Joining caterpillars and stability of the tree center

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Abstract

The path-table \( P(T) \) of a tree \( T \) collects information regarding the paths in \( T \): for each vertex \( v \), the row of \( P(T) \) relative to \( v \) lists the number of paths containing \( v \) of the various lengths. We call this row the path-row of \( v \) in \( T \).

Two trees having the same path-table (up to reordering the rows) are called path-congruent (or path-isomorphic).

Motivated by Kelly–Ulam’s Reconstruction Conjecture and its variants, we have looked for new necessary and sufficient conditions for isomorphisms between two trees.

Path-congruent trees need not be isomorphic, although they are similar in some respects. In [P. Dulio, V. Pannone, Trees with path-stable center, Ars Combinatoria, LXXX (2006) 153–175] we have introduced the concepts of trunk \( Tr(T) \) of a tree \( T \) and ramification \( \text{ram} v \) of a vertex \( v \in V(Tr(T)) \), and proved that, if the ramification of the central vertices attains its minimum or maximum value, then the path-row of a central vertex is “unique”, i.e. it is different from the path-row of any non-central vertex (in fact, this uniqueness property of a central path-row holds for all trees of diameter less than 8, regardless of the ramification values).

In this paper we prove that, for all other values of the ramification, and for all diameters greater than 7, there are trees in which the above uniqueness fails.

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1. Introduction and basic definitions

This paper is concerned with paths in trees, and it is an ideal continuation of [6]. Our original motivation comes from Reconstruction Theory and the Converse of Kelly’s Lemma (see [5]), but the study of paths of a graph \( G \) can be variously investigated (see, for instance [1,7,8,9]). In [10,11] (and also [1]) the attention is paid to the number of paths starting at each vertex of a tree. In [5,6] we have stressed that the number of paths passing through any vertex is also a meaningful notion. The path-table of a tree has been introduced in the same spirit as other tables previously appeared in the literature (e.g. in [2]).

In the present paper, we concentrate on the issue of characterizing the tree center by the paths-through, and Theorem 3.1 stands as a counterpart to Theorems 5.3 and 5.4 in [6].

Let \( G_1, G_2 \) be two (finite, simple, labelled) graphs. A path-congruence \( \Phi : G_1 \rightarrow G_2 \) is a bijection \( V(G_1) \rightarrow V(G_2) \) such that, for every \( l \in \mathbb{N} \), and every \( v \in V(G_1) \), the number of paths of \( G_1 \) of length \( l \) and containing \( v \) equals the
Theorem 1.3. Let \( T \) be a tree with diameter \( D \) even or odd, and \( |\text{Tr}(T)| \geq 1 \). If \( 0 \leq \text{ram} \leq \lfloor (D - 1)/2 \rfloor \).

For a tree \( T \), the pruned tree is the tree obtained by removing all the leaves (vertices of degree one) of \( T \). A caterpillar is a tree \( C \) such that its pruned tree is a path.

A set \( S \subseteq T \) is said to be path-stable (or simply stable) if \( \Phi(S) = S \) for every path-congruence \( \Phi : T \rightarrow T \).

In [6, Theorems 5.5 and 5.4] we discussed conditions on a tree \( T \) under which \( Z(T) \) is stable. In particular, we proved the following two theorems.

Theorem 1.2. Let \( T \) be a tree with \( \text{diam} \ T \leq 7 \). Then \( Z(T) \) is stable.

Theorem 1.3. Let \( T \) be a tree, \( Z(T) = \{c_L, c_R\} \) its center (possibly \( c_L = c_R \)), and \( D \) its diameter. Then \( Z(T) \) is stable if at least one of the following holds:

