Mobility Models based on Correlated Random Walks

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ABSTRACT
We provide an overview of models of vehicular motion that are based on continuous-time Markov chains. Of these models, we concentrate on the subset represented by correlated random walks because they are general enough to capture essential patterns of the mobility of vehicles and simple enough to allow the analytical study of special but still realistic cases. We review the analytical techniques available to obtain stochastic properties of correlated random walks in simple configurations and introduce the general problem of computing statistics of absorbing times. The numerical problem reduces to the solution of sparse linear systems for which we configure and evaluate an algebraic multi-grid technique. We apply the numerical method in a simple 2D example of a correlated random walk that models the mobility of vehicles on a grid of city streets. We consider different configurations of absorbing states and obtain approximations of the expected value of absorbing times for arbitrary initial conditions. The approach works directly with the analytic expression for expected values and thus does not rely on ergodic assumptions.

Categories and Subject Descriptors

Keywords
Mobility model, Correlated random walk, Algebraic multi-grid, Absorbing times

1. INTRODUCTION
Mobility models describe the location, velocity and other dynamic states of objects in motion. For many decades, the primary goal of these models was to accurately characterize the motion of particles in physical systems. The most prominent examples are: discrete-time, discrete-state Markovian random walks [7]; Gauss-Markov continuous-time, continuous-state Brownian motion [6] and Ornstein-Uhlenbeck processes [22]; and many variations/combinations of these processes, such as correlated random walks [8].

Significant recent activity in the development of mobility models has focused on applications in Mobile Ad hoc NETworks (MANETs). In these applications, the objects whose motions are being characterized represent users/devices that communicate wirelessly with each other and/or some fixed infrastructure as they move about [1]. Their mobility patterns typically play a significant role in determining the performance of the communication channels and protocols that are used. It is thus important for the mobility models to be able to mimic the aspects of users’ motions that have the greatest impact on performance; otherwise, the results of analysis and simulation studies will be misleading. Thus, the evaluation of MANET protocols requires a proper choice of the underlying mobility model.

In this paper we focus our attention on the mobility of vehicles in the streets of a city. Here, it becomes clear how the mobility model of vehicles is important for the design of MANETs as it determines mobility patterns such as the times between a vehicle’s contacts with other vehicles. These times have a significant impact on the network coverage and throughput-delay characteristics because in the absence of other infrastructure this is when most communication happens [2]. Thus, for this application it becomes useful to obtain the stochastic properties of such times. This analysis leads to the study of absorbing times in Markov chain models that represent the time needed for a vehicle to arrive at a certain destination, or the first time to reach some location, from either a random or a fixed starting point.

In a mobility model based on continuous-time Markov chains, the streets of a city are reduced to a grid where each point on the grid corresponds to an intersection of streets. Therefore, a vehicle moving in the city belongs to one point in the grid at a time and it jumps to other grid nodes at random times. The time instants at which the vehicle takes its steps are governed by a Poisson process, which means that it moves from one intersection to a neighboring intersection with a randomly chosen velocity. In the simplest scenario, the transitions occur only between neighboring grid nodes and thus are similar to those of a random walk.

In a Correlated Random Walk (CRW) mobility model, the
probability of a vehicle moving between grid nodes in the same direction as its previous step is different than the probability of moving in the opposite direction. Thus, a model based on correlated random walks is able to capture the directionality of motion. Such models thus more accurately describe mobility because they can account for time dependency (correlation of motion in time), geographical restrictions (correlation of motion in space), and nonzero drift [2].

Besides the benefits of using correlated random walk mobility models for the accurate description of mobility patterns, it is also possible to apply analytical techniques in simple cases to obtain explicit expressions for their stochastic properties. In [2] a matrix-analytic approach lead to a complete transient and steady state analysis of CRW mobility models on finite and semi-infinite grids and the derivation of the first-entry time distributions for 1-D CRW mobility models. Even for these simplified cases, the analytical results do not have a simple form and simulation techniques must be used in more complicated scenarios.

In this paper, we study the configuration of multi-grid methods for computing statistics of absorbing times in CRW mobility models. The statistics of absorbing times are difficult to obtain since they require the solution of sparse linear systems. The multi-grid algorithm represents an optimal choice for this purpose because its complexity scales optimally. Nevertheless, the configuration of parameters needs special care to optimize convergence. We try an algebraic multi-grid configuration that has a natural and efficient structure for this problem in a fairly simple scenario [15].

