j} \phi_{j-i}(z_i + c_i, z_j + c_j),
\]

where \( \phi_k(u, v) = \phi_{kk+1}(u, v) \). Then the assertion follows recalling the following well-known formula, affording a reduced expression of \( \sigma_0 \)

\[
\sigma_0 = \prod_{1<i<j<n} (j-i) = \sigma^{(n-1)} \cdots \sigma^{(1)}, \quad \sigma^{(i)} = s_1 \cdots s_i
\]

and observing that the coefficient of this element in (2.3) is 1.

Let us now prove (2.3). If we set

\[
\Phi_i(z_1, \ldots, z_n) = \psi_{i+1}(z_i + c_i, z_{i+1} + c_{i+1}) \cdots \psi_n(z_i + c_i, z_n + c_n)
\]

we can write

\[
\Phi_\lambda(z_1, \ldots, z_n) \sigma_0 = \Phi_1 \cdots \Phi_{n-1} \sigma^{(n-1)} \cdots \sigma^{(1)}.
\]

It is easy to see that (2.3) follows from

\[
\Phi_i \sigma_{n-1} \cdots \sigma_{n-i} = \sigma^{(n-1)} \cdots \sigma^{(n-i+1)} \psi_1 \cdots \psi_{n-i};
\]

the above identity is a straightforward verification. \( \square \)

References


Received 27 March 1998 and accepted 8 June 1998

V. GUZZI AND P. PAPI
Dipartimento di Matematica Istituto G. Castelnuovo,
Universitá di Roma "La Sapienza",
Piazzale Aldo Moro 5,
00185 Rome, Italy
E-mail: guazzi@mat.uniroma1.it
E-mail: papi@mat.uniroma1.it