(i) \( |\text{Tr}(T)| = 1 \).
(ii) \( \min\{\text{ram}_L, \text{ram}_R\} \geq \text{ram} \) for all \( x \in \text{Tr}(T) \).
(iii) \( \text{ram}_L = \text{ram}_R = \lfloor D/2 \rfloor - 1 \).
(iv) \( \deg c_L = \deg c_R = 2 \).
Remark 1.4. Note that, when \( D \) is odd, \( Z(T) \) is stable even if \( \text{ram}_{cL} = \text{ram}_{cR} = \lfloor (D - 1)/2 \rfloor \), since in this case \( Z(T) = \text{Tr}(T) \). Note also that, in a tree \( T \) with at least three vertices, the condition \( \deg cL = 2 \) (resp. \( \deg cR = 2 \)) is equivalent (since central vertices do not have degree one) to the condition \( \text{ram}_{cL} = 0 \) (resp. \( \text{ram}_{cR} = 0 \)). Thus, in view of Proposition 1.1, Theorem 1.3 states that path-stability of \( Z(T) \) is ensured (among others) if the ramification of the central vertices takes the minimum or maximum possible value. In other words, if the ramification of the central vertex (both central vertices if \( |Z(T)| = 2 \)) attains its minimum or maximum value, then the row in \( P(T) \) of a central vertex is unique, i.e. cannot be equal to the row in \( P(T) \) of a non-central vertex.

2. Locally path-congruent caterpillars

The notion of path-congruence is worth studying also from a local point of view. Given two graphs \( G_1 \) and \( G_2 \), possibly of different orders, \( v_1 \in V(G_1) \) and \( v_2 \in V(G_2) \), then we say that \( v_1, v_2 \) are path-congruent vertices if \( P(l)(v_1) = P(l)(v_2) \) for all \( l \in \mathbb{N} \). In other words, \( v_1, v_2 \) are path-congruent if their rows in \( P(G_1) \) and \( P(G_2) \) coincide.

Before proving Theorem 2.4, we remark that a formula for the number of paths of length \( l \) passing through a vertex of a caterpillar can easily be found (we restrict our attention to vertices of degree different from 1). Indeed, let \( C \) be a caterpillar of diameter \( d \). Let \( P = (v_1, \ldots, v_{d+1}) \) be a path of \( C \) of length \( d \). The vertices of \( C \) of degree different from 1 are \( v_2, \ldots, v_d \). For \( k \in \mathbb{Z} \), let \( x_k \) be the integer defined by

\[
x_k = \begin{cases} 
\deg v_k - 1 & \text{if } 2 \leq k \leq d, \\
0 & \text{otherwise}.
\end{cases}
\]

Proposition 2.1. Let \( P_l(v_i) \) be the number of paths of length \( l \) passing through \( v_i \). For all \( i, \ 2 \leq i \leq d \), the following equalities hold:

\[
\begin{align*}
P_1(v_i) &= x_i + 1, \\
P_2(v_i) &= \left(\frac{x_i + 1}{2}\right) + x_{i-1} + x_{i+1}, \\
P_l(v_i) &= x_{i-l+1} + \sum_{\substack{h,k \in \mathbb{Z} \\
2 \leq h,k \leq i \\
k-h+2l \leq d}} x_h x_k + x_{i+l-1} & \text{for } 3 \leq l \leq d.
\end{align*}
\]

Proof. The first and second equalities are obvious. For the third equality, note that all paths of length \( l \), passing through \( v_i \), have endpoints in vertices which are adjacent to \( v_h \) and \( v_k \) for some integer \( h \leq i \) and \( k \geq i \) respectively, such that \( k - h = l - 2 \) (see Fig. 2).

Consequently, the number of these paths is

\[
\sum_{\substack{h,k \in \mathbb{Z} \\
2 \leq h,k \leq i \\
k-h+2l \leq d}} x_h x_k.
\]

To this number we must add the number of paths with an endpoint in \( v_i \), that is \( x_{i-l+1} + x_{i+l-1} \), which proves the third equality. \( \square \)

Let \( \mathcal{P}_+ \) be the class of polynomials with positive integer coefficients and constant term 1. Note that \( \mathcal{P}_+ \) is closed under multiplication.