The structure of the paper is organized as follows. In Section 2, we review the analytical techniques available to obtain stochastic properties of correlated random walks in simple configurations and we introduce the general problem to compute statistics of absorbing times in a general scenario. In Section 3, we introduce a multi-grid method and its configuration for problems in correlated random walk mobility models. In Section 4, we show simulation results to obtain the statistics of absorbing times for different configurations of correlated random walk mobility models.

2. MOBILITY MODEL
In this section we review the definition and properties of correlated random walks mobility models.

2.1 Markov Chain Mobility Models and CRWs
The motion of a vehicle in a city is neither completely deterministic nor completely random. At one hand, someone unaware of a driver’s plan to arrive at a certain destination will consider each change of direction or speed as a random decision. And, at the other hand, even if a driver has a plan to arrive at a certain point and keep a certain speed, the traffic conditions will request to take random decisions. Thus, the natural randomness in the motion of vehicles requires the use of a probability model.

In a discrete description of a city, where only the intersections of streets are considered as possible locations, a Random Walk (RW) [7] would be the simplest model for the mobility of vehicles. In this model, the vehicles move with random speed since the transitions from one intersection to another occur at random times. The time when the steps occur is controlled by a Poisson process with rate $\lambda$ which represents the expected value of the speed. Each state of the continuous-time Markov chain corresponds to a geographical location and the transitions only occur between neighboring states. For example, a one-dimensional RW only connects a state with its left or right neighbor (also known as a death and birth process) as shown in Figure 1. Thus, in a RW the continuous-time Markov chain model incorporates the discretization of space and the notion of local interactions. The advantage of this model is its simplicity, which admits analytical results for many of its stochastic properties. Unfortunately, the model generates highly random motion because of the Markovian property, which makes each local decision independent of any other. This is seen in its sample paths, which would correspond to a vehicle that tends to be confined to small regions for long times. In practice, vehicles maintain motion in single direction for a number of transitions, depending on its unknown schedule or traffic conditions, before changing its direction randomly.

![Figure 1: State transition diagram of the continuous-time Markov chain representing a 1D random walk with reflecting boundary conditions.](image)

The reason the RW fails to generate the directed paths that one would expect in reality is because it does not include any notion of inertia within the model. The probability of a vehicle changing direction does not depend on its state of motion or, in other words, on the direction followed in its last step. On the other hand, the directionality of motion means that a vehicle is more likely to keep moving in a certain direction. The difficulty in accounting for this behavior is that a step taken at a given location has to be correlated with its previous step, which would violate the Markov property of the random walk model. In order to keep a Markov chain model and incorporate the directionality of motion, it is necessary to sacrifice the one-to-one correspondence of the states of the Markov chain and the geographical location of the vehicle (grid nodes).

In the Correlated Random Walks (CRW), which were initially studied as an extension of the RW for models of diffusion in turbulent media [8, 23], the correspondence between the states of the Markov chain and the location of the vehicle is many-to-one. For each grid node there are many states corresponding to the different directions in which a vehicle can move. We will consider a CRW model with the following rules:

- a vehicle takes a step in the same direction as its previous step with probability $p$, and
- it takes a step in the opposite direction with probability $q$. 
Note that, in general, \( p + q \leq 1 \) for a given direction of motion (e.g., a vehicle could move in orthogonal directions). In a one-dimensional problem, for each grid node there are two possible directions: left or right, as seen in Figure 2. Here, the vehicle takes a step in the same(opposite) direction as its previous step with probability \( p_1 \) or \( p_2 \) (\( q_1 = 1 - p_1 \) or \( q_2 = 1 - p_2 \)) depending on whether its previous step was to the right or left.

![Figure 2: State transition diagram of the continuous-time Markov chain representing a 1D correlated random walk with reflecting boundary conditions.](image)

In general, a CRW can account for many complicated scenarios. The RW is recovered when the sum of the probabilities to keep a direction from each grid node is equal to one (\( p_1 + p_2 = 1 \) in the 1D case) and therefore the CRW accounts for all the mobility behavior that a RW can model plus the directionality of motion. In particular, it can account for a notion of potential in which the vehicles are more likely to move to certain geographical region (\( p_1 \neq p_2 \) in the 1D case). We refer to this situation as an Asymmetric Correlated Random Walk (ACRW) and otherwise the model is referred as a symmetric CRW (sometimes called a persistent random walk [4]).

