Nilpotent Ordinary Differential Operators
with Polynomial Coefficients

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INTRODUCTION

We prove that nilpotent elements in the Weyl algebra have unique normal operator closures, and give a spectral representation.

We consider the Schwartz space $\mathcal{S}$ as a dense subspace of $L^2(\mathbb{R})$, and the operators $(Pf)(x) = -if'(x)$, $(Qf)(x) = xf(x)$, satisfying $(\text{ad} P)(Q)f = [P, Q]f = PQf - QPf = -if$, $f \in \mathcal{S}$, in the usual Schrödinger representation. Let $W$ denote the Weyl algebra which is the associative algebra over $\mathbb{C}$ with unit generated by $P$ and $Q$. Hence, $W$ is represented as an algebra of unbounded operators with $\mathcal{S}$ as the common invariant domain.

We assume that some $A \in W$ satisfies the local nilpotency condition: $(LN)$. There is an integer $n$ such that

$$(\text{ad} A)^n(P) = (\text{ad} A)^n(Q) = 0.$$ 

We prove that there is a unitary operator $U$ on $L^2(\mathbb{R})$, and a complex polynomial in one variable $\phi$, such that: (i) $U(\mathcal{S}) = \mathcal{S}$; (ii) $\alpha = \text{Ad}(U)$ implements an automorphism of $W$; (iii) $A = U \circ \phi(P) \circ U^* = \alpha(\phi(P))$ holds as an operator identity on $\mathcal{S}$. It follows from (i) and (iii) that $A$ is essentially normal, i.e., that the operator closure $\overline{A}$ has a spectral resolution since this is known to hold for $\overline{\phi(P)}$. As a by-product of the proof, we also show that every $\alpha \in \text{Aut}(W)$ satisfying $\alpha(B^*) = \alpha(B)^*$, $B \in W$, is unitarily implemented as in (i) and (ii). The result was conjectured by R. T. Powers.

It is not clear, a priori, that the nilpotency condition on a polynomial differential operator $A$ implies that $A$ is formally normal, i.e., that the real, and the imaginary, parts of $A$ commute formally. There are then three parts to the conclusion: If $A$ satisfies $(LN)$, then the two operators $A_1 = (A + A^*)2^{-1}$, and $A_2 = (A - A^*)(2i)^{-1}$ commute on the Schwartz

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space $\mathcal{H}$. Secondly, the closed operators $\tilde{A}_j$, $j = 1, 2$, have commuting spectral resolutions [32]; and, thirdly, the spectral type of $A$ is that of a constant coefficient ordinary differential operator on $\mathbb{R}$.

If $A$ is nilpotent, then the inner derivation, $\text{ad} A$, generates a one-parameter group of automorphisms. We consider one-parameter groups of *-automorphisms for a wider class of elements $A$ in $\mathcal{W}$, and prove that these one-parameter groups are implemented by strongly continuous one-parameter groups of unitary operators on $L^2(\mathbb{R})$.

In Section 5, we apply this to give a new integrability theorem for the Heisenberg commutation relation, $[P, Q] = -i$.

1. **The Algebra**

The Weyl algebra is defined, purely algebraically, as the associative algebra with unit and generators $p, q$ satisfying $[p, q] = pq - qp = 1$, where $1$ denotes the unit [4]. This algebra $\mathcal{W}$ has been considered over an arbitrary field, and a long list of remarkable properties are known for $\mathcal{W}$: [1–3, 12, 15, 25, 29]. In this paper, we shall consider only the field of complex numbers $\mathbb{C}$, and, moreover, we shall consider only a particular representation of $\mathcal{W}$, the so-called Schrödinger representation. The representation space will be the Hilbert space $L^2(\mathbb{R})$ of all square-integrable $\mathbb{C}$-valued functions. The identity operator will be denoted $I$. It is customary in quantum mechanics to work with slightly different generators, viz., $P$, resp...$Q$, where

$$(Pf)(x) = -if'(x) \quad \text{(derivative)},$$

and

$$(Qf)(x) = xf(x) \quad \text{(multiplication)}.$$ We have the Heisenberg commutation relation, $[P, Q] = -iI$. The Heisenberg–Lie algebra $\mathfrak{g}$ is the three-dimensional real Lie algebra with basis

$$\{iP, iQ, iI\}. \quad (1)$$

We shall think of $\mathcal{W}$ as the complex enveloping algebra [4] of $\mathfrak{g}$. If $(a, b, c)$ denote the three real parameters for the Heisenberg group, then we may regard $\mathfrak{g}$ as the infinitesimal operator Lie algebra [10] of the strongly continuous unitary irreducible representation $\pi$, given by

$$(\pi(a, b, c)f)(x) = e^{ic}e^{ibx}f(x + a).$$
This is the Schrödinger representation \([18]\) with Planck's constant normalized to one. It is well known \([16]\) that the space of \(C^\infty\)-vectors for \(\pi\) coincides with the usual Schwartz space \(S\) of rapidly decreasing smooth functions on \(\mathbb{R}\). It is a deeper fact that this space also coincides with the Gårding space for \(\pi\) \([6]\).

This means that we may regard \(W\) as an algebra of unbounded operators in \(L^2(\mathbb{R})\) having \(S\) as a common invariant domain. Moreover, \(W\) inherits the structure of a \(*\)-algebra from the \(*\)-operation of operator adjoint, 

\[ \langle Af, g \rangle = \langle f, A^* g \rangle, \quad f, g \in S, \]

where \(\langle \cdot, \cdot \rangle\) denotes the inner product, \(\langle f, h \rangle = \int f(x) \overline{h(x)} \, dx\) of \(L^2(\mathbb{R})\) \([24]\).