Let \( [p(x)]_1 \) be the coefficient of \( x \) in \( p(x) \in \mathcal{P}_+ \). We shall use later the fact that \( [f(x)g(x)]_1 = [f(x)]_1 + [g(x)]_1 \) for all \( f(x), g(x) \in \mathcal{P}_+ \).
Let $C_1, C_2$ be (labelled) caterpillars. We say that $C_1$ and $C_2$ are equivalent, writing $C_1 \sim C_2$, if

- the vertex sets $V(C_1), V(C_2)$ can be identified (put $V(C_1) = V(C_2) = \{u_1, \ldots, u_n\}$).
- whenever $u_i, u_j$ have degrees $> 1$, $\{u_i, u_j\}$ is an edge of $C_1$ if and only if $\{u_j, u_i\}$ is an edge of $C_2$.

Note that the equivalence classes so defined refine the isomorphism classes of caterpillars.

We can set up a bijection between the ordered pairs $(p(x), q(x))$ of polynomials in $\mathcal{P}_+$ such that $[p(x)]_1 = [q(x)]_1$ and the equivalence classes defined above, by mapping the ordered pair

$$f(x) = 1 + a_0 x + \ldots + a_k x^{k+1},$$
$$g(x) = 1 + b_0 x + \ldots + b_h x^{h+1} \quad (b_0 = a_0)$$

to the equivalence class of the caterpillars in Fig. 3, where $\deg v_j = a_j + 1$ for $j = 0, \ldots, k$ and $\deg v_i = b_i + 1$ for $i = 0, \ldots, h$. For example, the pair $(1 + x + 2x^2, 1 + x + x^2 + x^3)$ is mapped to the equivalence class represented on the left in Fig. 4, whereas the pair $(1 + x + x^2 + x^3, 1 + x + 2x^2)$ is mapped to the equivalence class represented on the right.

Note that the eccentricity $e(v_0) = \max\{\deg f(x), \deg g(x)\}$.

**Proposition 2.2.** Let $(f_1(x), g_1(x))$ and $(f_2(x), g_2(x))$ be such that $[f_1(x)]_1 = [g_1(x)]_1$ and $[f_2(x)]_1 = [g_2(x)]_1$, with $f_i(x), g_i(x) \in \mathcal{P}_+, i = 1, 2$. Let $C_1, C_2$ be (labelled) caterpillars belonging to the equivalence classes associated to $(f_1(x), g_1(x))$ and $(f_2(x), g_2(x))$, as described above, with distinguished vertices $v_0 \in C_1$ and $w_0 \in C_2$, where $[f_1(x)]_1 = [g_1(x)]_1$ and $[f_2(x)]_1 = [g_2(x)]_1$. Then $v_0$ and $w_0$ have identical path-rows if and only if $f_1(x)g_1(x) = f_2(x)g_2(x)$.

**Proof.** From the equalities for $l \geq 3$ in Proposition 2.1 one can see that the number of paths $p_l(v_0)$ is, for $l \geq 3$, the coefficient of $x^l$ in $f_1(x)g_1(x)$. The numbers $p_l(v_0)$ and $p_l(w_0)$ are not the coefficients of $x$ and $x^2$, but the claim follows easily from the first and second equality in Proposition 2.1. \[\Box\]

**Lemma 2.3.** Let $d, e_1, e_2$ be integers, $d \geq 5$, $[(d - 1)/2] < e_1 \leq e_2 < d$. There are integers $p, q, r, s$ such that

$$p \geq 1, \quad q \geq 0, \quad r \geq 2, \quad s \geq 1,$$
$$p + r = e_1, \quad r + s = e_2,$$
$$p + q + r + s = d.$$