In [2] a CRW model was introduced to model the mobility of cars in a city. Here, the streets form a 2D grid as shown in Figure 3 and the parameters of the 2D CRW are given by a uniform rate \( \lambda \) (i.e. the average speed is constant) and the transition probabilities correspond to each possible direction. As shown in Figure 3, the grid structure allows four possible neighbors for each interior grid node: north-east (NE), north-west (NW), south-east (SE) and south-west (SW). For each grid node, there are also four possible directions of motion, leading to the four transition probabilities: forward (\( p \)), backward (\( q \)), left (\( l \)) and right (\( r \)). Therefore, in the most general scenario there could be as many as 16 different transition probabilities per grid node. In order to simplify the structure of the transition probabilities, the model in [2] considers a combination of two independent 1D CRWs moving in directions North/South and East/West with simultaneous steps. In this way, the parameters of the model are reduced to the uniform rate \( \lambda \) and only four transition probabilities: \( p_{NE}, p_{SE}, p_{E} \) and \( p_{W} \). Then, for example, the transition probabilities of a vehicle moving in direction NE are given by: \( p_{NE} = p_{N} p_{E} \) to move forward, \( q_{NE} = (1 - p_{N})(1 - p_{E}) \) to move back, \( l_{NE} = p_{N}(1 - p_{E}) \) to move left and \( r_{NE} = (1 - p_{N})p_{E} \) to move right. When reaching the boundaries, the transition probabilities of the 1D CRWs are adjusted to reflect back with probability one as shown for the 1D CRW in Figure 2. In this mobility model an ACRW is obtained if any of the parameters \( p_{N}, p_{S}, p_{E} \) and \( p_{W} \) are different from any other.

![Figure 3: Model of streets in a city. The horizontal and vertical motions of the node combine to create diagonal movements along the solid lines that are shown. These diagonal solid lines form the grid of city streets.](image)

2.2 Overview of Analytical Results for CRWs

In the context of continuous-time Markov chains, the probability of finding a vehicle at time \( t \) at grid node \( n \) with direction \( m \) is called the transient distribution of the chain, and it will be denoted by \( \pi_{n,m}(t) \). As a function of \( n \) the transient distribution is an infinite sequence of probabilities if the grid is infinite (sometimes referred as unbounded or unrestricted) or a vector of probabilities if the grid is finite. In many cases the transient distribution converges for large times to a sequence of probabilities that, if it exists, is called the steady-state distribution given by \( \pi_{n,m} = \lim_{t \to \infty} \pi_{n,m}(t) \). Research on CRWs has also considered the case of discrete-time CRWs where the times when the steps occur are deterministic and given by an integer \( k \). The distributions in this case can be denoted by \( \pi_{n,m}(k) \) for the transient distribution and \( \pi_{n,m} = \lim_{k \to \infty} \pi_{n,m}(k) \) for the steady-state distribution.

The transient distributions of a CRW lead to all the stochastic properties that we are interested in for the study of the mobility of vehicles. For the steady-state distribution the most interesting information are the statistics given by the moments of the distribution. The limitation of this study is that the information about specific time events is not available. The analysis of the transient distribution lead to important information about time events such as the first, second,..., \( j \text{th} \) passage times which represent the expected time to visit a grid node for the first or further times. Additional behavior appears if absorbing boundary conditions are included in the grid, which means that at certain states there is a nonzero probability that the vehicle will leave the grid (e.g., if a vehicle leaves the city or parks). In this case it is important to compute the absorbing times [5], which represent the expected time for a vehicle to be absorbed; to leave the grid. The study of first passage times and absorbing times are closely related: the first passage time to a grid...
node can be computed as an absorbing time if the node has probability one of leaving the grid.