In addition to the Schrödinger representation, we shall make use of the Shale–Weil representation \([11, 20, 22, 23]\). This is a unitary representation \(R\) of a two-sheeted metaplectic covering group \(G\) of \(SL_2(\mathbb{R})\). The corresponding infinitesimal Lie algebra \(\mathfrak{g}_S\) is a copy of \(\mathfrak{sl}_2(\mathbb{R})\) and has a basis 

\[ \{ i(P^2 + Q^2), i(PQ + QP), i(P^2 - Q^2) \}, \]

(2)

corresponding to the three matrices \(\{X_0, X_1, X_2\}\), where

\[
X_0 = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad X_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_2 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

The Hermite functions will be denoted \(\{h_n\}_{n = 0, 1, \ldots}\). They form an orthonormal basis of eigenfunctions for the harmonic oscillator Hamiltonian \([25, 27]\). More precisely, \((P^2 + Q^2)h_n = (1 + 2n)h_n, n = 0, 1, 2, \ldots, \{h_n\} \subset S\), and \(\{h_n\}\) consists of analytic vectors for each of the four operators, \(dR(X_0), dR(X_1), dR(X_2),\) and \(A = dR(X_0^2 + X_1^2 + X_2^2)\). It follows then from Nelson's lemma \([14, \text{Lemma 6.2}]\) that \(\{h_n\}\) consists of analytic vectors for each second order polynomial

\[ \alpha + \beta P + \gamma Q + \delta P^2 + \varepsilon PQ + \zeta Q^2 \]

(3)

with complex coefficients \(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta\). (However, the examples \(PQ\), \(P^2\), and \(P^2 - Q^4\), show that the corresponding result is false for third, or fourth, order noncommutative polynomials in \(P\) and \(Q\). For a systematic analysis of a wider class of polynomial ordinary differential operators, the reader is referred to \([30, 31]\).}
2. NILPOTENT ELEMENTS

Let $A$ be an element in $W$, and define the subset

$$N(A) = \{ B \in W : (\text{ad } A)^n(B) = 0 \text{ for some } n \}.$$ 

It is essential that the positive integer $n$ is allowed to depend on the element $B$. It is well known, and easy to see, that $N(A)$ is a subalgebra of $W$ for each $A$. It is immediate from Leibniz’ formula,

$$(\text{ad } A)^n(B, B_2) = \sum_{i=0}^{n} \binom{n}{i} (\text{ad } A)^i(B_1)(\text{ad } A)^{n-i}(B_2),$$

that, if $(\text{ad } A)^n(B_j) = 0$, $j = 1, 2$, then the right-hand side vanishes when $n \geq n_1 + n_2$. We say that $A$ is nilpotent if $N(A) = W$, and this nilpotency condition is satisfied when $(\text{ad } A)^n(P) = (\text{ad } A)^n(Q) = 0$. We say that the smallest integer $n$ for which this holds is the degree of nilpotency.

Two observations are immediate and easy to check. For every polynomial $\phi$ in a single variable, each of the elements $\phi(P)$ and $\phi(Q)$ is nilpotent of degree 2. Moreover, if $\alpha$ is an automorphism of $W$, and $A$ is nilpotent, then it follows that $\alpha(A)$ is also nilpotent. Of course, the degree of nilpotency of $\alpha(A)$ is generally different from that of $A$.

A theorem in pure algebra, due to Dixmier, furnishes a converse to these two facts.

**Theorem (D [3]).** For every nilpotent element $A$ in $W$, there is an automorphism $\alpha$, and a polynomial $\phi$ in a single variable, such that $A = \alpha(\phi(P))$.

A different way of putting this is that, up to (algebraic) automorphism, the constant coefficient ordinary differential operators are the only polynomial differential operators which satisfy the nilpotency condition $(LN)$ of the Introduction.

An independent reason for the interest in the condition $(LN)$ is its relationship to one-parameter groups of automorphisms of $W$.

Indeed, if $A$ in $W$ satisfies $(LN)$, then

$$\beta_t(B) = e^{t \text{ad } A}(B)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} (\text{ad } A)^n(B)$$

defines a one-parameter group of automorphisms $\{\beta_t : t \in \mathbb{R}\} \subset \text{Aut}(W)$, where the right-hand side is a finite power series for each $B$ in $W$. But, of
course, the number of terms depends on $B$. It is immediate that $\beta_\epsilon$ is a group of $*$-automorphisms, i.e.,

$$\beta_\epsilon(B^*) = \beta_\epsilon(B)^*, \quad t \in \mathbb{R}, B \in W,$$

if and only if $iA$ is formally Hermitian, i.e., $\langle Af, g \rangle = -\langle f, Ag \rangle$, $f, g \in \mathcal{S}$.

There are many formally Hermitian elements in $W$ which are not essentially self-adjoint in the sense of the spectral theorem [27]; i.e., they have nontrivial deficiency subspaces so that, when they are closed up from the Schwartz space, the corresponding closed operators do not have spectral resolutions. The two operators, $PQP$ and $P^2 - Q^4$, for example, are formally Hermitian, but $PQP$ has deficiency indices $(1, 1)$ while the second operator has indices $(2, 2)$. The question of deficiency indices is purely analytic: The functions in the deficiency subspaces are solutions to ordinary differential equations. But the nilpotency condition in turn is purely algebraic. The interest in our result (see the Introduction) lies perhaps in the observation that the purely algebraic assumption $(LN)$ has a fundamental analytic consequence; certain differential equations have the trivial zero solution as its only square integrable solution.

Our main result was conjectured by R. T. Powers in a conversation with the author, and Powers also supplied us with motivating details behind the conjecture. We are pleased to acknowledge our indebtedness.

