A note on weak Sidon sequences

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Dedicated to Professors Brian Alspach and Bernt Lindström on the occasions of their milestone birthdays in 2003

Abstract

A sequence \((a_i)\) of integers is weak Sidon or well-spread if the sums \(a_i + a_j\), for \(i < j\), are all different. Let \(f(N)\) denote the maximum integer \(n\) for which there exists a weak Sidon sequence \(0 \leq a_1 < \cdots < a_n \leq N\). Using an idea of Lindström [An inequality for \(B_2\)-sequences, J. Combin. Theory 6 (1969) 211–212], we offer an alternate proof that \(f(N) < N^{1/2} + O(N^{1/4})\), an inequality due to Ruzsa [Solving a linear equation in a set of integers I, Acta. Arith. 65 (1993) 259–283]. The present proof improves Ruzsa’s bound by decreasing the implicit constant, essentially from 4 to \(\sqrt{3}\).

Keywords: Weak Sidon; Well-spread

A sequence \((a_i)\) of integers is well-spread (resp. Sidon) if the sums \(a_i + a_j\), for \(i < j\) (resp. \(i \leq j\)), are all different. Such sequences, especially Sidon sequences, have received considerable attention since Erdős and Turán [2] initiated their study in 1941; see, e.g., [8]. Kotzig [5] suggested the term ‘well-spread’—‘weak Sidon’ is a common synonym—but obtaining this reference requires some digging; [7] covers the highlights. For a nonnegative integer \(N\), let \(f(N)\) denote the maximum integer \(n\) for which there exists a well-spread sequence \(0 \leq a_1 < \cdots < a_n \leq N\). Our purpose is to present an alternate proof of the following result of Ruzsa [9].

Theorem. \(f(N) < N^{1/2} + O(N^{1/4}).\)

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After the proof, we indicate how our approach improves Ruzsa’s bound. We begin with a cruder estimate:

**Lemma.** If $N$ is sufficiently large, then $f(N) < 2.001N^{1/2}$.

**Proof.** Let $n := f(N)$ and $0 \leq a_1 < \cdots < a_n \leq N$ be a well-spread sequence. Since the sums $a_i + a_j$, for $i \neq j$, are distinct and lie in the set $\{1, 2, \ldots, 2N - 1\}$, we have $\binom{n}{2} < 2N$, from which the assertion follows easily. □

**Proof of Theorem.** Let $N$ be large enough to invoke the lemma, set $n := f(N)$, and consider a well-spread sequence $0 \leq a_1 < \cdots < a_n \leq N$. The key is to study the positive differences $a_j - a_i$. By obtaining both upper and lower bounds for the sum of a certain subset of these differences—the ‘small’ ones—we shall deduce the desired bound.

Following [6], for $1 \leq i < j \leq n$, we call $j - i$ the order of the difference $a_j - a_i$. Since the differences of order $\alpha > 0$ can be arranged into sequences of the form

$$a_\alpha - a_\beta, a_\beta - a_\gamma, a_\gamma - a_\delta, \ldots,$$

where $\alpha - \beta = \beta - \gamma = \gamma - \delta = \cdots = v$, by ‘telescoping’, we see that the sum of all these differences is at most $vN$ (and less than $vN$ for $v > 1$). Thus, for $m \geq 2$, the sum $\mathcal{S}$ of all the positive differences of order at most $m$ is less than $m(m + 1)N/2$.

We call $a_i$ a mean-point if $2a_i = a_j + a_k$ for some $j, k \in \{1, \ldots, n\}$; notice that then $a_i - a_k = a_j - a_i$. Except for the values $a_j - a_i$, for mean points $a_i$ (or $a_j$), the differences $a_k - a_\ell$, for $1 \leq \ell < k \leq n$, are all distinct since $(a_i)$ is well-spread. Now the only candidates for mean-points are $a_2, \ldots, a_{n-1}$, so we have at most $t := n - 2$ differences occurring with higher multiplicity, and the well-spread property implies that this multiplicity is 2. If $1 \leq m < n$ and $s := n - (m + 1)/2$, then the number of positive differences of order at most $m$ is $mn - m(m + 1)/2 = ms$. Thus,

$$\mathcal{S} \geq \sum_{i=1}^{t} 2i + \sum_{j=1}^{ms-2t} (t + j) = \frac{ms(ms + 1)}{2} - t(ms - t).$$

For $1 < m < n$, it follows that

$$\frac{ms(ms + 1)}{2} - t(ms - t) < \frac{m(m + 1)}{2} N,$$

so that

$$\frac{(ms)^2}{2} < \frac{m(m + 1)}{2} N + mst.$$

Since $s, t < n$, the second term on the right side is less than $mn^2$, which by the lemma is at most $(2.001)^2 mN < 4.5mN$. Thus, $s^2 < N(1 + 10/m)$, and since $(1 + x)^{1/2} < 1 + x/2$
for $x = 10/m$, we have

$$n = \frac{m + 1}{2} + s < \frac{m + 1}{2} + N^{1/2} \left(1 + \frac{5}{m}\right).$$

(1)

With $m := \lceil N^{1/4} \rceil$, this gives the bound in the statement of the theorem. □

Closing remarks. Our proof uses the main idea of Lindström [6], as adapted to well-spread, constant-parity sequences in [4]. Ruzsa [9] also based his proof on the idea of studying the ‘small’ differences $a_j - a_i$ in a “somewhat hidden” fashion (his quote). Here we compare the resulting implicit constants.

To optimize ours, we first perform another iteration of the proof. Instead of applying the lemma (to bound $mn^2$ from above), we apply the theorem itself. This allows us to replace ‘10’ by ‘$3 + O(N^{-1/4})$’. To minimize the right side of (the adjusted) inequality (1), we now choose $m$ to be $\lceil cN^{1/4} \rceil$, for $c := \sqrt{3}$. These modifications reduce our upper bound on $f(N)$ to $N^{1/2} + cN^{1/4} + O(1)$. Ruzsa’s proof essentially delivers the value 4 in place of our $\sqrt{3}$: he shows that a weak Sidon sequence contained in the set $\{1, \ldots, N\}$ contains at most $N^{1/2} + 4N^{1/4} + 11$ terms. Thus, aside from being more transparent, our proof yields an (however slight) improvement to the bound.

It should be noted that the theorem compares favourably with the best-known lower bounds for $f(N)$. Using Singer difference sets (see [10]), it is easy to show that $f(N) > N^{1/2}$ for infinitely many integers $N$; additionally, prime density results (e.g. [1]) imply that $f(N) > N^{1/2} - N^{21/80}$ if $N$ is sufficiently large.

Finally, we add a word on an application. In [4], our present theorem was used to determine the growth rate of the maximum label $A(n)$ in a ‘most-efficient’ metric, injective edge-labelling of the complete graph $K_n$ for which every Hamilton cycle has identical length. We proved that $2n^2 - O(n^{5/2}) < A(n) < 2n^2 + O(n^{61/40})$, thus settling the main conjecture in [3].

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References