In what appears to be the earliest work on CRWs, Goldstein [9] obtained steady-state distributions under various conditions for the discrete-time, one-dimensional CRW on the infinite grid \( n \in (-\infty, \infty) \). He also derived expressions for the moments and obtained an asymptotic estimate of the distribution function in terms of hyper-geometric functions. In [3, 10, 17, 18] different researchers have obtained generating functions or exact combinatorial expressions for the distribution of a discrete-time CRW on a one-dimensional grid that is either infinite or has one or two absorbing boundaries. Zhang [24] obtained explicit expressions for the absorbing probability and expected duration of the discrete-time ACRW in the presence of elastic or absorbing barriers. In [11, 20], the authors obtained generating functions of the probabilities for the time to the first and further passage to the origin or any other grid node in a discrete-time, infinite, ACRW. In [12], the authors proposed the use of transition probability matrices associated with discrete-time CRWs to obtain various probability distributions for CRWs. The authors in [10] used the matrix-geometric results [16] to obtain the transient and steady-state distributions for both an ACRW on the semi-infinite grid \( \mathbb{N} \) and a symmetric CRW on the infinite grid. They further obtained the probability of first passage to the boundaries at time \( k \) in a discrete-time CRW on a one-dimensional grid with two absorbing boundaries. Gillis [8] studied the discrete-time CRW on a \( d \)-dimensional grid in which the vehicle moves in the same direction with probability \( p \), in the opposite direction with probability \( q \), and in one of the orthogonal directions with probability \( r \) such that \( p + q + 2(d - 1)r = 1 \). He obtained a generating function for the probability that the vehicle is at a particular grid node at the \( n^{th} \) step and the probability of return to the origin at the \( n^{th} \) step for the case \( p = r \).

In the latest research to our knowledge, the authors in [2] used a matrix-analytic approach [16] to obtain the transient and steady-state distributions for both an ACRW on the semi-infinite grid \( n \in [0, \infty) \) and a symmetric CRW on the finite grid \( n \in [0, N] \). They also obtained the distribution of the first passage time for any of the previous cases. The results for transient and steady-state distribution can be extended to the 2D CRW introduced in section 2.1 by combining the results of two independent 1D CRWs. The results on first passage time were not obtained, so simulation techniques were used to study those cases.

It is important to note that even for the simplified scenarios where analytic results can be obtained, the results do not have a simple form and even their numerical evaluation could be challenging. For example, the transient distribution of an ACRW on the semi-infinite grid \( n \in [0, \infty) \) is obtained from a recursion that involves several continuous-time convolutions of Bessel functions. Similar results for a symmetric CRW in a finite grid \( n \in [0, N] \) cannot be obtained as close analytic expressions in time-domain and need the numerical inversion of Laplace transforms [2].

### 2.3 Statistics of Absorbing Times

In a mobility model of cars in a city an absorbing node is a node that is not connected to the grid. Thus, if a vehicle jumps into an absorbing node then it will disappear from the grid. In applications this could be interpreted in different ways, for instance an absorbing node could be a parking space or an exit from the city. Also, a grid node can be artificially treated as an absorbing node to study the time of first-passage to it. This is of particular interest in the study of MANETs since a set of absorbing nodes can be interpreted as the connectivity region where vehicles can communicate with each other. In this case, the time it gets to enter the connectivity region is in direct relation with the throughput-delay characteristics of the network.

Given a vehicle with some initial condition (location and direction) the time spent on the grid before disappearing in one of the absorbing nodes is called the absorbing time [5]. Since the absorbing nodes do not belong to the grid, it is convenient to aggregate all of them in a single absorbing node. Then, in a grid with \( N \) possible states and one absorbing node the transition generation matrix can be written as:

\[
Q = \begin{bmatrix}
S & \mathbf{s} \\
0^T & 0
\end{bmatrix},
\]

where \( S \in \mathbb{R}^{N \times N} \) considers all the possible states and excludes the absorbing node, \( \mathbf{s} \) is a vector of zeros and \( s \in \mathbb{R}^N \) is such that \( S1 + s = 0 \) where \( \mathbf{1} \in \mathbb{R}^N \) is a vector of ones. The absorbing time is then a random process with parameters \( S \), called the sub-generator [16], and a vector of probabilities \( \alpha \in \mathbb{R}^N \) of being at any initial state. The distribution associated with absorbing times is also called a phase-type distribution and are important in the study of continuous-time Markov chains since they can be used to approximate any positive-value distribution [16].