The Weyl algebra has received a great amount of attention recently, both from the point of view of pure algebra [3, 4, 7, 15], and from the point of view of applications: representation theory [2, 5, 15, 25], analysis [1, 10, 12, 28, 29], and quantum mechanics [13, 17, 20, 24].

Results for the Weyl algebra have consequences for arbitrary enveloping algebras of nilpotent Lie groups. A remarkable special instance of this is the following result of Dixmier [2], which has been generalized in different directions, most notable [15]: Let $\pi$ be a unitary irreducible representation of a nilpotent Lie group $N$, and let $E$ be the complex enveloping algebra of the Lie algebra of $N$. Let $d\pi$ denote the infinitesimal representation of $E$, and let $I$ be the kernel of $d\pi$. Then Dixmier showed that the quotient, $E/I$, is isomorphic to a Weyl algebra over the complex numbers.

Using this, in combination with our theorem, we get that for every unitary irreducible representation $\pi$ of a nilpotent Lie group $N$, and for any $X$ in the Lie algebra of $N$, the operator $d\pi(X)$ has the same spectral representation as a constant coefficient partial differential operator on $L^2(\mathbb{R}^n)$. To see this, we note that, under the isomorphism

$$E/I \simeq W_n \simeq W_1 \otimes \cdots \otimes W_1,$$

the operator $d\pi(X)$ gets mapped into a nilpotent element in $W_n$, and our theorem applies mutatis mutandis to this element.
Since the present paper is addressed to a mixed audience, we list the following background references: [9] to Lie algebras, [8] to Lie groups, [4] to enveloping algebras, [27] to differential equations, and [10] to representations of Lie groups, and commutation relations for operators. We shall use analytic vectors and analytic domination for families of operators, and we refer to Nelson's paper [14] for this.

Finally, we note that the nilpotency condition occurs in a different context [21] as well.

3. The Metaplectic Group Acting on $W$

We saw in Section 1 that the metaplectic group $G$ (which is a two-fold covering group for $SL_2(\mathbb{R})$) acts via the Shale–Weil representation $R$ as a group of unitary operators on $L^2(\mathbb{R})$. Using the Lie algebra basis (2), it can be checked that $L^2(\mathbb{R})$ decomposes under $R$ into the two irreducible subspaces which are spanned by the Hermite functions,

\[ \{ h_n: n = 0, 2, 4, \ldots \}, \quad \text{resp.} \quad \{ h_n: n = 1, 3, 5, \ldots \}. \]

**Lemma 3.1.** The Shale–Weil representation implements an action of the metaplectic group $G$ as a group of $*$-automorphisms of the Weyl algebra $W$, i.e., $\rho_g(B) = R(g) BR(g)^* \in W$ for all $g \in G$, and $B \in W$.

Moreover, the map, $g \rightarrow \langle \rho_g(B) f_1, f_2 \rangle$ is $C^\infty$ on $G$ for all $B \in W$, and $f_1, f_2 \in \mathcal{S}$ (the Schwartz space).

**Remark 3.2.** It is known, and easy to check, that the formulas

\[ \alpha_t(B) = \exp \left( i t P^2/2 \right) B \exp \left( - i t P^2/2 \right) \]

and

\[ \beta_t(B) = \exp \left( i t Q^2/2 \right) B \exp \left( - i t Q^2/2 \right), \]

for $t \in \mathbb{R}$, $B \in W$, define one-parameter groups of $*$-automorphisms of $W$ such that $t \rightarrow \langle \alpha_t(B) f_1, f_2 \rangle$, and $t \rightarrow \langle \beta_t(B) f_1, f_2 \rangle$, is smooth for all $B \in W$, $f_1, f_2 \in \mathcal{S}$. In fact, this is immediate from formula (4) applied to $A = iP^2/2$, resp., $A = iQ^2/2$, since these two elements act nilpotently. As noted by Dixmier, a more general pair of one-parameter subgroups of $\text{Aut}(W)$ result if we take instead $A = iP^{n+1}/n + 1$, resp., $A = iQ^{n+1}/n + 1$. 


We shall denote the corresponding one-parameter subgroups, \( \{\alpha(t, n) : t \in \mathbb{R}\} \), resp., \( \{\beta(t, n) : t \in \mathbb{R}\} \), and we have the formulas,

\[
\begin{align*}
\alpha(t, n)(P) &= P \\
\alpha(t, n)(Q) &= Q + tP^n \\
\beta(t, n)(P) &= P - tQ^n \\
\beta(t, n)(Q) &= Q.
\end{align*}
\]

The two directions in the Lie algebra \( \mathfrak{g}_S \approx \mathfrak{sl}(2) \) resulting from taking \( n = 1 \) play a special role. It is known that \( \mathfrak{g}_S \) is generated as a Lie algebra by these two special directions. But the process of taking linear combinations and Lie commutators involves certain limit considerations where the convergence depends on regularity properties of various Trotter products [26]. This works well in the case \( n = 1 \) because of the algebraic properties of the Shale–Weil representation, but singularities arise for \( n > 1 \); cf. Example 4.3.