**Proof.** It is enough to prove that there are 4-ples $p, q, r, s$ with $p = 1$, which are solutions of the stated equalities and inequalities. Let $p = 1$, $r = e_1 - 1$, $s = e_2 - r$ and $q = d - p - r - s$. The equalities are thus satisfied. We next verify the inequalities. We have $r \geq 2$, since $d \geq 5$, and $s \geq 1$ since $s = e_2 - e_1 + 1$. Also, $q \geq 0$, since $q = d - 1 - e_2$ ($e_2 < d$). Moreover, from $[(d - 1)/2] < e_1 \leq e_2$, it follows $d \leq 2e_2$, i.e. $d - e_2 \leq 2e_2$. Thus $p + q = d - r - s = d - e_2 \leq e_2 = r + s$. Finally, from $[(d - 1)/2] < e_1$, it follows $d - e_1 \leq e_1$. Therefore $s + q = d - (p + r) = d - e_1 \leq e_1 = p + r$. \[\Box\]
The following result, which we shall use in the proof of Theorem 3.1, is interesting in its own right. It says that two vertices, taken in two caterpillars of the same diameter, may have the same path-row although they have very different eccentricities. It can also be employed in problems of Geometric Tomography, see [3] for more details.

**Theorem 2.4.** For any integers \( d, e_1, e_2 \) such that \( d \geq 5 \) and \( [(d - 1)/2] < e_1 \leq e_2 < d \), there are caterpillars \( C_1, C_2 \) of diameter \( d \), and vertices \( x \in C_1, y \in C_2 \) of degree different from 1, such that \( e_{C_1}(x) = e_1, e_{C_2}(y) = e_2 \) and \( p_{C_1}(x) = p_{C_2}(y) \).

**Proof.** Let \( p, q, r, s \) be any 4-ple as in the statement of Lemma 2.3. Define polynomials \( p(x), q(x), r(x), s(x) \) as follows:

\[
\begin{align*}
    p(x) &= 1 + x + \cdots + x^p, \\
    q(x) &= \begin{cases} 
    1 & \text{if } q = 0, \\
    1 + x & \text{if } q = 1, \\
    1 + x^2 + \cdots + x^q & \text{if } q > 1, 
\end{cases} \\
    r(x) &= \begin{cases} 
    1 + x^2 + \cdots + x^r & \text{if } q = 0 \text{ or } q > 1, \\
    1 + x + \cdots + x^r & \text{if } q = 1, 
\end{cases} \\
    s(x) &= 1 + x + \cdots + x^s.
\end{align*}
\]

Define also \( f_1(x) = p(x)r(x), g_1(x) = q(x)s(x), f_2(x) = p(x)q(x), \) and \( g_2(x) = r(x)s(x) \).

Note that \( f_1(x), g_1(x), f_2(x), g_2(x) \in \mathcal{P}_+, \deg f_1(x)g_1(x) = \deg f_2(x)g_2(x) = d, \max(\deg f_1(x), \deg g_1(x)) = e_1, \) and \( \max(\deg f_2(x), \deg g_2(x)) = e_2 \).

Moreover, we can easily see that \( [f_1(x)]_1 = [g_1(x)]_1 \) and \( [f_2(x)]_1 = [g_2(x)]_1 \). Consequently, the ordered pairs \( (f_1(x), g_1(x)) \) and \( (f_2(x), g_2(x)) \) define two (equivalence classes of) caterpillars for which, by Proposition 2.2, the statement of the theorem is verified. \( \square \)

**Remark 2.5.** There are many choices of polynomials \( p(x), q(x), r(x), s(x) \) which lead to the result stated in Theorem 2.4. For instance, we could replace \( x \) with \( kx \), for some integer \( k > 1 \) in the definition of \( p(x) \) and \( s(x) \), obtaining that the degree of the distinguished vertices in \( C_1 \) and \( C_2 \) is greater than \( k \).

### 3. Joining caterpillars to obtain trees with unstable center

Now we employ two locally path-congruent caterpillars (as described in Theorem 2.4) to get a tree \( T \) and a path-congruence \( \Phi : T \to T \) which does not leave the center \( Z(T) \) invariant.

**Theorem 3.1.** Let \( D, m \) be positive integers, with \( D \geq 8 \), and \( 1 \leq m \leq [D/2] - 2 \).

There exists a tree \( T \) with \( \diam T = D \), a vertex \( c \in Z(T) \) with \( \text{ram } c = m \), and a path-congruence \( \Phi : T \to T \) such that \( \Phi(c) \notin Z(T) \).