The moments of absorbing times can be expressed in closed form as a function of its parameters \( S \) and \( \alpha \). This is, given an absorbing time \( T \in \mathbb{R}_+ \) then \( E[T^n] = (-1)^n n! \alpha^T S^{-n} \mathbf{1} \) for \( n > 0 \). The expected value of an absorbing time is then given by \( E[T] = -\alpha^T S^{-1} \mathbf{1} \), and similarly \( \text{Var}[T] = 2\alpha^T S^{-2} \mathbf{1} - (\alpha^T S^{-1} \mathbf{1})^2 \). The difficulty to compute these statistics is that all of them depend on the inverse of the sub-generator \( S \). Since \( S \) is a sparse matrix, the problem reduces to the solution of sparse linear systems. In particular, we will focus our attention in the numerical solution of the system

\[
-S\alpha = \mathbf{1}
\]

where the components of the unknown vector \( t \in \mathbb{R}^N \) represent the expected value \( t_k = E[T_k] \) of the absorbing time with initial condition \( \alpha_i = \delta_{ik}, i = 1, \ldots, N \). A vector of variances \( \nu \in \mathbb{R}^N \) can be obtained similarly by solving for \( y \) in the system \(-Sy = 2t \) (which has the same system matrix and different right hand side) and then \( \nu_k = y_k - t_k^2 \).

### 3. NUMERICAL METHODS

The numerical problem to solve is the solution of a sparse linear system where the system matrix is \( A = -S \). The matrix \( A \) is sparse, its diagonal values are all positive and equal to \( \lambda \), and its off-diagonals are all negative and equal to \(-\lambda \rho_{ij} \) where \( \rho_{ij} \) is the transition probability between states \( i \) and \( j \). In the 2D grid introduced in section 2.1 a vehicle in a given state can jump at most to 4 other states and therefore each row of the matrix \( A \) has at most 4 nonzero elements. The system matrix is also positive-definite, which
is justified by the fact that the expected values of absorbing times have to be positive. Thus, the system matrix $A$ fulfills the conditions of an M-matrix (positive definite, positive diagonal entries and negative off-diagonal entries).

For the solution of the linear system we have decided to use multi-grid methods. This choice is justified by the efficiency of the algorithm and the structure of the system. Multi-grid methods are among the fastest methods for solving sparse linear systems and the M-matrix structure of the system matrix is well suited for the Algebraic Multi-grid (AMG) algorithm [21]. We also consider this approach because the hierarchical structure of these methods is well-suited for potential applications in distributed networks. Multi-grid methods also show optimal scalability since the number of computations carried out by the algorithm increases as a linear function of the number of nodes in the system. On the other hand, the main drawback of multi-grid methods is the lack of a strong convergence theory for many practical systems. This is how techniques such as AMG rely on heuristic assumptions that make it difficult to study the convergence properties of the algorithm in most practical systems.

The purpose of the multi-grid algorithm is to solve a discrete linear system of the form $Au = f$. For a general problem, the system can always be associated with a graph in which the unknowns of the system are the nodes of the graph. In the context of multi-grid methods, the nodes of the graph are associated with a set of labels $\Omega$ called a grid. In a CRW mobility model, we will call $\Omega$ an algebraic grid in order to differentiate it from the geometric grid formed by the set of intersections of streets. In the 2D mobility model mentioned in section 2.1, each geometric grid node (intersection of streets) is associated with 4 algebraic grid nodes (vehicle moving in 4 possible directions). The algebraic grid associated with the original problem is referred to as the fine grid and is denoted by $\Omega_1$. Algebraic coarse grids $\Omega_j$, $j > 1$, are defined as sets of nodes such that $\Omega_j \subset \ldots \subset \Omega_2 \subset \Omega_1$.

The most common multi-grid scheme is called the full multi-grid and it combines a scheme to determine an initial approximation, called nested iteration, and an iterative scheme to improve the accuracy of the approximation, called the correction scheme. The algorithm uses linear transformations to transfer information among different grids. These transformations are called the inter-grid operators: the interpolation operator transfers information from coarse to fine grids and the restriction operator transfers information from fine to coarse grids. Figure 4 shows a diagram of how the full multi-grid algorithm transfers information between different grids. It starts transferring the source vector $f$ from the fine grid down to the coarsest grid; then it obtains an initial approximation for problems at finer grids and it corrects the approximation by using $\nu_1$ and $\nu_2$ smoothing iterations (such as Gauss-Seidel, Jacobi, Richardson, etc.) and solving an equation for the approximation error in the coarse grids. The configuration of the algorithm has to determine: the selection of nodes to coarse grids, the weights for the inter-grid operators, the smoothing operators and the number of smoothing iterations.