**Proof.** Let \( G \) be an element in \( G \), and assume \( g \) covers some \([a, b] \in SL_2(\mathbb{R})\), then the basic property of the Shale–Weil representation is the covariance relation

\[
R(g)(c_1 P + c_2 Q + c_3 I) = (ac_1 + bc_2)P + (cc_1 + dc_2) Q + c_3 I. \tag{6}
\]

Moreover, \( \rho(g) = Ad(R(g)) \) acts as a Lie isomorphism of the Heisenberg operator Lie algebra \( \mathfrak{g} \) which is specified in terms of the basis (1). Since the Weyl algebra \( \mathcal{W} \), which is simple [4], is the enveloping algebra of \( \mathfrak{g} \), each Lie automorphism, \( \rho_g = Ad(R(g)) \in \text{Aut}(\mathfrak{g}) \), extends canonically to an associative algebra-automorphism, also denoted by \( \rho_g \), of the \(*\)-algebra \( \mathcal{W} \). By the Poincaré–Birkhoff–Witt theorem, the elements \( \sum_{k, r \geq 0} c_{k, r} P^k Q^r \) (finite sum) span \( \mathcal{W} \), \( c_{k, r} \in \mathbb{C} \), and the monomials \( P^k Q^r \) are linearly independent. If \( B \in \mathcal{W} \) is given by such a polynomial expression, then

\[
\rho_g(B) = \sum c_{k, r} \rho_g(P)^k \rho_g(Q)^r
\]

\[
= \sum c_{k, r} (Ad(R(g))P)^k (Ad(R(g))Q)^r
\]

\[
= \sum c_{k, r} Ad(R(g))(P^k) Ad(R(g))(Q^r)
\]

\[
= Ad(R(g)) \sum c_{k, r} P^k Q^r
\]

\[
= Ad(R(g))(B),
\]

where we have used formula (6) above in the second identity in the chain.
The proof of formula (6), in turn, can be derived from formulas (5) and standard Lie theory. We sketch the details since some essential points should be noted. It is enough to check (6) for \( g \) in a coordinate neighbourhood of \( e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and we may assume \( g = \exp X \), where \( X \in \mathfrak{sl}_2(\mathbb{R}) \). We claim that the set of Lie algebra elements \( X \), such that (6) holds for \( g = \exp X \), is a Lie subalgebra. To see this, assume that \( X_1 \) and \( X_2 \) both have the stated property. Clearly, then each of the maps

\[
t \to \langle R(\exp tX_1)(c_1 P + c_2 Q + c_3) R(\exp tX_2)^* f_1, f_2 \rangle
\]

is smooth for all \( f_1, f_2 \in \mathcal{S} \). Inserting the formulas

\[
\left( \exp \frac{t}{n} X_1 \exp \frac{t}{n} X_2 \right)^n = \exp \left\{ nt(X_1 + X_2) + O\left( \frac{1}{n} \right) \right\}
\]

and

\[
\left\{ \exp \left( -\frac{t}{n} X_1 \right) \exp \left( -\frac{t}{n} X_2 \right) \exp \left( \frac{t}{n} X_1 \right) \exp \left( \frac{t}{n} X_2 \right) \right\}^n = \exp \left\{ n^2 [X_1, X_2] + O\left( \frac{1}{n} \right) \right\}
\]

into \( \langle R(\cdot)(c_1 P + c_2 Q + c_3) R(\cdot)^* f_1, f_2 \rangle \), the desired conclusion follows for \( X_1 + X_2 \), resp., \([X_1, X_2] \), when passing to the limit \( n \to \infty \).

On the other hand, we have the conclusion for a set of Lie generators, viz., \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \), by virtue of formulas (5) above.

Once the formula

\[
\rho_g(B) = R(g) BR(g)^*
\]

is established for \( g \in G \) and \( B \in \mathcal{W} \), as an operator identity on \( \mathcal{S} \), it is immediate that \( g \to \langle \rho_g(B) f_1, f_2 \rangle \) is smooth for \( f_1, f_2 \in \mathcal{S} \), since \( \mathcal{S} \) is known to coincide with the space of smooth vectors for the Shale–Weil representation \( R \).

**Remark 3.3.** Lemma 3.1 is stronger than saying just that every automorphism \( \rho(g) \), \( g \in G \), is unitarily implemented. Lemma 3.1 states that the specific Shale–Weil representation provides a unitary implementation. As a consequence, we get that for every one-parameter subgroup \( \rho(\exp tX) \), \( t \in \mathbb{R} \), we have an implementing unitary one-parameter group, viz., \( R(\exp tX) \). The weaker assertion, that every \( \rho(g) \) is implemented by some unitary operator would not give that. If \( \rho(g) = \text{Ad}(U(g)) \) for a family of unitaries \( \{ U(g) : g \in G \} \), then we do have a projective representation, i.e.,

\[
U(g_1) U(g_2) = \beta(g_1, g_2) U(g_1 \cdot g_2).
\]
where $\beta(\cdot, \cdot)$ is a scalar valued function on $G \times G$ satisfying

$$\beta(g_1, g_2) \beta(g_1 g_2, g_3) = \beta(g_1, g_2) \beta(g_2, g_3).$$

This is easy to derive directly from the fact that the Schrödinger representation is irreducible.

This weaker conclusion, that every $\rho(g)$ is implemented, may be derived from purely algebraic considerations. It can be shown (details in Example 4.3) that every $\rho(g)$ factors as a finite product of automorphisms picked from the set \{ $\alpha(t, 1), \beta(s, 1)$: $t, s \in \mathbb{R}$ \}.

4. Automorphisms of the $\ast$-algebra $W$

Let $\alpha$ be an automorphism of the Weyl algebra $W$ satisfying $\alpha(B^*) = \alpha(B)^*$, $B \in W$. We shall prove in this section that $\alpha$ is unitarily implemented. Moreover, it is possible to choose an implementing unitary $U$ which leaves the Schwartz space $\mathcal{S}$ invariant and has a particular factorization. This will then be true for all implementing unitaries since $U$ is unique up to a scalar multiple, by virtue of irreducibility of the Schrödinger representation.

Turning now to the details, we recall the following three kinds of unitary operators: $R(g)$, $g \in G$, $U(t, n)$, and $V(s, m)$, for $t, s \in \mathbb{R}$, $n, m = 0, 1, \ldots$ which were introduced in Sections 1 and 3. The purpose of this section is to show that $U$ exists and that $U = U_1 U_2 \cdots U_r$, where the unitary factors $U_i$ in the finite product may be chosen from the set of operators \{ $R(g)$, $U(t, n)$, $V(s, m)$ \} described above. (It is essential that it is possible to factor $U$ as a finite product of the above type because various limit operators tend to easily "escape" the "polynomial" algebra $W$.)