**Proof.** If \( D \) is even, let \( d = m + D/2 \). From \( m \geq 1 \) and \( D \geq 8 \), it follows \( d \geq 5 \). Since \( m \leq D/2 - 2, d \leq D - 2 \). Thus \( [(D - 3)/2] \geq [(d - 1)/2] \). Let \( e_1 = D/2 - 1, e_2 = D/2 \). Then \( d > e_2 > e_1 > [(D - 3)/2] > [(d - 1)/2] \). By Theorem 2.4, there are two caterpillars \( C_1, C_2 \) of diameter \( d \), such that \( C_1 \) contains a vertex \( x \) of degree \( > 1 \) and eccentricity \( e_1 \), \( C_2 \) contains a vertex \( y \) of degree \( > 1 \) and eccentricity \( e_2 \), and \( p_{C_1}(x) = p_{C_2}(y) \) (Fig. 5 only shows, of \( C_1 \) and \( C_2 \), a path of maximum length).

![Fig. 5. Joining of caterpillars (diam \( C_1 = \diam C_2 = d \)).](image-url)
Join $C_1$ and $C_2$ by adding an edge $xy$. The tree $T$ so obtained has $\text{diam } T = D/2 − 1 + D/2 + 1 = D$, $\text{ram}_T y = m$, and $y \in Z(T)$, whereas $x \notin Z(T)$.

If $D$ is odd, let $d = m + \lfloor D/2 \rfloor$. As before, $d \geq 5$, and since $m \leq \lfloor D/2 \rfloor − 2$, then $d \leq D − 3$. Thus $\lfloor (D − 4)/2 \rfloor \geq \lfloor (d − 1)/2 \rfloor$. Let $e_1 = \lfloor D/2 \rfloor − 1$, $e_2 = \lfloor D/2 \rfloor$. Then $d > e_2 \geq e_1 > \lfloor (D − 4)/2 \rfloor \geq \lfloor (d − 1)/2 \rfloor$. From Theorem 2.4 (supplemented with Remark 2.5) it follows that there are caterpillars $C_1$, $C_2$ of diameter $d$, such that $C_1$ contains a vertex $x$ of degree $> 2$ and eccentricity $e_1$, $C_2$ contains a vertex $y$ of degree $> 2$ and eccentricity $e_2$, and $p_{C_1}(x) = p_{C_2}(y)$ (Fig. 6 only shows, of $C_1$ and $C_2$, a path of maximum length and one leaf connected to $x$ and $y$).

Splice $C_1$ and $C_2$ by identifying $x'$ and $y'$. The tree $T$ so obtained has $|T| = |C_1| + |C_2| − 1$, $\text{diam } T = |D/2| − 1 + |D/2| + 2 = D$, $\text{ram}_T y = m$, and $y \in Z(T)$, whereas $x \notin Z(T)$.

In either of the above cases ($D$ even or odd) consider the map $\Phi : T \rightarrow T$ which exchanges $x$ and $y$ and fixes all other vertices. Note that $\Phi(y) \notin Z(T)$. For each $l \in \mathbb{N}$, denote by

1. $p_l(x, y)$, the number of paths in $T$, of length $l$, passing through both $x$ and $y$,
2. $p_l(x, \overline{y})$, the number of paths in $T$, of length $l$, passing through $x$ and missing $y$,
3. $p_l(\overline{x}, y)$, the number of paths in $T$, of length $l$, passing through $y$ and missing $x$.

Note that $p_l(x, \overline{y})$ is also equal to the value of $p_l(x)_{C_1}$, and $p_l(\overline{x}, y)$ is also equal to the value of $p_l(y)_{C_2}$. Consequently, by construction, it is $p_l(x, \overline{y}) = p_l(\overline{x}, y)$, and thus we have

\[ p_l(x)_T = p_l(x, y) + p_l(x, \overline{y}) = p_l(x, y) + p_l(\overline{x}, y) = p_l(y)_T. \]

Thus $\Phi$ is in fact a path-congruence. □

**Remark 3.2.** The case $D$ odd and $m < \lfloor D/2 \rfloor − 2$ could also be treated by a joining construction.

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**References**