In a so-called geometric multi-grid algorithm the inter-grid operators are determined based on the geometry of the discretization whereas the algebraic multi-grid algorithm automatically configures these matrices based on a heuristic definition of smoothing error. The most standard algorithm obtained by the algebraic approach is the Ruge Stüben Algebraic Multi-grid (RS-AMG) [19]. The heuristics of RS-AMG are based on a notion of strong dependence between grid nodes. Given a threshold value $0 < \theta < 1$, an algebraic grid node $i$ is said to depend strongly on a node $j$ if $-A_{ij} \geq \theta \max_{k \neq i} \{-A_{ik}\}$. From this definition, the automatic configuration follows. Thus, the parameter $\theta$ determines the configuration of inter-grid operators in the RS-AMG algorithm.

The assumptions of RS-AMG are fulfilled by the system matrix in the computation of absorbing times in CRW mobility models. Nevertheless, the algorithm is not guaranteed to work efficiently and it can become hard to configure the parameter $\theta$ in some cases. In section 4 we will show situations in which RS-AMG cannot be configured efficiently. The problem then is that the use of heuristics in AMG methods makes it difficult to apply strong convergence properties. In certain cases of geometric multi-grid algorithms, Fourier analysis can be applied and the configuration of multi-grid algorithms can be based on strong convergence results. Fourier analysis applied to the configuration of multi-grid parameters is known as Local Fourier Analysis (LFA). When the grid nodes do not represent spatial locations (algebraic grid nodes), the concepts of high- and low-frequency components of the error need to be defined. This is done by the heuristics of AMG methods, but LFA is still limited to cases where the row elements along the diagonal are constant – which is not the case for CRW mobility models. The alternative is to use an extension of LFA called LFA for systems of PDEs [21] or the more general extension of LFA for non-geometric problems introduced in [15]. In [14], we used the results in [15] to analyze CRW mobility models in simple cases. For a 1D problem, such as the one shown in Figure 2, the RS-AMG method can be studied using strong convergence results and it shows it to validate the following assumptions: $\nu_1 \geq 1$ and $\nu_2 \geq 1$.

Figure 4: Diagram of the Full Multi-grid algorithm using one iteration of the correction scheme per level. Each box represents a number of pre or post smoothing iterations.
be more efficient than a different configuration. In this paper, we focus on more realistic cases to test the performance of RS-AMG in CRW mobility models.

4. SIMULATIONS

We simulate a CRW mobility model in a 2D grid of size 50 × 50, as described in section 2.1. The continuous-time Markov chain is controlled by a Poisson process of rate \( \lambda = 10 \) [1/min], which means that a vehicle moves at an average speed of 10 [blocks/min]. As described in section 2.1, the remaining parameters are the set of transition probabilities: \( p_N, p_S, p_E \) and \( p_W \). Then, we consider three configurations as follows:

- a Random Walk (RW) model in which \( p_N = p_S = p_E = p_W = 0.5 \),
- a symmetric Correlated Random Walk (CRW) model in which \( p_N = p_S = p_E = p_W = 0.9 \), and
- an Asymmetric Correlated Random Walk (ACRW) model in which \( p_N = p_S = 0.9 \) and \( p_E = p_W = 0.7 \).

We use our own implementation of RS-AMG initially developed in [13] configured with: Gauss-Seidel smoothing iterations, \( \nu_1 = 2 \) and \( \nu_2 = 2 \), \( \theta = 0.01 \), and coarser grids are created until the size of the algebraic grid has less than 300 nodes or if the reduction of nodes by AMG is less than 10%. At each grid level, the system matrix is obtained using the Galerkin condition [21] and preconditioned by dividing each row by the magnitude of the diagonal element. At the coarsest level, we use a direct solver that uses a sparse LU factorization without reordering of nodes.