We first need a preliminary

**Lemma 4.1.** Each of the unitary operators, $R(g)$, $U(t, n)$, and $V(s, m)$, leaves the Schwartz space $\mathcal{S}$ invariant.

**Proof:** For $R(g)$, this follows from the known fact that $\mathcal{S}$ coincides with the space of all $C^\infty$-vectors for the Shale–Weil representation $R$ [16]. The special case, $g_1 = \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$, is of particular interest. The automorphism $\rho(g_1) = \text{Ad}(R(g_1))$ satisfies $\rho(g_1)(P) = Q$ and $\rho(g_1)(Q) = -P$, and an easy argument shows that, in fact, $R(g_1)$ is a multiple of the usual Fourier transform, regarded as a unitary operator, $F$ say, in $L^2(\mathbb{R})$. In fact, $F = \sqrt{i} R(g_1)$. An independent proof of the invariance of $\mathcal{S}$ under the Fourier transform results.

Since we have $\rho(g_1)(U(t, n)) = FU(t, n) F^* = V(t, n)$, it is enough to show that each of the unitary operators $V(t, n)$ leaves $\mathcal{S}$ invariant.
But \((V(t, n)f)(x) = \exp (it x^n + 1/n + 1) f(x)\), and we are left with proving that multiplication by a function of the form \(\exp (i\phi (x))\) leaves \(\mathcal{S}\) invariant when \(\phi\) is a real valued polynomial in \(x\). The proof of this is easy: It can be based on the commutation relation

\[
P^n \exp(i\phi(x)) = \exp(i\phi(x))(P - \phi'(x))^n,
\]

which in turn is immediate from (5).

We are now ready to prove the main

**Lemma 4.2.** Let \(\alpha\) be a \(*\)-automorphism of \(W\). Then \(\alpha\) is implemented by a unitary \(U\), and it is possible to factor \(U\) as a finite product of operators, \(R(g)\), \(U(t,n)\), and \(V(s,m)\), where \(g\) ranges over the metaplectic group, \(t\) and \(s\) are real numbers, and \(n\) and \(m\) are nonnegative integers.

**Proof.** The proof is based on the algebraic steps in Dixmier's paper. We shall go through a sequence of reductions where each reduction is based on an induction argument. In each step we shall make use of the three types of unitary operators which are listed in the lemma above.

We shall consider first the special case when \(\alpha\) is assumed to satisfy \(\alpha(P) = P + cI\) for \(c \in \mathbb{R}\) (see attached footnote). Then \([P, \alpha(Q) - Q]\) = \(\alpha([P, Q]) - [P, Q] = \alpha(-il) + il = 0\). Let \(C(P) = \{B \in W: [P, B] = 0\}\). It is known [3, Proposition 5.2] and easy to check that \(C(P)\) consists of elements of the form \(\phi(P)\), where \(\phi\) is a complex polynomial in a single variable. In other words, \(C(P) = C[P]\). It follows that \(\alpha(Q) - Q = t_0 P^n + t_1 P^{n-1} + \cdots + t_{n-1} P + t_n\). Let \(v\) be the product automorphism \(v = \alpha(t_n, 0) \alpha(t_{n-1}, 1) \cdots \alpha(t_0, n)\). Using the first two formulas in (5), we get \(v(P) = P\) and \(v(Q) = \alpha(Q)\). Since an automorphism is uniquely determined by its value or the generators \(P\) and \(Q\), it follows that \(v = \alpha\). Since \(v\) is implemented by the product unitary \(\exp(it_0 P) \exp(it_{n-1} P^2/2) \cdots \exp(it_0 P^{n+1}/n + 1) = U(t_0, 0) \cdots U(t_n, n)\), so is \(\alpha\), and the conclusion in the lemma follows.

Hence, it remains to consider the reduction to the case \(\alpha(P) = P + cI\). Let \(\alpha\) be a given \(*\)-automorphism. Since \(C(P) = C[P]\), it follows that \(X = \alpha(P)\) has the same property, viz., \(C(X) = C[X]\). We show first there is an automorphism \(\beta\) which is implemented by a product unitary (as described in the conclusion of the lemma) such that \(\beta(X)\) is a polynomial in \(P\). But \(P\) then commutes with \(\beta(X)\), and \(P\) must also be a polynomial in \(\beta(X)\). It follows that \(\beta(X) = a_0 I + a_1 P\), for \(a_1, a_0 \in \mathbb{R}\), \(a_1 \neq 0\), is the only possible

1 Using the automorphism \(\beta(c, 0)\), considering \(\alpha \circ \beta(c, 0)\), one may further reduce to the case \(c = 0\).
polynomial. Since \( \rho(G) \) acts as a group of automorphisms on the Heisenberg–Lie algebra (cf. (6)) there is a \( g \in G \) such that \( \rho(g)(\beta(X)) = P + cI \). Hence, we have proved that \( \rho(g) \circ \beta \circ \alpha \) sends \( P \) to \( P + cI \). It follows from the first part of the proof that it is implemented, and we conclude that \( \alpha \) is implemented as well.

It remains to construct the automorphism \( \beta \) with the stated properties, i.e., \( \beta \) is implemented and \( \beta(X) \in \mathbb{C}[P] \). We will point out that \( \beta \) may be constructed as a finite product of the three types of automorphisms, \( \rho(g) \), \( \alpha(t, n) \), and \( \beta(s, m) \). Since each of these automorphisms is implemented, it will follow that the product \( \beta \) is implemented. The existence of this product automorphism \( \beta \) can be established as in Dixmier’s paper [3], Sections 7 and 8. Suppose \( X = \sum c_{ij}P^iQ^j \). According to [3, Lemma 8.7], only the case \( X = c_{r,0}P^r \cdot \cdots \cdot c_{0,s}Q^s \) agrees with the nilpotency assumption on \( X \). Using the Fourier transform, we may reduce to the case \( r > s \). If further \( s \leq 1 \), then [3, Lemma 8.5] provides us with an automorphism \( \nu \) which is a finite product of automorphisms \( \alpha(t, n) \) for different values of \( t, n \) such that \( \nu(X) \) is in the Heisenberg–Lie algebra. We may then (cf.(6)) adjust with \( g \in G \) such that \( \rho(g)(\nu(X)) = P + cI \).

The case \( s = 2 \) is ruled out by the nilpotency condition on \( X \); cf. [3, Proposition 5.3 and Lemma 8.4] or section 6 below.

Finally, the case \( s > 2 \) requires a special induction argument. In this case, [3, Proposition 7.4] leaves us with only the following two possibilities:

(i) \( X = (P^{r/s} + aQ)^s \cdot \sum b_{ij}P^iQ^j \) with \( r \) divisible by \( s \), \( a, b_{ij} \in \mathbb{R} \), and the summation running only over indices \((i, j)\) such that \( si + rj < rs \); or

(ii) \( X = (P + a_1Q)^h (P + a_2Q)^{-h} \cdot \sum b_{ij}P^iQ^j \) now with integral \( h \), \( 0 \leq h \leq r \), \( a_i, b_{ij} \in \mathbb{R} \), and the summation running over indices with \( i + j < r \).

In case (i), a calculation shows that the element \( \alpha(-a^{-1}, r/s)(X) \) has lower degree. In case (ii), one checks that \( \alpha(-a_1^{-1}, 1)(X) \) has lower degree. Hence, induction applies, and the desired conclusion holds. We refer to [3, proof of Lemma 8.8] for further details. The arguments above suffice for checking that unitary implementability is preserved in each step; the important fact to note is that limit-considerations are involved only for products of automorphisms inside the closed subgroup \( \rho(G) \subset \text{Aut}(W) \), and here Lemma 3.1 above applies.

**Examples 4.3.** We show that it is possible to factor every unitary \( R(g) \), for \( g \in G \), as a finite product of unitaries of the form \( U(t, 1) \) and \( V(s, 1) \). As a consequence, every automorphism \( \rho(g) \) factors, \( \nu_1 \circ \nu_2 \circ \cdots \circ \nu_9 \), where the factors \( \nu_i \) in the string may be picked from the set \( \{ \alpha(t, 1) \cdot \beta(s, 1) \} \) for different values of the real parameters \( t, s \). (Presumably, nine is not best possible.)
Suppose \( g \in G \) covers the element \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) in \( SL_2(\mathbb{R}) \). If \( a \neq 0 \), we have the factorization
\[
\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 0 \end{bmatrix} \begin{bmatrix} 0 & k^{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},
\]
where \( k = a \), \( s = ba^{-1} \), \( t = ca^{-1} \), and \( d \) is determined from \( ad - bc = 1 \). The middle diagonal matrix, in turn, factors as
\[
\begin{bmatrix} k & 0 \\ 0 & k^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k & 1 \\ -k & 1 \end{bmatrix}.
\]
Hence, if \( a \neq 0 \), \( R(g) \) factors into a product of six unitaries, where three are of the form \( U(t, 1) \), and the other three, \( V(s, 1) \). The factors alternate.

If \( a = 0 \), then \( b \neq 0 \) by virtue of \( ad - bc = 1 \). Hence, the argument from above applies to the product matrix
\[
\begin{bmatrix} a & k \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & d \end{bmatrix}.
\]
The inverse of the matrix on the left is \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), and it is closely related to the Fourier transform. We shall show in the next paragraph that it factors into a product with three factors of the desired type. It follows that the asserted factorization holds true also when \( a = 0 \).

Let \( g_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in G \), and let \( F \) be the Fourier transform. We saw in the proof of Lemma 4.1 that \( R(g_1) = Fe^{-in/4} \), with suitable normalization. For the automorphism \( \rho(g_1) \) we therefore have \( \rho(g_1) = Ad(F) \). Using the basis (2), a direct calculation yields \( R(g_1) h_n = e^{in(-1/4 - n/2)} h_n \), and therefore,
\[
Fh_n = e^{in/4} e^{-in(1/4 + n/2)} h_n = e^{-i\pi n/2} h_n = (-i)^n h_n.
\]

We now record the fact that the Fourier transform factors. The matrix \( g_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \), factors as follows:
\[
g_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.
\]
Applying the representation \( \rho \) from Lemma 3.1 to this identity, we get \( \rho(g_1) = \alpha(-1, 1) \cdot \beta(-1, 1) \cdot \alpha(-1, 1) \). Since each of these three automorphisms is unitarily implemented, we get
\[
F = cU(-1, 1) V(-1, 1) U(-1, 1) = c \exp(-iP^2/2) \exp(-iQ^2/2) \exp(-iP^2/2) = c \exp(iD_x^2/2) \exp(-ix^2/2) \exp(iD_x^2/2).
\]
for some $c \in \mathbb{C}$, $|c| = 1$. We have not previously seen this factorization of the Fourier transform recorded in the literature.

5. COROLLARIES

In this section, we derive the result from the Introduction and examine representations of the Weyl algebra.

**Corollary 5.1.** Let $A$ be a nilpotent element in $W$. Then there is a unitary operator $U$ which leaves the Schwartz space $\mathcal{S}$ invariant, and a polynomial $\varphi$ such that $A = U\varphi(P)U^*$ on $\mathcal{S}$, and $W$ is covariant under $U$.