In Figure 5 we show the expected value of absorbing times for different absorbing regions and initial conditions at each location of the grid and all initial directions equally likely. In Figures 5(a), 5(b) and 5(c) we consider one absorbing region of size 7 × 7 at the center of the grid. When comparing the RW and CRW models we can see that the smaller absorbing times are more concentrated next to the absorbing region for the CRW model than for the RW model. At the same time the maximum absorbing time is much smaller for the CRW model than for the RW model. In the ACRW model the vehicles starting at the SW region are absorbed faster than those starting at other regions of the grid because of the asymmetry that attracts them to the NE corner. The expected value of absorbing time increases significantly with this asymmetry, and normally it takes hours or days for vehicles to reach the absorbing region. Similar simulations in [2] that generate sample paths of the vehicles show that they normally get stuck in the NE corner for long times. Then, even if a vehicle starts at the SW region and is moving in the NE direction (see Figure 6(a)), the unlikely paths in which the vehicles deviate from their primary direction and end in the NE corner make the major contributions to the expected value of absorbing times. In this case, it is more convenient to use the present technique to compute statistics of absorbing times because the numerical method does not depend on the nature of sample paths. Otherwise special techniques such as Monte Carlo simulations would be needed [5]. The problem with the asymmetry disappears if we consider three absorbing regions in which one of them is located near the NE corner, as seen in Figures 5(d), 5(e) and 5(f). The absorbing times for particular initial conditions are shown in Figure 6.

The RS-AMG algorithm is very stable and fast for all the models except the CRW and ACRW with one obstacle at the center of the grid. The parameter \( \theta = 0.01 \) was adjusted to efficiently reduce the size of the grids and to reach fast convergence. For the CRW and ACRW models with one obstacle at the center of the grid, it does not seem possible to find a value of \( \theta \) that reduces the size of the grids such that the total number of grids is \( O(\log N) \) (less than 10 coarse grids for this system) and such that the algorithm uses \( O(1) \) iterations per level. If these conditions are not fulfilled, the algorithm looses its optimal complexity and scalability.

5. CONCLUSIONS

In this paper we have reviewed the research on continuous-time Markov chain mobility models based on correlated random walks. In simple but still realistic cases, it is possible to obtain analytic expressions for the stochastic properties but they become complicated to analyze and evaluate in even simple scenarios. For the computation of statistics associated with absorbing times or times of first-passage, we propose to use multi-grid methods to directly obtain the expected values from analytic expressions. This procedure simplifies the use of application-specific techniques based on Monte Carlo simulations. The RS-AMG algorithm has been found to be stable and fast for most cases, but still fails to obtain fast solutions in certain scenarios.

6. REFERENCES


Figure 5: Expected time of absorption of vehicles moving according to a RW ($p_N = p_S = p_E = p_W = 0.5$), CRW ($p_N = p_S = p_E = p_W = 0.9$), ACRW ($p_N = p_S = 0.9, p_E = p_W = 0.7$) in a $50 \times 50$ grid with reflecting boundary conditions and one or three absorbing regions of size $7 \times 7$. The locations that the vehicles cannot reach (forming a checkerboard pattern) and the absorbing regions are shown in white. The gray intensities are proportional to the expected time of absorption for each initial location and all possible initial directions are equally likely. The intensities are normalized so that a black intensity corresponds to the minimum absorbing time and white intensity corresponds to the maximum absorbing time. The magnitude of the minimum and maximum absorbing times are indicated at each label.

Figure 6: Expected time of absorption of vehicles moving according to a RW ($p_N = p_S = p_E = p_W = 0.5$), CRW ($p_N = p_S = p_E = p_W = 0.9$), ACRW ($p_N = p_S = 0.9$, $p_E = p_W = 0.7$) in a $50 \times 50$ grid with reflecting boundary conditions and one or three absorbing regions of size $7 \times 7$. The locations that the vehicles cannot reach (forming a checkerboard pattern) and the absorbing regions are shown in white. The gray intensities are proportional to the expected time of absorption for each initial location and for initial direction according to the labels. The intensities are normalized so that a black intensity corresponds to the minimum absorbing time and white intensity corresponds to the maximum absorbing time. The magnitude of the minimum and maximum absorbing times are indicated at each label.