Since $\varphi(P)$ is essentially normal on $\mathcal{S}$, it follows that $A$ has this property as well.

**Proof.** According to Theorem D, there is a $*$-automorphism $\alpha$ such that $\alpha(A)$ is a polynomial in $P$. By Lemma 4.2, $\alpha$ is implemented by a unitary $U$ which leaves $\mathcal{S}$ invariant. It follows that $U^*$ satisfies the conditions in the conclusion of the corollary.

**Corollary 5.2.** Let $P$ and $Q$ be a pair of closed Hermitian operators which are defined on an invariant dense domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$. Assume

- (a) $[P, Q]f = -if$, $f \in \mathcal{D}$;
- (b) $P^2 + Q^2$ is essentially self-adjoint on $\mathcal{D}$;
- (c) the pair, $P$, $Q$, is irreducible; and
- (d) $\mathcal{D}$ is the largest space which is contained in the domain of both $P$ and $Q$, and invariant under $P$ and $Q$.

Then it follows that, for every $*$-automorphism $\alpha$ of the Weyl algebra on $P$, $Q$, the operator $\alpha(P^2 + Q^2)$ is also essentially self-adjoint on $\mathcal{D}$.

**Proof.** By the Rellich–Dixmier theorem [5, 19, 18], or alternatively, Nelson's theorem [14, Theorem 5], the representation is integrable by virtue of assumption (6). By the Stone–von Neumann uniqueness theorem [18], and (c), the representation above is unitarily equivalent to the Schrödinger representation on $L^2(\mathbb{R})$, such that $P$ corresponds to $-i\frac{d}{dx}$, and $Q$ to multiplication by $x$. By assumption (d), $\mathcal{D}$ must be equal to the space of all $C^\infty$-vectors for the integrated representation on $\mathcal{H}$. Hence, the unitary equivalence maps $\mathcal{D}$ onto the Schwartz space $\mathcal{S}$ in $L^2(\mathbb{R})$. By Lemma 4.2, the automorphism $\alpha$ is implemented by a unitary which leaves $\mathcal{S}$ invariant. The corresponding unitary $U$ on $\mathcal{H}$ will then leave $\mathcal{D}$ invariant, and $\alpha(P^2 + Q^2) = U(P^2 + Q^2)U^{-1}$ is essentially self-adjoint on $\mathcal{D}$. 
Remark 5.3. Let $P, Q, \mathcal{D}, \mathcal{H}$ be as in Corollary 5.2, and assume just (a). Then $\mathcal{D}$ is contained in a unique space $\tilde{\mathcal{D}}$ which satisfies the same conditions, and is maximal in the sense of (d).

We leave details to the reader; or see, for example, [10, Section 7.A].

**Corollary 5.4.** Let $P, Q, \mathcal{D}, \mathcal{H}$ be a quadruple satisfying the assumptions in Corollary 5.2, except for condition (b). Let $A$ be a second order polynomial expression in the $P$'s and $Q$'s with coefficients specified as in (3). Assume that $A$ has a dense space of analytic vectors, and that the coefficients $\delta, \epsilon$ and $\zeta$ are real satisfying $4\delta\xi - \epsilon^2 > 0$.

Then it follows that the system is integrable.

**Proof.** The assumption on the coefficients $\delta, \epsilon, \xi$ allows us to determine a transformation $g \in G$, a number $\alpha \neq 0$, and an operator $H$ in the complex span of $\{P, Q, I\}$ such that

$$\text{Ad}(g)(A) = \alpha(P^2 + Q^2) + H.$$  

Note that the pair, $P_1 = \text{Ad}(g^{-1})P$ and $Q_1 = \text{Ad}(g^{-1})Q$, also satisfies the Heisenberg relation (a) in Corollary 5.2. We have viewed $\text{Ad}(g)$ (and, of course, $\text{Ad}(g^{-1})$) as an automorphism in the unital enveloping algebra on $P$ and $Q$. In view of formulas (5) and the conclusion from Example 4.3, the elements $P_1$ and $Q_1$ are also operators which are both defined on $\mathcal{D}$ and leave $\mathcal{D}$ invariant. We have the following operator-identity on $\mathcal{D}$:

$$\begin{align*}
a(P_1^2 + Q_1^2) &= \text{Ad}(g^{-1})(\alpha(P^2 + Q^2)) \\
&= \text{Ad}(g^{-1})(\text{Ad}(g)(A) - H) \\
&= A - \text{Ad}(g^{-1})H.
\end{align*}$$

In view of [14, Lemma 6.2], the element $P_1^2 + Q_1^2$ analytically dominates $\text{Ad}(g^{-1})H$. Since $A$ is assumed to have a dense space of analytic vectors in $\mathcal{D}$, it follows that analytic vectors for $A$ are also analytic for $P_1^2 + Q_1^2$. Nelson's Theorem [14, Theorem 5] now applies to the $P_1, Q_1$-system, and we conclude integrability for this system. Once integrability is established, it follows from Section 1 that $P^2 + Q^2$ also has analytic vectors, so the original $P, Q$-system is integrable as well.

**Remark 5.5.** The above result, Corollary 5.4, is best possible in a strong sense: If $A$ is a second order element as on (3) which is formally Hermitian, and if $4\delta\xi - \epsilon^2 \leq 0$, then the remaining conditions in Corollary 5.4 may be satisfied for nonintegrable $P, Q$-systems such that, nonetheless, $A$ is essentially self-adjoint on $\mathcal{D}$. Such examples have been constructed recently by Fuglede [33].
6. INNER DERIVATIONS OF $W$

In this section, we discuss the formula

$$\beta_t(B) = e^{t \text{ad } A}(B) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\text{ad } A)^n (B)$$

for elements $A$ in $W$ which are generally not nilpotent.

**Proposition 6.1.** Suppose $A$ is given by formula

$$A = x + \beta P + \gamma Q + \delta P^2 + \varepsilon PQ + \zeta Q^2$$

with complex coefficients. Then the power series (4) is convergent for all $B \in W$, and the $t$-dependence is entire analytic. If an automorphism is applied to the right hand side of (3), the resulting element $\tilde{A}$ has the same property.

**Proof.** The second part is immediate. To prove the first part, we note that the set of all elements $A$ as in (3) form a six-dimensional Lie algebra $g_6$ over $\mathbb{C}$. It is a semidirect product of a nilpotent and simple one, both of dimension 3, and it is contained in $W$. It follows that the complex enveloping algebra of $g_6$ coincides with $W$. Since the adjoint representation of any Lie algebra is known to exponentiate to an analytic one-parameter group of automorphisms of the enveloping algebra, the proposition follows by application of this result to $g_6$. Specifically,

$$e^{t \text{ad } A}(B) = \text{Ad}(\exp(tA))(B),$$

for $A \in g_6$ and $B \in W$. The properties of $\beta_t$ follow from elementary facts about the filtering of $W$, when it is regarded as the enveloping algebra of $g_6$.

**Remark 6.2.** The special case of $A$ in the proposition, when $\beta = \gamma = \varepsilon = 0$ and $\delta \neq 0 \neq \zeta$, is called semisimple. The algebraic classification of the semisimple elements is contained in [3].

It is not known whether the elements from Proposition 6.1 and Corollary 5.1 exhaust all possibilities for generators of one-parameter groups of automorphisms of $W$. We give two examples below which illustrate two types of singular behaviour for the expansion (4).

**Corollary 6.3.** Let $A$ be a formally Hermitian element in $W$, and assume that there is a automorphism of $W$ which maps $A$ into an expression of the form (3), i.e., a second degree polynomial in the noncommuting variables $P$ and $Q$. Then it follows that $A$ is essentially self-adjoint on the
Schwartz space $\mathcal{S}$, and the automorphism group $e^{it\text{ad }A}$ is implemented by the corresponding unitary one-parameter group.

**Proof.** The automorphism is implemented by a unitary $U$ which leaves $\mathcal{S}$ invariant by Lemma 4.2, and a formally Hermitian element of the form (3) must be essentially self-adjoint on $\mathcal{S}$ since the Hermite-functions are analytic vectors. Recall that by virtue of [14, Lemma 6.2], the operator $A = dR(X_0^2 + X_1^2 + X_2^2)$ analytically dominates every element of the form (3), and we noted in Section 1 that the Hermite-functions are analytic for $A$. In fact, a direct calculation yields $\Delta h_n = \left[-16 - 2\left[\frac{1}{4} + \frac{n^2}{2}\right]^2\right] h_n$ for $n = 0, 1, \ldots$.

To see that the automorphism group $e^{it\text{ad }A}$ is implemented by the unitary one-parameter group, $U(t) = \exp(itA)$, we refer to Lemma 3.1. By the assumption in Corollary 6.3, we have an automorphism $\alpha$ such that $\alpha(A)$ is $i = \sqrt{-1}$ times an element in the Lie algebra $g_\Sigma$; cf. (2), of the Shale-Weil representation. It follows that the automorphism $t \mapsto \exp(it\text{ad }\alpha(A))$ is implemented by the unitary one-parameter group, $R(\exp tX)$ if $\alpha(A) = -iX$, for $X \in g_\Sigma$. The essential part of Lemma 3.1 is used here. Since $\alpha$ is implemented by the unitary $U$, and $\exp(it\text{ad }\alpha(A)) = \alpha(\exp(it\text{ad }A))$, it follows that $\exp(it\text{ad }A)$ is implemented by the unitary one-parameter group with generator

$$A = -i\alpha^{-1}(X) = -i\text{Ad}(U^{-1})(X).$$

**Examples 6.4.** We shall now calculate the power series expansion in (4), $\beta_t(B) = e^{it\text{ad }A}(B)$, for different pairs of elements $A, B$ in the Weyl algebra:

(a) If $A = iPQP$, then

$$\beta_t(P) = \sum_{n=0}^{\infty} (-t)^n P^n + \frac{1}{t} = P(I + tP)^{-1},$$

where the last term is a purely formal expression. An easy way to verify this is to recall that $B(t) = \beta_t(P)$ is the solution to the Cauchy problem: $dB(t)/dt = i[PQP, B(t)]$, and $B(0) = P$. It follows that $\beta_t(P)$ is not in $W$ for any nonzero values of $t$. However, $B(t)$ is a well-defined bounded operator on $L^2(\mathbb{R})$ when $t \neq 0$, since then the function $b_t(\lambda) = \lambda(1 + t\lambda)^{-1}$ is defined and bounded, $\lambda \in \mathbb{R}$. Hence, $B(t) = \beta_t(P)$ is given by functional calculus.

(b) With $A = iPQP$, we have

$$\beta_t(Q) = Q + t(PQ + QP) + t^2PQP,$$

for all $t \in \mathbb{R}$. It follows that $\beta_t(Q) \in W$ for all $t$ and $Q \in N(A)$. In view of (a), we conclude that $N(A) \neq W$. 


For any element $A$ in $W$, we have $C[A] \subseteq C(A) \subseteq N(A) \subseteq W$, and the example $A = iPQ P$ shows that these inclusions may be strict. The last two inclusions are strict. But it follows from [3, Proposition 5.2] that $C[A] = C(A)$.


REFERENCES

16. N. S. Poulsen, On $C^\infty$-vectors and intertwining bilinear forms for representations of Lie groups, J. Funct. Anal. 9 (1972), 87–120.